Bachelor Thesis Project/ Term Project

(Mathematics, Computational Science & Engineering)

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Dispersion Analysis of Higher Order Schemes for the 1D Wave Equation

1 Dispersion Analysis

We consider the Cauchy problem from the linear wave equation in one spatial dimension

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{on } \mathbb{R}\times]0,T[\ , \tag{1}$$

$$u(x,0) = u_0(x) \quad , \quad \frac{\partial u}{\partial t}(x,0) = v_0(x)\ , \quad x \in \mathbb{R}\ . \tag{2}$$

The spatial discretization of (1) is done by means of Lagrangian finite elements of degree p on an infinite equidistant spatial grid $\mathcal{M} := \{[(j-1)\Delta x, j\Delta x],\ j \in \mathbb{Z}\}$ with meshwidth $\Delta x > 0$. Temporal discretization will rely on explicit symplectic methods with unform timestep Δt. This leads to a fully discrete evolution problem.

Each cell of $\mathcal{M}$ "owns" p degrees of freedom corresponding to local shape functions. Let $\vec{\zeta}_j^{(k)}$ stand for the coefficient vector associated with cell $](j-1)\Delta x, j\Delta x[$. In order to investigate numerical dispersion, we use the "discrete plane wave" trial expression

$$\vec{\eta}_j^{(k)} = \vec{\xi} \exp(i(kj\Delta x - \omega k \Delta t))\ , \quad \vec{\zeta} \in \mathbb{C}^p,\ k \in \mathbb{R},\ \omega \in \mathbb{R}\ . \tag{3}$$

The requirement that (3) solves the discrete evolution yields an equation linking k and ω, the *discrete dispersion relation*, see [4, Sect. 1.9].

2 Task

1. Derive the discrete discpersion relation for linear finite elements and leapfrog timestepping and discuss its asymptotics for $\Delta t \to 0$, $\Delta x \to 0$.

2. Find the discrete dispersion relation for arbitrary p and simple leapfrog timestepping.

3. Determine the discrete dispersion relation for higher order symplectic timestepping schemes when used with spatial discretization of the same order.

4. Investigate the asymptotics, if possible.

In each case MATLAB can be used to obtain plots of the discrete dispersion curves $\omega = \omega(k)$.

3 Sources of Information

- [1]: dispersion analysis in frequency domain, study of asymptotics
- [2]: dispersion analysis of spatial semidiscretization
- [3]: introduction and detailed discussion of dispersion analysis
- [5]: symplectic higher order timestepping

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References

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