Maxwell Equations on S^2

A Bachelor's Thesis on Boundary Element Equations for Maxwell-Type Problems on the Sphere

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Abstract

This bachelor's thesis is concerned with finding boundary element methods for the solution of the following Maxwell-type problem

$$(\delta d - k^2)u = 0 \tag{0.1}$$

on the sphere S^2 where u is a zero or one form defined on a simply connected $\Omega \subset S^2$ and the boundary conditions are given on the boundary Γ of a simply connected $C \subset S^2$. This is a generalization of rot rot- and grad div-type problems in a differential geometric setting.
The flat ($\mathbb{R}^n$) case has already been discussed in [5].

To do this, first a Green's function for the Helmholtz problem is obtained

$$(\triangle - k^2)u = 0 \tag{0.2}$$

with $\triangle = \delta d + d\delta$ the Hodge Laplacian.

This Green's function (a Green's double form) is then used to construct a single layer potential for the Maxwell-type problem. With help of this single layer potential and a discretization of the appropriate function spaces, discretized boundary integral equations for the Maxwell-type problem are obtained.

Acknowledgements

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Contents

1 Introduction

The intent of this bachelor's thesis is to numerically solve the following problem

$$(\delta d - k^2)u = 0 \qquad \text{in } \Omega \subseteq S^2 \tag{1.1}$$

where u is a differential form, on the unit sphere S^2 with given Dirichlet boundary conditions on the boundary Γ of a simply connected $C \subset S^2$. An example of the geometry can be seen in Figure 1.1.

This problem is a generalization of a range of equations from electrodynamics. On one hand, if ω is a 1-form representing a vector, (1.1) generalizes the following problem from magnetics, with A as the vector potential:

$$(\operatorname{rot}\operatorname{rot} + k^2)A = 0 \tag{1.2}$$

if we use the relation $\operatorname{rot}\omega = \star d\omega$ (found for example in [7, p.22]).
On the other hand, if ω is a 0-form (i.e. a function), (1.1) generalizes the following electric potential problem, where V is the electric potential:

$$(\operatorname{grad}\operatorname{div} + k^2)V = 0 \tag{1.3}$$

if we use the relations $\operatorname{grad} f = df$, $\operatorname{div} X = \sqrt{\det g}^{-1}\partial_i(X^i\sqrt{\det g})$ and $\delta\omega = \sqrt{\det g}^{-1}\partial_i(g^{ij}\sqrt{\det g}\,\omega_j)$ (see [7, pp. 17ff]).

This problem will be approached by applying the differential geometric ideas from [5] to S^2 which has nontrivial curvature. The final goal is obtaining boundary integral equations which enable a solution via boundary element methods.
First, Green's fundamental solutions will be developed for the similar Helmholtz-type equation $(\triangle - k^2)\omega = 0$ in the case of 0- and 1-forms, then with help of a modified lemma from [5] boundary integral equations for (1.1) are obtained.

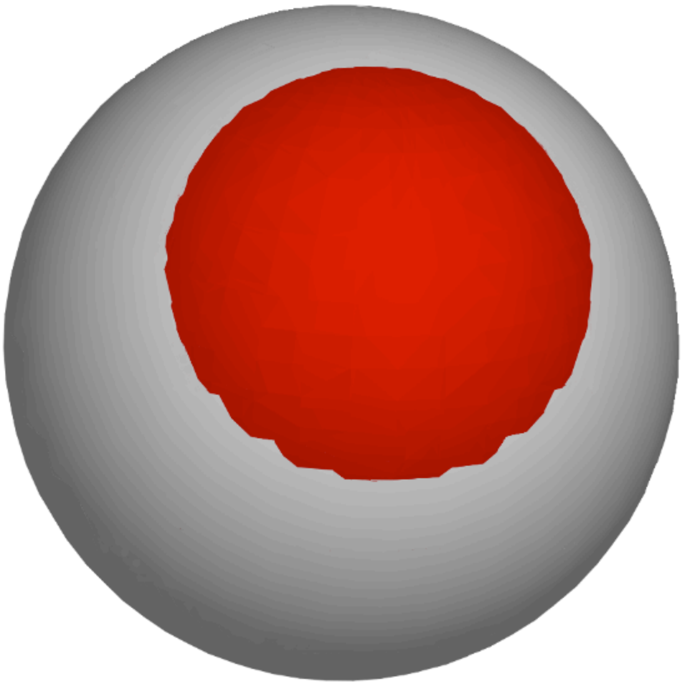

Figure 1.1: Example of the situation described above with Ω the red region and C the complement of Ω. Γ is the boundary of C.

2 Definitions

2.1 Basic Definitions

In this chapter the necessary definitions and notations are explained.

S^2 is understood as the unit sphere with its standard topology, atlas and metric coming from its canonical embedding into $\mathbb{R}^3$. If not further specified the standard coordinates on S^2 are given by φ, the longitude and θ the colatitude. This gives us the standard metric $g = \sin^2\theta\, d\varphi^2 + d\theta^2$.

The tangent space of a manifold M at a point $x \in M$ will be denoted by T_xM, the tangent bundle over M is TM. The space of p-forms over the manifold M is given by $\Omega^p(M)$.

Vectors and one-forms are related to each other via the metric. The metric g induces the musical isomorphisms $\sharp$ and $\flat$ such that the following holds:

$$\begin{aligned} A^\flat(\cdot) &= g(A, \cdot) \\ \omega(\cdot) &= g(\omega^\sharp, \cdot) \end{aligned} \tag{2.1}$$

for A a vector and ω a 1-form. This naturally extends to p-forms and p-vectors (the theory with $\sharp$ and $\flat$ is taken from [6, p.342], notation is inspired from [2]).

The metric also induces an L^2 product on forms on the manifold M which will be denoted by the following:

$$\langle \omega, \eta \rangle_M := \int_M g(\omega^\sharp, \eta^\sharp) \tag{2.2}$$

Sometimes, to avoid heavy usage of musical symbols, $\hat{g}(\cdot,\cdot) := g(\cdot^\sharp, \cdot^\sharp)$ will be used.

$\star$ denotes the Hodge star on manifolds induced by the metric. $d : \Omega^p(M) \to \Omega^{p+1}(M)$ denotes the exterior derivative, $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$ denotes the codifferential induced by the Hodge star. Note that d and δ are adjoint in the L^2 product in the sense that $\langle d\cdot, \cdot\rangle_M = \langle \cdot, \delta\cdot\rangle_M$.
$\triangle_p = d\delta + \delta d$ denotes the Hodge Laplacian acting on p-forms.

2.2 Sobolev Spaces

$H^k(U)$ denotes the L^2-Sobolev space of k-th order on a domain U, $H^{-k}(U)$ denotes its dual. $H^k\Omega^p(U)$ denotes the space of p-forms on the domain U with component functions in $H^k(U)$.

$H^k\Omega^p(d,U)$ denotes the space of p-forms on the domain U that are in $H^k\Omega^p(U)$ and have an exterior derivative in $H^k\Omega^p(U)$. $H^k\Omega^p(\delta,U)$ and $H^k\Omega^p(\delta d,U)$ are defined analogously.

$H_{\|}^{\frac{1}{2}}\Omega^p(\partial U)$ is defined as the trace space of $H^1\Omega^p(U)$ where ∂U is the boundary of U. $H_{\perp}^{\pm\frac{1}{2}}\Omega^p(\partial U)$ is defined as the hodge star of the space $H_{\|}^{\pm\frac{1}{2}}\Omega^{q-1}(\partial U)$ where q is the number adjoint to p in terms of the Hodge star.

All these definitions can be found (with slightly different notation) in [5].

2.3 Double Forms

A double p-form on the tangent spaces T_xM, T_yN is a map D of the following form:

$$D : (T_xM)^p \times (T_yN)^p \to \mathbb{R} \tag{2.3}$$

that is bilinear and alternating in $(T_xM)^p$, $(T_yN)^p$. This definition is motivated by [5, pp.13ff] This double form induces an operator on p-forms by the following identification;

$$\begin{aligned} D : \Omega^p(N) &\to \Omega^p(M) \\ \omega &\mapsto D[\cdot, \omega^\sharp] \end{aligned} \tag{2.4}$$

The operator notation and the double form notation will be used side by side and without further comment.[1]
By convention, the L^2 product defined in (2.2) is evaluated in the second part of the two-form, i.e. $\langle D, \eta\rangle_M(v_x) := g(D[v_x, \cdot]^\sharp, \eta^\sharp)$ is a p-form.

The identity double p-form I_p is defined as a double form such that the following holds:

$$\begin{aligned} I_p : (T_xM)^p &\times (T_yN)^p \to \mathbb{R} \\ I_p(P_y^x v) &= v^\flat \qquad \text{or equivalently} \\ I_p[u_x, v_y] &= g(u_x, P_x^y v_y) \end{aligned} \tag{2.5}$$

where P_x^y is the parallel transport from y to x along minimizing geodesics. In general one may have to worry about well-definedness, but on S^2 this is not an issue.

[1] D can also be turned into an operator by plugging $\omega^\sharp$ into its first slot, however this convention will only be used where specifically indicated

3 Solving the Helmholtz Equation

3.1 Theory

This section draws heavily on unpublished notes form S. Kurz and on his work with B. Auchmann [5].

Solving (1.1) will happen via solving a different problem first, the Helmholtz equation. The Helmholtz equation is defined as follows:

$$(\triangle_p - k^2)u = 0 \qquad \text{on } \Omega \subseteq S^2 \tag{3.1}$$

where u is a differential form, on the unit sphere S^2, boundary conditions given on Γ as in the introduction.

To construct boundary integral equations a Green's function is needed. Similar to [5] p.15 an ansatz is used where the Green's function is a double form:

$$\begin{aligned} G_p(x,y) &:= w_p(s(x,y))I \\ (\triangle_p - k^2)G_p(x,y) &= \delta_y(x)I_p \end{aligned} \tag{3.2}$$

where $s(x,y)$ is the geodesic distance between x and y, $\delta_y(x)$ is the Dirac delta function and the double form is interpreted as an operator as in (2.4). This ansatz makes sense, as in the flat case there is the same ansatz in [5], just with the standard euclidean distance instead of the geodesic distance.

Finding the Green's function is done using theory from [3]. Lemma 1 from p.111 states, with $\Sigma = S^2$:

$$\triangle_p \int_0^\pi w_p(s)(S_s + \hat{S}_s)\beta \, d\mu(s) = \int_0^\pi L_p w_p(s)(S_s + \hat{S}_s)\beta \, d\mu(s) \tag{3.3}$$

Identities (3a) and (3b) from p.108 give the following:

$$(S_s + \hat{S}_s)\beta(s) = \frac{1}{m(s)} \int_{S_{S^2}(x,s)} (\tau_p(x,x') + \hat{\tau}_p(x,x')) \cdot' \beta(x') \, dS(x') \tag{3.4}$$

Using the property from the top of p.108 allows to introduce parallel transport:

$$\begin{aligned} (S_s + \hat{S}_s)\beta(s) &= \frac{1}{m(s)} \int_{S_{S^2}(x,s)} g(\cdot, P_x^{x'} \beta^\sharp(x')) \, dS(x') \\ &= \frac{1}{m(s)} \int_{S_{S^2}(x,s)} I_p(\beta^\sharp(x')) \, dS(x') \end{aligned} \tag{3.5}$$

As this holds for an arbitrary form β, weakly one has:

$$(S_s + \hat{S}_s) = I_p \tag{3.6}$$

and similarly conclude from (3.3) that weakly one has:

$$\triangle_p (w_p(s) I_p) = L_p w_p(s) I \tag{3.7}$$

where the L_p are from [3] Lemma 1:

$$\begin{aligned} L_0 &= -\frac{\partial^2}{\partial s^2} - \cot s \frac{\partial}{\partial s} \\ L_1 &= -\frac{\partial^2}{\partial s^2} - \cot s \frac{\partial}{\partial s} + \frac{1}{\cos^2 \frac{s}{2}} \end{aligned} \tag{3.8}$$

Thus, to find a Green's function for (3.1) the following two ODEs have to be solved:

$$\begin{aligned} \left(-\frac{d^2}{ds^2} - \cot s \frac{d}{ds} - k^2\right) w_0(s) &= \delta_0(s) \\ \left(-\frac{d^2}{ds^2} - \cot s \frac{d}{ds} + \frac{1}{\cos^2\left(\frac{s}{2}\right)} - k^2\right) w_1(s) &= \delta_0(s) \end{aligned} \tag{3.9}$$

3.2 Solving the ODE for p=0

In this section the following ODE is solved:

$$\left(-\frac{d^2}{ds^2} - \cot s \frac{d}{ds} - k^2\right) w_0(s) = \delta_0(s) \tag{3.10}$$

First the ODE $= 0$ will be solved, then the function will be appropriately scaled to fit the singularity.

The problem (3.10) is transformed into a hypergeometric differential equation with the substitution $t = \sin^2 \frac{s}{2}$. The following identities then hold:

$$\begin{aligned} \frac{d}{ds} &= \frac{dt}{ds}\frac{d}{dt} = \frac{1}{2}\sin s \frac{d}{dt} \\ \frac{d^2}{ds^2} &= \frac{1}{2}\cos s \frac{d}{dt} + \frac{1}{4}\sin^2 s \frac{d^2}{dt^2} \\ \sin^2 s &= 4t(1-t) \\ \cos s &= 1 - 2t \end{aligned} \tag{3.11}$$

Thus the problem becomes

$$\begin{aligned} 0 &= \left(-t(1-t)\frac{d^2}{dt^2} - (1-2t)\frac{d}{dt} - k^2\right) w_0(t) \\ &= (1-t)tw''(t) + (1-2t)w'(t) + k^2 w_0(t) \end{aligned} \tag{3.12}$$

This is the hypergeometric equation. A solution to this in the form of hypergeometric functions is not very convenient though, so a further substitution is made, $x = 2t - 1$. The following then holds:

$$\begin{aligned} \frac{d}{dt} &= \frac{dx}{dt}\frac{d}{dx} = 2\frac{d}{dx} \\ \frac{d^2}{dt^2} &= 2\frac{d}{dx}2\frac{d}{dx} = 4\frac{d^2}{dx^2} \end{aligned} \tag{3.13}$$

This gives

$$\begin{aligned} 0 &= \left((1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + k^2\right) w_0(x) \\ &= (1-x^2)w_0''(x) - 2xw_0'(x) + k^2 w_0(x) \end{aligned} \tag{3.14}$$

This is the Legendre differential equation with $\nu(\nu+1) = k^2$ (as of [1, pp.332ff]). Defining $\kappa = \sqrt{1+4k^2}$ this means $\nu = -\frac{1 \pm \kappa}{2}$. Without loss of generality choose $\nu = \frac{-1+\kappa}{2}$. As degeneracies might develop if κ is an integer, exclude all these cases from the analysis.

The Legendre ODE has two linear independent solutions consisting of Legendre functions of the first (P_l) and second (Q_l) kind. As the second kind functions have a singularity near 1, which is unwanted, discard them. The solution is thus of the form:

$$\begin{aligned} w_0(x) &= CP_{\frac{\kappa-1}{2}}(x) \\ w_0(t) &= CP_{\frac{\kappa-1}{2}}(2t-1) \\ w_0(s) &= CP_{\frac{\kappa-1}{2}}\left(2\sin^2\frac{s}{2} - 1\right) \end{aligned} \tag{3.15}$$

It is left to determine C in (3.15). In the flat case the fundamental solution is $\frac{i}{4}H_0(ks)$, with H_0 Hankel function of the first kind (as of [5, p.15]). Near 0, this behaves similarly to $\frac{1}{2\pi}\log ks = K - \frac{1}{2\pi}\log\frac{s}{2} \sim -\frac{1}{2\pi}\log\frac{s}{2}$ if one disregards all terms not of order log.

As the behavior near the singularity is determined by the partial derivative of highest order, the terms of order log should be the same for the curved and the flat case where the ODE just reads $\left(-\frac{d^2}{ds^2} - k^2\right) w_0(s) = \delta_0(s)$.

Near -1 one has $P_l(z) \sim \frac{1}{\pi}\sin(l\pi)\log\frac{z+1}{2}$ (again ignore terms not of order log).[1] Thus:

[1] series expansion taken from `http://functions.wolfram.com/HypergeometricFunctions/LegendrePGeneral/06/01/05/`, [9]

$$
\begin{aligned}
w_0(s) = CP_{\frac{\kappa-1}{2}}\left(2\sin^2\frac{s}{2}-1\right) \sim C\frac{1}{\pi}\sin\left(\pi\frac{\kappa-1}{2}\right)\log\left(\sin^2\frac{s}{2}\right) \\
C\frac{1}{\pi}\sin\left(\pi\frac{\kappa-1}{2}\right)\log\left(\frac{s}{2}\right)^2 \sim C\frac{2}{\pi}\sin\left(\pi\frac{\kappa-1}{2}\right)\log\frac{s}{2}
\end{aligned}
\tag{3.16}
$$

This gives the following equation:

$$
\begin{aligned}
C\frac{2}{\pi}\sin\left(\pi\frac{\kappa-1}{2}\right)\log\frac{s}{2} &= -\frac{1}{2\pi} \\
C &= -\frac{1}{4\sin\left(\pi\frac{\kappa-1}{2}\right)}
\end{aligned}
\tag{3.17}
$$

which is well-defined as κ can't be an integer.
This comparison of series terms is legitimate as the Taylor series is unique after subtracting the log-like singularity and the singularity is exactly of order log in both cases.

3.3 Solving the ODE for p=1

In this section the following ODE is solved:

$$
\left(-\frac{d^2}{ds^2}-\cot s\frac{d}{ds}+\frac{1}{\cos^2\frac{s}{2}}-k^2\right)w_1(s)=\delta_0(s) \tag{3.18}
$$

First the ODE $=0$ will be solved, then the function will be appropriately scaled to fit the singularity.

To try to come to a hypergeometric problem consider the substitution $t=\sin^2\frac{s}{2}$. Then:

$$
\begin{aligned}
\frac{d}{ds} &= \frac{dt}{ds}\frac{d}{dt} = \frac{1}{2}\sin s\frac{d}{dt} \\
\frac{d^2}{ds^2} &= \frac{1}{2}\sin s\frac{d}{dt}\frac{d}{ds} = \frac{1}{2}\sin s\frac{d}{dt}\frac{1}{2}\sin s\frac{d}{dt} \\
&= \frac{1}{4}\sin s\left(\frac{d\sin s}{dt}\frac{d}{dt}+\sin s\frac{d^2}{dt^2}\right) \\
&= \frac{1}{4}\sin s\left(\cos s\frac{ds}{dt}\frac{d}{dt}+\sin s\frac{d^2}{dt^2}\right)
\end{aligned}
\tag{3.19a}
$$

$$
\begin{aligned}
\sin s\cos s\frac{ds}{dt} &= 2-4t \\
\sin^2 s &= 4t(1-t) \\
\cos s &= 1-2t \\
\cos^2\frac{s}{2} &= 1-t
\end{aligned}
\tag{3.19b}
$$

Thus the problem becomes

$$\begin{aligned}\left(-\frac{1}{4}(2-4t)\frac{d}{dt} - t(1-t)\frac{d^2}{dt^2} - \frac{1}{2}(1-2t)\frac{d}{dt} + \frac{1}{1-t} - k^2\right) w_1(s) = 0 \\ \left(t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{1-t} + k^2\right) w_1(s) = 0\end{aligned} \tag{3.20}$$

As the next step in the hypergeometric approach substitute $w_1(t) = (1-t)v(t)$. The problem changes to:

$$\begin{aligned}0 &= \left(t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{1-t} + k^2\right)(1-t)v(t) \\ &= t(1-t)\left(-2v'(t) + (1-t)v''(t)\right) + (1-2t)\left(-v(t) + (1-t)v'(t)\right) - v(t) + k^2(1-t)v(t) \\ &= t(1-t)^2 v''(t) + ((1-2t)(1-t) - 2t(1-t))v'(t) + (k^2(1-t) - (1-2t) - 1)v(t) \\ &= (1-t)\left((1-t)tv''(t) + (1-4t)v'(t) + (k^2-2)v(t)\right) \\ 0 &= (1-t)tv''(t) + (1-4t)v'(t) - (2-k^2)v(t)\end{aligned} \tag{3.21}$$

This is the hypergeometric ODE with $c = 1$, $a + b = 3$, $ab = 2 - k^2$ (notation here and the following solutions are taken from [4, pp.56ff]). A solution for this is $2a = 3 - \kappa$, $2b = 3 + \kappa$ with $\kappa := \sqrt{1 + 4k^2}$.
Assuming κ is no integer (to avoid degeneracies of the hypergeometric ODE) leads to the following two linearly independent solutions (as of [4, p.75]):

$$\begin{aligned}v^{(1)}(t) &= F\left(\frac{3-\kappa}{2}, \frac{3+\kappa}{2}, 1, t\right) \\ v^{(2)}(t) &= F\left(\frac{3-\kappa}{2}, \frac{3+\kappa}{2}, 3, 1-t\right)\end{aligned} \tag{3.22}$$

with $F(a, b, c, z)$ the standard Gauss hypergeometric function.[2]

Again here the first solution will lead to a singularity at $s = 1$ which is unwanted. As it was done in the $p = 0$ case, the first solution is discarded. The solution is thus of the form:

$$\begin{aligned}w_1(t) &= C(1-t)F\left(\frac{3-\kappa}{2}, \frac{3+\kappa}{2}, 3, 1-t\right) \\ w_1(s) &= C\left(1 - \sin^2\frac{s}{2}\right)F\left(\frac{3-\kappa}{2}, \frac{3+\kappa}{2}, 3, 1 - \sin^2\frac{s}{2}\right)\end{aligned} \tag{3.23}$$

It is left to determine C in (3.23). Again in the flat case $w_1(s) = \frac{i}{4}H_0(ks)$, with H_0 Hankel function of the first kind (as of [5, p.15]). Similarly to last section, disregarding all terms not or order log this behaves similarly to $-\frac{1}{2\pi}\log\frac{s}{2}$. By the same reasoning as

[2] This function is sometimes also written as ${}_2F_1(a; b, c, z)$ in literature.

in the last sections this is compared to the logarithmic terms in the series expansion of (3.23).
Near 1 one has that $F(a, b, a+b, z) = -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \log(1-z)$. This gives:

$$\begin{aligned} F\left(\frac{3-\kappa}{2}, \frac{3+\kappa}{2}, 3, 1-\sin^2\frac{s}{2}\right) &\sim -\frac{\Gamma(3)}{\Gamma\left(\frac{3-\kappa}{2}\right)\Gamma\left(\frac{3+\kappa}{2}\right)} \log\sin^2\frac{s}{2} \\ &\sim -\frac{2}{\Gamma\left(\frac{3-\kappa}{2}\right)\Gamma\left(\frac{3+\kappa}{2}\right)} \log\left(\frac{s}{2}\right)^2 \\ &\sim -\frac{4}{\Gamma\left(\frac{3-\kappa}{2}\right)\Gamma\left(\frac{3+\kappa}{2}\right)} \log\frac{s}{2} \end{aligned} \tag{3.24}$$

Inserting into $w_1(s)$ gives:

$$\begin{aligned} w(s) &= C\left(1-\sin^2\frac{s}{2}\right) F\left(\frac{3-\kappa}{2}, \frac{3+\kappa}{2}, 3, 1-\sin^2\frac{s}{2}\right) \\ &\sim -C\left(1-\left(\frac{s}{2}\right)^2\right) \frac{4}{\Gamma\left(\frac{3-\kappa}{2}\right)\Gamma\left(\frac{3+\kappa}{2}\right)} \log\frac{s}{2} \\ &\sim -C\frac{4}{\Gamma\left(\frac{3-\kappa}{2}\right)\Gamma\left(\frac{3+\kappa}{2}\right)} \log\frac{s}{2} \end{aligned} \tag{3.25}$$

For C this gives:

$$C = \frac{1}{2\pi}\frac{\Gamma\left(\frac{3-\kappa}{2}\right)\Gamma\left(\frac{3+\kappa}{2}\right)}{4} = \frac{\Gamma\left(\frac{3-\kappa}{2}\right)\Gamma\left(\frac{3+\kappa}{2}\right)}{8\pi} \tag{3.26}$$

which is well-defined as κ can't be an integer.

3.4 Green's Functions for the Helmholtz Equation

(3.15) and (3.17) give the following Green's function for the case $p = 0$:

$$\begin{aligned} G_0(x, y) &= w_0(s(x, y))I_0 \\ w_0(s) &= -\frac{1}{4\sin\left(\pi\frac{\kappa-1}{2}\right)} P_{\frac{\kappa-1}{2}}\left(2\sin^2\frac{s}{2} - 1\right) \end{aligned} \tag{3.27}$$

a plot of w_0 can be seen in Figure 3.1.

(3.23) and (3.26) give the following Green's function for the case $p = 1$:

$$\begin{aligned} G_1(x, y) &= w_1(s(x, y))I_1 \\ w_1(s) &= \frac{\Gamma\left(\frac{3-\kappa}{2}\right)\Gamma\left(\frac{3+\kappa}{2}\right)}{8\pi}\left(1-\sin^2\frac{s}{2}\right) F\left(\frac{3-\kappa}{2}, \frac{3+\kappa}{2}, 3, 1-\sin^2\frac{s}{2}\right) \end{aligned} \tag{3.28}$$

a plot of w_1 can be seen in Figure 3.2.

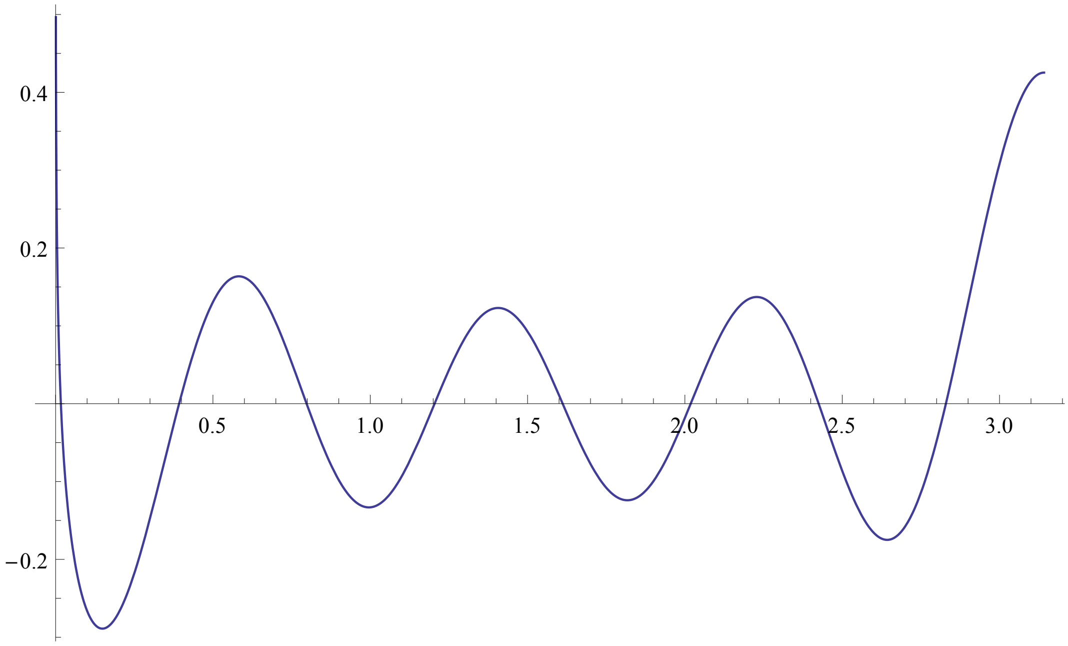

Figure 3.1: A plot of $w_0(s)$ from 0 to π with $\kappa = 15.4$.

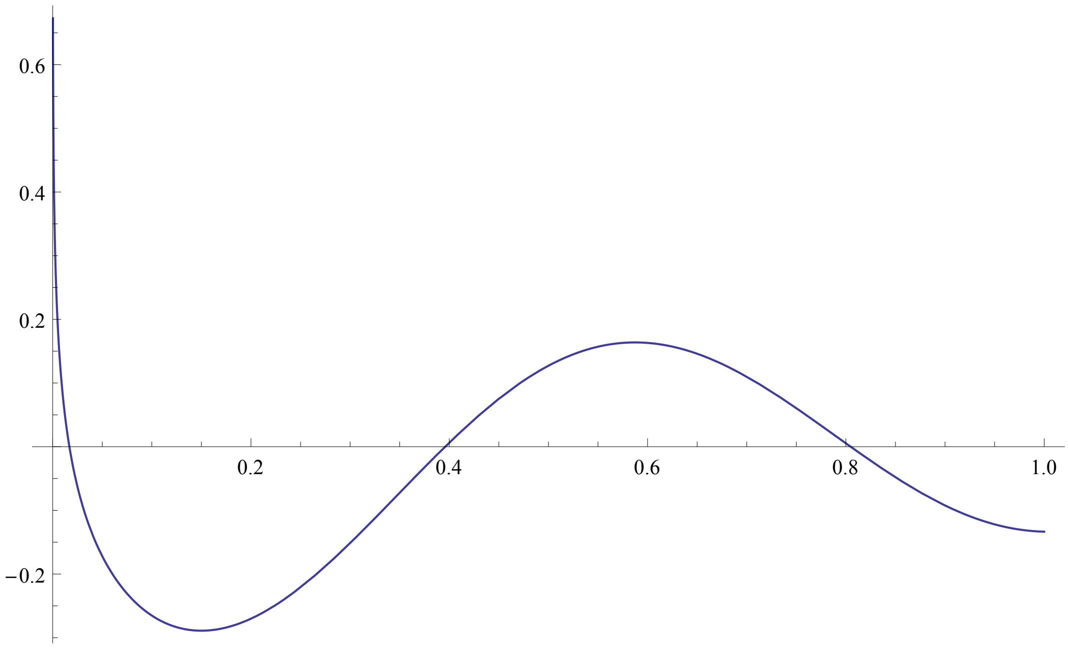

Figure 3.2: A plot of $w_1(s)$ from 0 to π with $\kappa = 15.4$.

In both cases $s(x, y)$ denotes the geodesic distance between x and y. This solves the problem of finding Green's functions for the Helmholtz equation. In the following chapters the functions from (3.27) and (3.28) will be used without further comment.

One could construct boundary integral equations for the Helmholtz problem now, but as the goal is to tackle the Maxwell problem in the following chapter these Green's functions are used to construct a Green's function for the Maxwell problem.

4 Solving the Maxwell Equation

4.1 Theory

The Green's double form for the Helmholtz problem can now be used to construct a Green's double form for the Maxwell problem. But first one needs to set up a clear formulation of the problem using the right Sobolev spaces. A more extensive formulation of (1.1):

$$\begin{aligned} (\delta d - k^2)u &= 0 \qquad \text{in } \Omega \subseteq S^2 \\ \operatorname{tr} u &= \beta \qquad \text{on } \Gamma \subseteq S^2 \end{aligned} \tag{4.1}$$

where Γ is the boundary of a simply connected $C \subset S^2$ and $u \in L^2\Omega^p(\delta d, \Omega)$ (the definition here is inspired from [5, p.27]). In the weak formulation later this is reduced to $L^2\Omega^p(d, \Omega)$. Thus the boundary condition $\beta \in H_\perp^{-\frac{1}{2}}\Omega(d, \Gamma)$ makes sense. One has $H_\perp^{-\frac{1}{2}}\Omega(d, \Gamma) = \operatorname{tr} L^2\Omega^p(d, \Omega)$ as of [5, p.23], so this is well-defined.

A gauge-type condition as in [5, p.27], as for $k \neq 0$ (and we excluded all the cases where κ is not an integer) the following holds:

$$\begin{aligned} \delta d u &= k^2 u \\ \delta^2 d u &= 0 = k^2 \delta u \end{aligned} \tag{4.2}$$

and thus $\delta u = 0$.

4.2 Finding a Fundamental Solution

The Green's double form for the Helmholtz problem can now be used to construct a Green's double form for the Maxwell problem (1.1). A naive approach (described in [5, p.42]) would be:

$$\tilde{G}_p = \left(1 - \frac{1}{k^2} d\delta\right) G_p \tag{4.3}$$

This is a Green's function to the Maxwell problem as the following calculation shows:

$$\begin{aligned} (\delta d - k^2)\tilde{G}_p(x, y) &= (\delta d - k^2)\left(1 - \frac{1}{k^2} d\delta\right) G_p(x, y) = (\delta d + d\delta - k^2) G_p(x, y) \\ &= (\triangle_p - k^2) G_p(x, y) = \delta_y(x) I_p \end{aligned} \tag{4.4}$$

However this is not a very good approach: (4.3) contains two derivatives, and as G_p already contains a logarithmic singularity this creates an even worse singularity. This makes numerical methods more complicated.

A smarter alternative is done in [5] p.41 for the flat case if one is going to use the single layer potential anyway. For this however first a generalization of Lemma 1 from [5] needs to be proven:[1]

Lemma 4.1 *The following identities hold for the Green's functions G_p of the Helmholtz problem, where d,δ act on the first slot and d',δ' act on the second slot of the double form:*[2]

$$(1) \qquad dG_p = \delta' G_{p+1} \tag{4.5a}$$
$$(2) \qquad \delta G_p = d' G_{p-1} \tag{4.5b}$$
$$\tag{4.5c}$$

provided that k is chosen such that $\triangle_p - k^2$ has trivial kernel.

Note that the restriction on k isn't very bad: resonance cases were excluded anyways when solving the relevant ODEs.

Proof First note that the following hold:

$$\begin{aligned} (\triangle_{p+1} - k^2)d &= d(\triangle_p - k^2) \\ (\triangle_{p-1} - k^2)\delta &= \delta(\triangle_p - k^2) \end{aligned} \tag{4.6}$$

This is true as of following calculation:

$$\begin{aligned} (\triangle_{p+1} - k^2)d &= d\delta d + \delta dd - k^2 d = d\delta d - k^2 d \\ &= d\delta d + dd\delta - dk^2 = d(\triangle_p - k^2) \end{aligned} \tag{4.7}$$

The calculation is analogous for δ.

Now for any $u \in L^2\Omega^p(\delta d, \Omega)$ one has:

$$\begin{aligned} u(x) &= \langle \delta_y(x) I_p(x,y), u(y) \rangle_{S^2} = \langle (\triangle_p - k^2) G_p(x,y), u(y) \rangle_{S^2} \\ &= (\triangle_p - k^2) \langle G_p(x,y), u(y) \rangle_{S^2} \end{aligned} \tag{4.8}$$

This gives the following the equivalent formulations, once by pulling δ into the L^2 product, once by inserting $\delta u(x)$ into the above identity:

$$\begin{aligned} \delta u(x) &= \langle \delta(\triangle_p - k^2) G_p(x,y), u(y) \rangle_{S^2} \\ \delta u(x) &= \langle (\triangle_{p-1} - k^2) G_{p-1}(x,y), \delta' u(y) \rangle_{S^2} \end{aligned} \tag{4.9}$$

[1] This proof comes mainly from notes by Stefan Kurz
[2] As specified in the note after (2.4)

Using the fact that d' and δ' are adjoint:

$$\begin{aligned}\langle \delta(\triangle_p - k^2)G_p(x,y), u(y)\rangle_{S^2} &= \langle (\triangle_{p-1} - k^2)G_{p-1}(x,y), \delta' u(y)\rangle_{S^2} \\ &= \langle d'(\triangle_{p-1} - k^2)G_{p-1}(x,y), u(y)\rangle_{S^2}\end{aligned} \tag{4.10}$$

As this holds for any test form u, one has weakly:

$$\delta(\triangle_p - k^2)G_p(x,y) = d'(\triangle_{p-1} - k^2)G_{p-1}(x,y) \tag{4.11}$$

Using (4.6) and the fact that $(\triangle_p - k^2)$ is injective gives:

$$\begin{aligned}(\triangle_{p-1} - k^2)\delta G_p(x,y) &= (\triangle_{p-1} - k^2)d' G_{p-1}(x,y) \\ \delta G_p(x,y) &= d' G_{p-1}(x,y)\end{aligned} \tag{4.12}$$

which proves (2). The proof of (1) is analogous.

This proves the lemma. □

4.3 Single Layer Potential

With this lemma the single layer potential can now be formulated. The single layer potential is now defined in the spirit of [5] pp.17ff. Define the Helmholtz single layer potential:

$$(\Psi_p \omega)(x) := \langle \omega(y), \mathrm{tr'}\, G_p(x,y)\rangle_\Gamma \tag{4.13}$$

This is the normal single layer potential as used in boundary integral equations (seen for example in [8, p.118]) where the L^2 product has been changed into a L^2 product for differential forms.

Lemma 4.2 *The following properties hold for the Helmholtz single layer potential:*

$$(1) \qquad (\triangle_p - k^2)\Psi_p = 0 \quad on \quad \Omega \tag{4.14a}$$

$$(2) \qquad \delta\Psi_p - \Psi_{p-1}\delta = 0 \quad on \quad \Omega \tag{4.14b}$$

Proof (1) is shown by the following calculation:

$$\begin{aligned}(\triangle_p - k^2)\Psi_p\omega &= \langle \omega(y), (\triangle_p - k^2)\,\mathrm{tr'}\, G_p(x,y)\rangle_\Gamma \\ &= \langle \omega(y), \mathrm{tr'}(\triangle_p - k^2)G_p(x,y)\rangle_\Gamma = 0\end{aligned} \tag{4.15}$$

Similarly, (2) is shown by:

$$\begin{aligned}\delta\Psi_p\omega &= \delta\langle \omega(y), \mathrm{tr'}\, G_p(x,y)\rangle_\Gamma = \langle \omega(y), \mathrm{tr'}\, \delta G_p(x,y)\rangle_\Gamma \\ &= \langle \omega(y), \mathrm{tr'}\, d' G_{p-1}(x,y)\rangle_\Gamma = \langle \omega(y), d'\,\mathrm{tr'}\, G_{p-1}(x,y)\rangle_\Gamma \\ &= \delta'\langle \omega(y), \mathrm{tr'}\, G_{p-1}(x,y)\rangle_\Gamma = \Psi_{p-1}\delta'\omega\end{aligned} \tag{4.16}$$

(the dash left at the end is cosmetics, it means that the codifferential of $\omega(y)$ is taken in terms of y. This is clear, so one can just omit it as it is done in the formulation of the

lemma). It was also used that the exterior derivative commutes with the trace operator, but this follows from the fact that exterior differentiation commutes with the pullback of differential forms.[3]
This proves the lemma. $\square$

It is worth noting that as in Lemma 7 of [5] $\Psi : H_{\|}^{-\frac{1}{2}}\Omega^p(\Gamma) \to H^1\Omega^p(\Omega)$, or by restricting the domain $\Psi : H_{\|}^{-\frac{1}{2}}\Omega^p(\delta,\Gamma) \to L^2\Omega^p(\delta d,\Omega)$.

With the Helmholtz single layer potential it is now possible to define a Maxwell single layer potential that won't exhibit the same problems as (4.3):

$$\tilde{\Psi}_p\omega := \left(\Psi_p - \frac{1}{k^2} d\Psi_{p-1}\delta\right)\omega \tag{4.17}$$

Lemma 4.3 *The following property holds for the Maxwell single layer potential:*

$$(\delta d - k^2)\tilde{\Psi}_p = 0 \quad on \quad \Omega \tag{4.18}$$

Proof This is verified by the following calculation:

$$\begin{aligned}(\delta d - k^2)\tilde{\Psi}_p &= (\delta d - k^2)\left(\Psi_p - \frac{1}{k^2} d\Psi_{p-1}\delta\right) = (\triangle - k^2)\Psi_p - d\delta\Psi_p + d\Psi_{p-1}\delta \\ &= d(\Psi_{p-1}\delta - \delta\Psi_p) = 0\end{aligned} \tag{4.19}$$

where Lemma 4.2 was used.
This proves the lemma. $\square$

From the spaces discussion for Ψ_p it can be seen that $\tilde{\Psi}_p : H_{\|}^{-\frac{1}{2}}\Omega^p(\delta,\Gamma) \to L^2\Omega^p(\delta d,\Omega)$

In the spirit of [8] p.119 the single layer operator is now defined:

$$\tilde{V}_p := \operatorname{tr}\tilde{\Psi}_p \tag{4.20}$$

Now as [5] p.23 states that $\operatorname{tr} : L^2\Omega^p(d,\Omega) \to H_{\perp}^{-\frac{1}{2}}\Omega^p(d,\Gamma)$, it is possible to conclude that $\tilde{V}_p : H_{\|}^{-\frac{1}{2}}\Omega^p(\delta,\Gamma) \to H_{\perp}^{-\frac{1}{2}}\Omega^p(d,\Gamma)$.

4.4 Boundary Integral Equations

With the single layer potential set up and working one can now set up boundary integral equations. Here the indirect method will be used which only needs the single layer

[3] Muss ich hierzu etwas zitieren?

potential. Theory for this mainly can be seen e.g. in [8, pp.171ff].

Plugging the Dirichlet condition into the boundary integral equation for the single layer operator gives the following integral equation:

$$\tilde{V}_p \omega = \beta \tag{4.21}$$

where $\beta \in H_{\perp}^{-\frac{1}{2}} \Omega^p(d, \Gamma)$ is the Dirichlet boundary condition and $\omega \in H_{\|}^{-\frac{1}{2}} \Omega^p(\delta, \Gamma)$ is the unknown boundary density. A weak formulation for this is:

$$\langle \tilde{V}_p \omega, \eta \rangle_\Gamma = \langle \beta, \eta \rangle_\Gamma \tag{4.22}$$

with $\eta \in H_{\|}^{-\frac{1}{2}} \Omega^p(\delta, \Gamma)$.

Writing this out in integral form gives:

$$\begin{aligned}
\langle \tilde{V}_p \omega, \eta \rangle_\Gamma &= \int_\Gamma \hat{g} \left(\operatorname{tr} \left(\Psi_p \omega - \frac{1}{k^2} d \Psi_{p-1} \delta \omega \right), \eta \right) d\Gamma \\
&= \int_\Gamma \hat{g} \left(\Psi_p \omega, \eta \right) d\Gamma - \frac{1}{k^2} \int_\Gamma \hat{g} \left(d \Psi_{p-1} \delta \omega, \eta \right) d\Gamma \\
&= \int_\Gamma \hat{g} \left(\Psi_p \omega, \eta \right) d\Gamma - \frac{1}{k^2} \int_\Gamma \hat{g} \left(\Psi_{p-1} \delta \omega, \delta \eta \right) d\Gamma \\
&= \int_\Gamma \int_\Gamma \hat{g} \left(\hat{g}'(\omega(y), \text{tr' } G_p(x,y)), \eta(x) \right) d\Gamma_y d\Gamma_x \\
&\quad - \frac{1}{k^2} \int_\Gamma \int_\Gamma \hat{g} \left(\hat{g}'(\delta\omega(y), \text{tr' } G_{p-1}(x,y)), \delta\eta(x) \right) d\Gamma_y d\Gamma_x \\
&= \int_\Gamma \int_\Gamma \hat{g} \left(G_p(x,y)[\omega^\sharp(y), \cdot], \eta(x) \right) d\Gamma_y d\Gamma_x \\
&\quad - \frac{1}{k^2} \int_\Gamma \int_\Gamma \hat{g} \left(G_{p-1}(x,y)[(\delta\omega)^\sharp(y), \cdot]), \delta\eta(x) \right) d\Gamma_y d\Gamma_x \\
&= \int_\Gamma \int_\Gamma G_p(x,y)[\omega^\sharp(y), \eta^\sharp(x)] d\Gamma_y d\Gamma_x \\
&\quad - \frac{1}{k^2} \int_\Gamma \int_\Gamma G_{p-1}(x,y)[(\delta\omega)^\sharp(y), (\delta\eta)^\sharp(x)] d\Gamma_y d\Gamma_x
\end{aligned} \tag{4.23}$$

where the trace can be ignored after the first and fourth lines because of the scalar product with a form that lies on the boundary. For the right side:

$$\langle \beta, \eta \rangle_\Gamma = \int_\Gamma g(\beta^\sharp(x), \eta^\sharp(x)) d\Gamma_x \tag{4.24}$$

The boundary integral equation that has to be solved is thus the weak problem

$$\begin{aligned}
\int_\Gamma \int_\Gamma G_p(x,y)[\omega^\sharp(y), \eta^\sharp(x)] d\Gamma_y d\Gamma_x - \frac{1}{k^2} \int_\Gamma \int_\Gamma G_{p-1}(x,y)[(\delta\omega)^\sharp(y), (\delta\eta)^\sharp(x)] d\Gamma_y d\Gamma_x \\
= \int_\Gamma g(\beta^\sharp(x), \eta^\sharp(x)) d\Gamma_x
\end{aligned} \tag{4.25}$$

for $\eta \in H_{\|}^{-\frac{1}{2}}\Omega^p(\delta,\Gamma)$ test function.

4.5 Representation Formula

Once the boundary density ω from the last section has been found the representation formula can be used to obtain a solution u to (1.1). As of Lemma 4.2 one can obtain the solution from the following representation formula:

$$u(x)[v_x] = \int_\Gamma \tilde{G}_p(x,y)[v_x, \omega^\sharp(y)]d\Gamma_y \tag{4.26}$$

where $\tilde{G}_p$ is the Green's function for the Maxwell problem.

Again the direct use of the Green's function for the Maxwell problem is not advisable as of the double differentiation of the logarithmic singularity. Using Lemma 4.2 and the fact that for the representation formula $x \notin \Gamma$, (4.26) can be rewritten as:

$$u(x)[v_x] = (\tilde{\Psi}_p\omega)[v_x] \tag{4.27}$$

This is equivalent to (4.26) because of the following calculation:

$$\begin{aligned}
\tilde{\Psi}_p\omega(y) &= (\Psi_p - \frac{1}{k^2}d\Psi_{p-1}\delta)\omega(y) \\
&= \int_\Gamma \hat{g}'(\omega(y), \text{tr' } G_p(x,y))d\Gamma_y - \frac{1}{k^2}d\int_\Gamma \hat{g}'(\delta'\omega(y), \text{tr' } G_{p-1}(x,y))d\Gamma_y \\
&= \int_\Gamma \hat{g}'(\omega(y), \text{tr' } G_p(x,y))d\Gamma_y - \frac{1}{k^2}\int_\Gamma \hat{g}'(\omega(y), \text{tr' } dd'G_{p-1}(x,y))d\Gamma_y \\
&= \int_\Gamma \hat{g}'(\omega(y), \text{tr' } G_p(x,y) - \frac{1}{k^2}\text{tr' } d\delta G_{p-1}(x,y))d\Gamma_y \\
&= \int_\Gamma \tilde{G}_p(x,y)[\cdot, \omega^\sharp(y)]d\Gamma_y
\end{aligned} \tag{4.28}$$

where Lemma 4.1 was used.

Inserting the definition for $\tilde{\Psi}_p$ and calculating gives:

$$\begin{aligned}
u(x)[v_x] &= \left(\Psi_p - \frac{1}{k^2}d\Psi_{p-1}\delta\right)\omega[v_x] \\
&= \int_\Gamma \text{tr' } G_p(x,y))[v_x, \omega^\sharp(y)] - \frac{1}{k^2}d\,\text{tr' } G_{p-1}(x,y)[v_x, \delta\omega(y)]d\Gamma_y
\end{aligned} \tag{4.29}$$

Notice that here it is not possible to get rid of the last derivative.

5 Discretization

Where the last sections have been rather theoretical, here the case $p = 1$ (i.e. u is a 1-form) is further explored and an explicit discretization is proposed.

5.1 Theory

The weak equation that has to be discretized is (4.22). For the case of 1-forms the spaces simplify:

$$H_{\|}^{-\frac{1}{2}}\Omega^1(\delta,\Gamma) = \star H_{\perp}^{-\frac{1}{2}}\Omega^0(d,\Gamma) = \star \operatorname{tr} L^2\Omega^0(d,\Omega) \tag{5.1}$$

where [5], pp.22f was used. Now 0-forms on Ω with exterior derivative in L^2 are just functions with all partial derivatives in L^2. One can thus write:

$$H_{\|}^{-\frac{1}{2}}\Omega^1(\delta,\Gamma) = \star \operatorname{tr} H^1(\Omega) = \star H^{\frac{1}{2}}(\Gamma) \tag{5.2}$$

A similar trick works for the other space which appears in the weak formulation:

$$H_{\perp}^{-\frac{1}{2}}\Omega^1(d,\Gamma) = \star H_{\|}^{-\frac{1}{2}}\Omega^0(\delta,\Gamma) = \star \operatorname{tr} L^2\Omega^0(\delta,\Omega) \tag{5.3}$$

Now all 0-forms for which the codifferential is defined have a codifferential in L^2, as it is 0 for 0-forms. This gives all L^2 functions, and thus:

$$H_{\perp}^{-\frac{1}{2}}\Omega^1(d,\Gamma) = \star \operatorname{tr} L^2(\Omega) = \star H^{-\frac{1}{2}}(\Gamma) \tag{5.4}$$

With this simplification of the spaces it is possible to first discretize the normal scalar Sobolev space and then simply $\star$ it. This will be done in the next section.

5.2 Galerkin's Method for the Boundary Integral Equations

The discretization for the space $H^{\frac{1}{2}}(\Gamma)$ which will be used consists of piecewise linear functions.

Assume the boundary Γ is parametrized by arc length via $\chi : [0, d] \to \Gamma$. Consider the partition of $[0,d]$ into N intervals $[x_i, x_{i+1}]$, $h_i := x_{i+1} - x_i$. On this partition it is possible to define hat functions:

$$\varphi_i(\xi) := \begin{cases} 0 & \text{if } \xi \notin [x_{i-1}, x_{i+1}) \\ \frac{\xi - x_{i-1}}{h_i} & \text{if } \xi \in [x_{i-1}, x_i) \\ \frac{x_{i+1}-\xi}{h_{i+1}} & \text{if } \xi \in [x_i, x_{i+1}) \end{cases} \tag{5.5a}$$

with respective derivative:

$$\Phi(\xi) := \varphi_i'(\xi) = \begin{cases} 0 & \text{if } \xi \notin [x_{i-1}, x_{i+1}) \\ \frac{1}{h_i} & \text{if } \xi \in [x_{i-1}, x_i) \\ -\frac{1}{h_{i+1}} & \text{if } \xi \in [x_i, x_{i+1}) \end{cases} \tag{5.5b}$$

With $\frac{\partial}{\partial l}$ the unit tangent vector to Γ coming from the parametrization χ and dl the corresponding 1-form it is possible to define the hat forms $L_i(x) := \varphi_i(\chi^{-1}(x))dl(x)$ on Γ. This is a valid discretization of $\star H^{\frac{1}{2}}(\Gamma)$.
Project ω, the boundary density, onto the subspace to get:

$$\omega^{(N)} := \sum_i \omega_i^{(N)} L_i \tag{5.6}$$

Insert into the boundary equations from last chapter and construct Galerkin equations:

$$\begin{aligned} &\int_\Gamma \int_\Gamma G_1(x,y)[\sum_i \omega_i^{(N)} L_i^\sharp(y), L_j^\sharp(x)] d\Gamma_y d\Gamma_x \\ &\quad - \frac{1}{k^2} \int_\Gamma \int_\Gamma G_0(x,y)[\sum_i \omega_i^{(N)} (\delta L_i)^\sharp(y), (\delta L_j)^\sharp(x)] d\Gamma_y d\Gamma_x = \int_\Gamma g(\beta^\sharp(x), L_j^\sharp(x)) d\Gamma_x \\ &\sum_i \omega_i^{(N)} \left(\int_\Gamma \int_\Gamma G_1(x,y)[L_i^\sharp(y), L_j^\sharp(x)] - \frac{1}{k^2} G_0(x,y)[(\delta L_i)^\sharp(y), (\delta L_j)^\sharp(x)] d\Gamma_y d\Gamma_x \right) \\ &\quad = \int_\Gamma g(\beta^\sharp(x), L_j^\sharp(x)) d\Gamma_x \end{aligned} \tag{5.7}$$

Inserting results from (7.1 - 7.4) in the appendix the equations read:

$$\begin{aligned} &\sum_i \omega_i^{(N)} \Bigg(\int_\Gamma \int_\Gamma w_1(s(x,y)) \varphi_i(\chi^{-1}(x)) \varphi_j(\chi^{-1}(x)) g(dl^\sharp(x), P_x^y dl^\sharp(y)) \\ &\quad - \frac{1}{k^2} w_0(s(x,y)) \Phi_i(\chi^{-1}(x)) \Phi_j(\chi^{-1}(x)) d\Gamma_y d\Gamma_x \Bigg) = \int_\Gamma \varphi_j(\chi^{-1}(x)) b(x) d\Gamma_x \end{aligned} \tag{5.8}$$

with $\beta(x) = b(x)dl(x)$.

Define the following coefficients:

$$
\begin{aligned}
K_{i,j} &:= \int_0^d \int_0^d \varphi_i(\zeta)\varphi_j(\xi) w_1(s(\chi(\xi),\chi(\zeta))) g(dl^\sharp(\chi(\xi)), P_x^y dl^\sharp(\chi(\zeta)))\, d\zeta d\xi \\
L_{i,j} &:= -\frac{1}{k^2} \int_0^d \int_0^d w_0(s(\chi(\xi),\chi(\zeta))) \Phi_i(\zeta)\Phi_j(\xi)\, d\zeta d\xi \\
R_j &:= \int_0^d b(\chi(\xi))\varphi_j(\xi)\, d\zeta
\end{aligned}
\tag{5.9}
$$

Then the Galerkin equations reduce to the following linear system:

$$
\sum_i \omega_i^{(N)} (K_{i,j} + L_{i,j}) = R_j \tag{5.10}
$$

5.3 Discretizaton of the Representation Formula

The only thing left on the way to a complete numerical method is the discretization of the representation formula (4.29).
First write out the representation formula. Inserting results from (7.5 - 7.6) in the appendix the equations read:

$$
\begin{aligned}
g(u^\sharp(x), v_x) &= \int_\Gamma \text{tr'}\, G_1(x,y))[v_x, \omega^\sharp(y)] - \frac{1}{k^2} dG_0(x,y)[v_x, \delta\omega(y)] d\Gamma_y \\
&= \int_\Gamma w_1(s(x,y)) g(v_x, \operatorname{tr} P_x^y \omega^\sharp(y)) - \frac{1}{k^2} dw_0(s(x,y))[v_x] \delta\omega(y) d\Gamma_y
\end{aligned}
\tag{5.11}
$$

Now, inserting $\omega = \sum_i \omega_i^{(N)} L_i$ gives:

$$
\begin{aligned}
u^\sharp(x)[v_x] &= \sum_i \omega_i^{(N)} \int_\Gamma \Bigg(w_1(s(x,y)) g(v_x, \operatorname{tr} P_x^y L_i^\sharp(y)) \\
&\quad - \frac{1}{k^2} dw_0(s(x,y))[v_x] \delta L_i(y) \Bigg) d\Gamma_y \\
&= \sum_i \omega_i^{(N)} \int_\Gamma \Bigg(w_1(s(x,y)) \varphi_i(\chi^{-1}(y)) g(v_x, \operatorname{tr} P_x^y dl^\sharp(y)) \\
&\quad + \frac{1}{k^2} dw_0(s(x,y))[v_x] \Phi_i^{-1}(\chi^{-1}(y)) \Bigg) d\Gamma_y
\end{aligned}
\tag{5.12}
$$

Defining the following coefficient:

$$
\begin{aligned}
P_i(v_x) = \int_0^d & w_1(s(\chi(x),\chi(y))) \varphi_i(\zeta) g(v_x, \operatorname{tr} P_x^y dl^\sharp(\chi(\zeta))) \\
&+ \frac{1}{k^2} dw_0(s(\chi(x),\chi(y)))[v_x] \Phi_i^{-1}(\zeta) d\zeta
\end{aligned}
\tag{5.13}
$$

then gives the following solution to the Maxwell problem:

$$u^\sharp(x)[v_x] = \sum_i \omega_i^{(N)} P_i(v_x) \tag{5.14}$$

This is the representation formula which serves as a general solution to (1.1) for the case $p = 1$.

5.4 Implementing the Parallel Transport Function

The parallel transport function can be implemented via embedding into $\mathbb{R}^3$. For some vector v_y based at $y \in S^2$ transported to $w_x := P_x^y v_y$ based at $x \in S^2$ this happens like this: consider S^2 embedded into $\mathbb{R}^3$ where the equator is the great circle containing y and x and the segment from y to x is positively oriented. Pick local spherical coordinates with α longitude, β latitude. Then the α and β components of a v_x are the same as the α and β components of w_x, as the unit vectors corresponding to these spherical coordinates form an orthonormal frame along the great circle.

Let r_y and r_x be the position vectors of y and x in $\mathbb{R}^3$. Then the unit latitude vectors a_x, a_y are orthogonal to v_y and w_x and can thus be represented using the $\mathbb{R}^3$ cross product:

$$a_x = a_y = \frac{r_y \times r_x}{\|r_y \times r_x\|} = \frac{r_y \times r_x}{\sin s} \tag{5.15}$$

as the angle between r_y and r_x is exactly s due to the radius of S^2 being 1. The vector points up as a consequence of the right-hand rule and the fact the line segment from y to x is positively oriented.

The unit longitude vectors b_x, b_y are orthonormal to a_x, a_y and r_x, r_y and point the same direction as the great circle: from y to x. They can thus be represented by another cross product:

$$\begin{aligned} b_y &= a_y \times r_y \\ b_x &= a_x \times r_x \end{aligned} \tag{5.16}$$

A sketch of the situation can be seen in Figure 5.1.

Using the normal $\mathbb{R}^3$ dot product one can now simply pick out the right components and the parallel transport reduces to:

$$w_x = P_x^y v_y = (v_y \cdot b_y)\, b_x + (v_y \cdot a_y)\, a_x \tag{5.17}$$

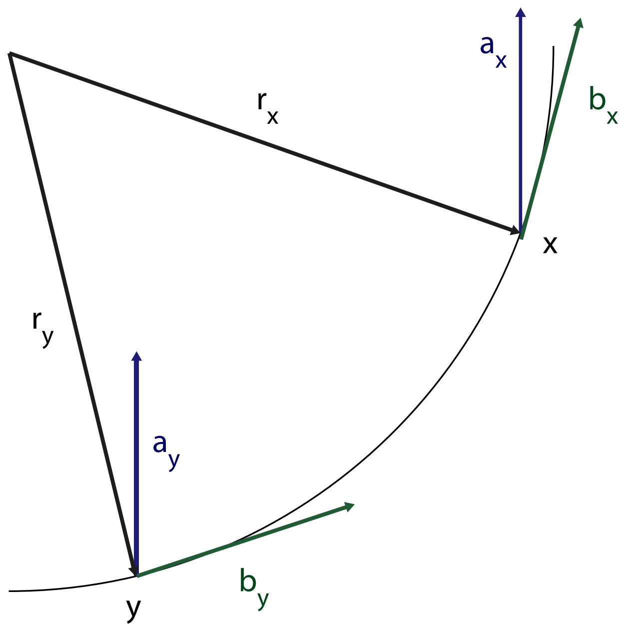

Figure 5.1: Calculating the unit vectors of the orthonormal frame along the great circle from y to x. r_x, r_y are position vectors, a_x, a_y are latitude vectors, b_x, b_y are longitude vectors.

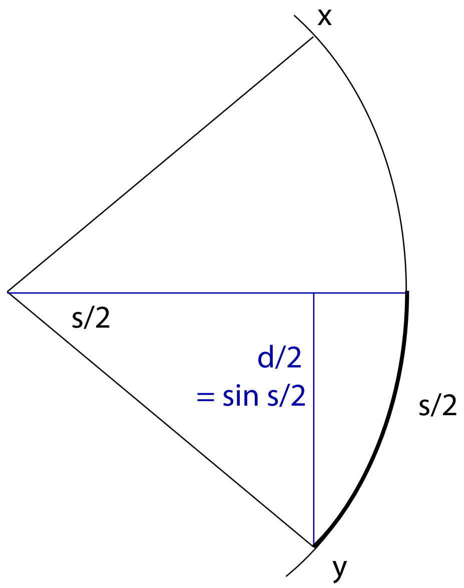

Figure 5.2: Calculating the geodesic distance s between x and y given the $\mathbb{R}^3$ distance d. The figure represents a cross-section through the sphere along a great circle going through x and y.

5.5 Implementing the Geodesic Distance Function

The geodesic distance function $s = s(x, y)$ can be easily implemented using standard trigonometry on the $\mathbb{R}^3$ distance between x and y that one quickly obtains after the embedding into $\mathbb{R}^3$.
The geodesic distance is the arc length of a great circle centered at 0 passing through x and y. Figure 5.2 shows the situation with s geodesic distance, d $\mathbb{R}^3$ distance.

This gives $\sin \frac{s}{2} = \frac{d}{2}$.

6 Summary and Outlook

In this bachelor's thesis the Maxwell-type problem $(\delta d - k^2)u = 0$ on some $\Omega \subseteq S^2$ with Dirichlet boundary conditions on the boundary Γ of a simply connected $C \subset S^2$ was solved for u a 1-form. First the Green's double form for 0- and 1-forms was calculated. This was then used to construct the single layer operator. The single layer operator was discretized with a Galerkin method to obtain the boundary density. With an indirect approach the representation formula was then used to calculate u.

Still much remains to be done: the quadrature which has to be used to calculate (5.9) and (5.13) is not trivial as it involves a singularity.
After this, an actual working implementation in MATLAB would be a next step.

7 Appendix

Several facts which are not important to the main work but still need to be shown are shown in this appendix.

7.1 Calculations for the Galerkin Discretization

The following calculations are referenced throughout the section about Galerkin discretization. All notation and definitions are the same as in that section.

$$\begin{aligned} \star L_i(x) &= \varphi_i(\chi^{-1}(x)) \star dl(x) = \varphi_i(\chi^{-1}(x)) \\ d \star L_i(x) &= \frac{\partial}{\partial x}\varphi_i(\chi^{-1}(x))dl = \Phi_i(\chi^{-1}(x))dl \\ \delta L_i(x) &= - \star^{-1} d \star L_i(x) = -\Phi_i(\chi^{-1}(x)) \end{aligned} \tag{7.1}$$

$$\begin{aligned} G_0(x,y)[\alpha^\sharp(x),\gamma^\sharp(y)] &= w_0(s(x,y))I_0[\alpha^\sharp(x),\gamma^\sharp(y)] = w_0(s(x,y))\alpha(x),\gamma(y) \\ G_0(x,y)[\delta L_i^\sharp(x),\delta L_j^\sharp(y)] &= w_0(s(x,y))\Phi_i(\chi^{-1}(x))\Phi_j(\chi^{-1}(x)) \end{aligned} \tag{7.2}$$

$$\begin{aligned} G_1(x,y)[\alpha^\sharp(x),\gamma^\sharp(y)] &= w_1(s(x,y))I_0[\alpha^\sharp(x),\gamma^\sharp(y)] = w_1(s(x,y))g(\alpha^\sharp(x),P_x^y\gamma^\sharp(y)) \\ G_1(x,y)[L_i^\sharp(x),L_j^\sharp(y)] &= w_1(s(x,y))\varphi_i(\chi^{-1}(x))\varphi_j(\chi^{-1}(x))g(dl^\sharp(x),P_x^y dl^\sharp(y)) \end{aligned} \tag{7.3}$$

$$g(b(x)dl^\sharp(x),L_j^\sharp(x)) = \varphi_j(\chi^{-1}(x))b(x)g(dl^\sharp(x),dl^\sharp(x)) = \varphi_j(\chi^{-1}(x))b(x) \tag{7.4}$$

7.2 Calculations for the Discretization of the Representation Formula

The following calculations are referenced throughout the section about the Discretization of the Representation Formula. All notation and definitions are the same as in that section.

$$\text{tr'}\, G_1(x,y))[v_x,\omega^\sharp(y)] = w_1(s(x,y))g(v_x,P_x^y\omega^\sharp(y)) \tag{7.5}$$

$$dG_0(x,y))[v_x,\delta\omega^\sharp(y)] = dw_0(s(x,y))[v_x]\delta\omega(y) \tag{7.6}$$

8 Bibliography

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