Goals:

- Devise space-time DG-method for the wave equation:

  \[ u_{tt} - u_{xx} = f \quad \text{in } Q := \Omega \times [0, T] \]

- Implement method and test it
- Investigate stability of method

Weak formulation:

- \( \mathcal{M} = \{K\} \) is a mesh that covers the space-time domain \( Q \)
- Testfunction \( v \in V_h(\mathcal{M}) = \bigotimes_{K \in \mathcal{M}} P_p(K) \)
- Notation: \( \diamondsuit u = (u_x, -u_t) \) and \( \nabla u = (u_x, u_t) \).

\[
- \sum_{K \in \mathcal{M}} (\nabla \cdot \diamondsuit u, v)_K \, d\mathbf{x} = \sum_{K \in \mathcal{M}} (f, v)_K \, d\mathbf{x}
\]

i.b.p. \( \Rightarrow \)

\[
\sum_{K \in \mathcal{M}} (\diamondsuit u, \nabla v)_K - \sum_{K \in \mathcal{M}} \langle \diamondsuit u \cdot n, v \rangle_{\partial K} = \sum_{K \in \mathcal{M}} (f, v)_K
\]

DG-magic \( \Rightarrow \)
\[ \sum_{K \in \mathcal{M}} (\diamond u, \nabla v)_K - \langle \{\diamond u\}, [v]\rangle_{\mathcal{E}} - \langle \{\diamond v\}, [u]\rangle_{\mathcal{E}} = \sum_{K \in \mathcal{M}} (f, v)_K \]

Notation: \( \{v\} = (v^+ + v^-)/2 \) and \([v] = v^+ n^+ + v^- n^-\).
Numerical scheme is then:
\[ \forall K \in \mathcal{M}, \text{ seek } u_h \in V_h \text{ such that} \]
\[ a_K(u_h, v_h) = \ell_K(v_h), \quad \text{for all } v_h \in P_p(K) \]

where \( \ell_K(v) = (f, v)_K \) and
\[ a_K(u, v) = (\diamond u, \nabla v)_K - \langle \{\diamond u\}, [v]\rangle_{\partial K} - \langle \{\diamond v\}, [u]\rangle_{\partial K} + \langle \alpha [u], [v]\rangle_{\partial K} \]
Choose basis, and plug in:
\[ a_K(\sum_i \mu_i b_i, \sum_j q_j b_j) = \ell_K(\sum_j q_j b_j), \quad \text{for all } v_h = \sum_j q_j b_j \text{ local b.f.} \]
\[ \tilde{\mu}_j^{(n+1)} = \left( A_j^{(n+1)} \right)^{-1} \left( \tilde{\ell} - A_j^{(n-1)} \tilde{\mu}_j^{(n-1)} - \sum_{i=j-1}^{j+1} A_i^{(n)} \mu_i^{(n)} \right) \]
Possible if we choose *locally* supported basis functions

Now we have *explicit* scheme, suited for implementation

- Determined entries in matrices analytically in Maple
- Then implemented method in Matlab
Unluckily:

- Numerical experiments: Blow up in solutions
- Von Neumann analysis showed *unconditional* instability
- Possible remedy? Perhaps: Use mesh that avoids vertical edges