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The variational counterpart of the equation above is

$$a(u, v) + \lambda m(u, v) = f(v)$$

for all $v \in H$, where

$$a(u, v) :=< \mathcal{G}u, v >_{H' \times H} \; and \quad m(u, v) :=< u, v >_{H \times H} .$$

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In typical situations, the bilinear form $a(\cdot, \cdot)$ representing the integral operator can be written as

$$a(u, v) = \int_{\Omega} v(x) \int_{\Omega} g(x, y) u(y) dy dx \tag{3}$$

for a kernel function $g(\cdot, \cdot)$ and a domain or manifold Ω.

The equation (2) is discretized by choosing an n-dimensional subspace H_n of H and considering the problem of finding a function $u_n \in H_n$ s.t.

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For any basis $(\varphi_i)_{i \in \mathcal{I}}$ of H_n, this is equivalent to

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Since the solution u_n is an element of H_n, there is a coefficient vector $(x_i)_{i\in\mathcal{I}}$ satisfying

$$u_n = \sum_{j\in\mathcal{I}} x_j \varphi_j,$$

s.t. the coefficients satisfy the equation

$$\sum_{j\in\mathcal{I}} x_j a(\varphi_j, \varphi_i) + \lambda \sum_{j\in\mathcal{I}} x_j m(\varphi_j, \varphi_i) = f(\varphi_i)$$

for all $i \in \mathcal{I}$.

This is a system of linear equations and can be written in matrix form

$$Gx + \lambda Mx = b$$

by introducing matrices $G, M \in \mathbb{R}^{\mathcal{I} \times \mathcal{I}}$ and a vector $b \in \mathbb{R}^{\mathcal{I}}$ with

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If we use standard finite element basis functions $(\varphi_i)_{i\in\mathcal{I}}$, the matrix M will be sparse, but G will be densely populated, since typical kernel functions have global support.

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Storing G directly doesn't lead to efficient algorithms. Therefore we approximate G by a matrix that can be treated efficiently, by replacing the kernel function $k(\cdot,\cdot)$ by local degenerate approximations, and this leads to a hierarchical matrix.

Recall: Degenerate approximation

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The idea is to approximate the kernel function $g(\cdot, \cdot)$ by using interpolation instead of Taylor expansion, and thus avoiding the need of being able to evaluate the derivatives of g efficiently.

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As mentioned before, we don't need any derivative of g for that approximation.

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We can now replace the matrix G with $\tilde{G}$ defined by

$$\tilde{G}_{ij} := \int_\Omega \varphi_i(x) \int_\Omega \tilde{g}(x,y)\varphi_j(y)dydx = (AB^T)_{ij},$$

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and

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Part II: Boundary Element Method in 2D

We consider a closed curve in 2-dimensional space, given as an array vertex of n points. We use piecewise constant basis functions and choose the characterizing point to be the middle of the corresponding interval. We will now solve integral equations on this curve.

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We are interested in a boundary integral problem, i.e. the set Ω is a submanifold. Here Ω is a a curve. Since we'll be talking about integral equations, I will recall the meaning of an integral on a curve:

1 Curve integrals

Let $\gamma : [0,1] \to \mathbb{R}^2$ be injective in [0,1[, $\gamma \in C^1, \gamma' \in C^0$. We write $\Gamma := \gamma([0,1])$. Let $u \in C^0(\Gamma)$. We introduce a partition $0 = x_0 < x_1 < ... < x_n = 1$ of [0,1] and consider the sum

$$I_x := \sum_{i=1}^{n} u(\gamma(x_i))\|\gamma(x_i) - \gamma(x_{i-1})\|.$$

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Lemma (Curve integral) : *Let $\epsilon \in \mathbb{R}_{>0}$. There is a $\delta \in \mathbb{R}_{>0}$ s.t. $\forall$ partitions $0 = x_0 < x_1 < ... < x_n = 1$ with $x_i - x_{i-1} < \delta (i \in 1, ..., n)$ we have*

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Proof: elementar analysis

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Proof: elementar analysis

Definition: We define the *curve integral* : let $(\gamma_i)_{i=1}^m$ be a tuple of injective functions in $C^1([0,1], \mathbb{R}^2)$. For all $i \in 1, ..., m$, we set $\Gamma_i := \gamma_i([0,1])$. The *curve integral* over the piecewise differentiable curve $\Gamma := \cup_{i=1}^m \Gamma_i$ is given by

$$\int_\Gamma u(x)dx := \sum_{i=1}^{m} \int_0^1 u(\gamma_i(y))\|\gamma_i'(y)\|dy.$$

2 Single layer potential

We fix n points $p_0, ..., p_{n-1} \in \mathbb{R}^2$, set $p_n := p_0$ and define the affine parametrizations

$$\gamma_i : [0,1] \to \mathbb{R}^2, \quad y \mapsto p_{i-1}(1-y) + p_i y,$$

for $i \in 1, ..., n$. As long as $p_i \neq p_j$ holds $\forall i, j \in 0, ..., n-1$ with $i \neq j$, this defines a polygonal curve $\Gamma := \cup_{i=1}^{m} \gamma_i([0,1])$.

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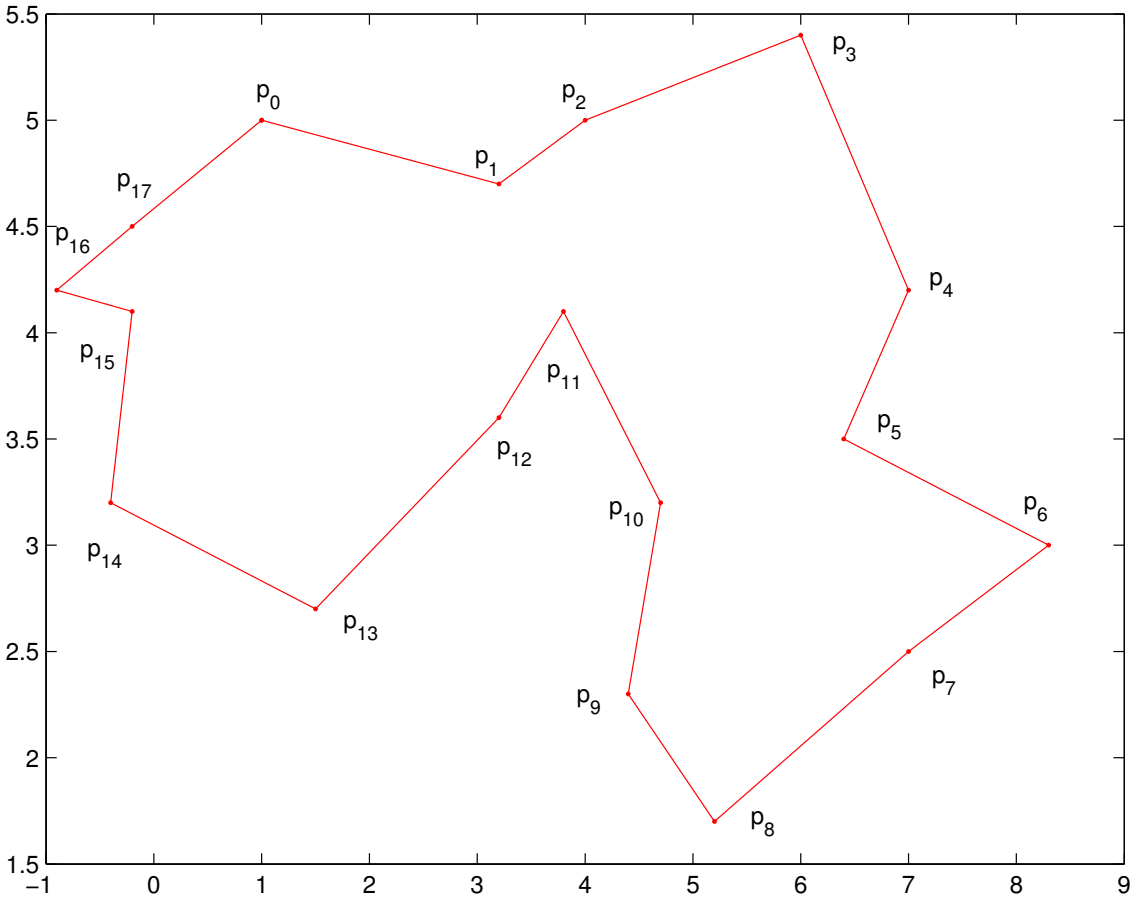
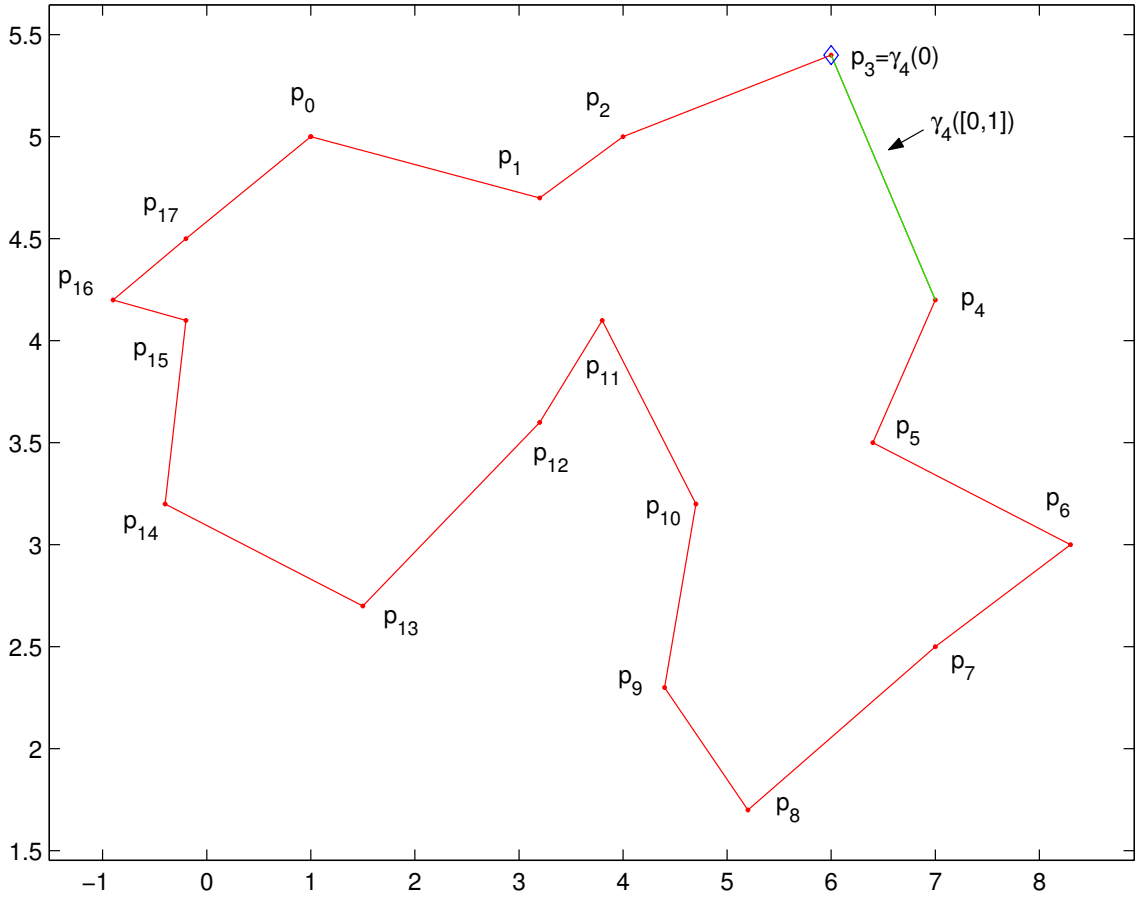
Example:

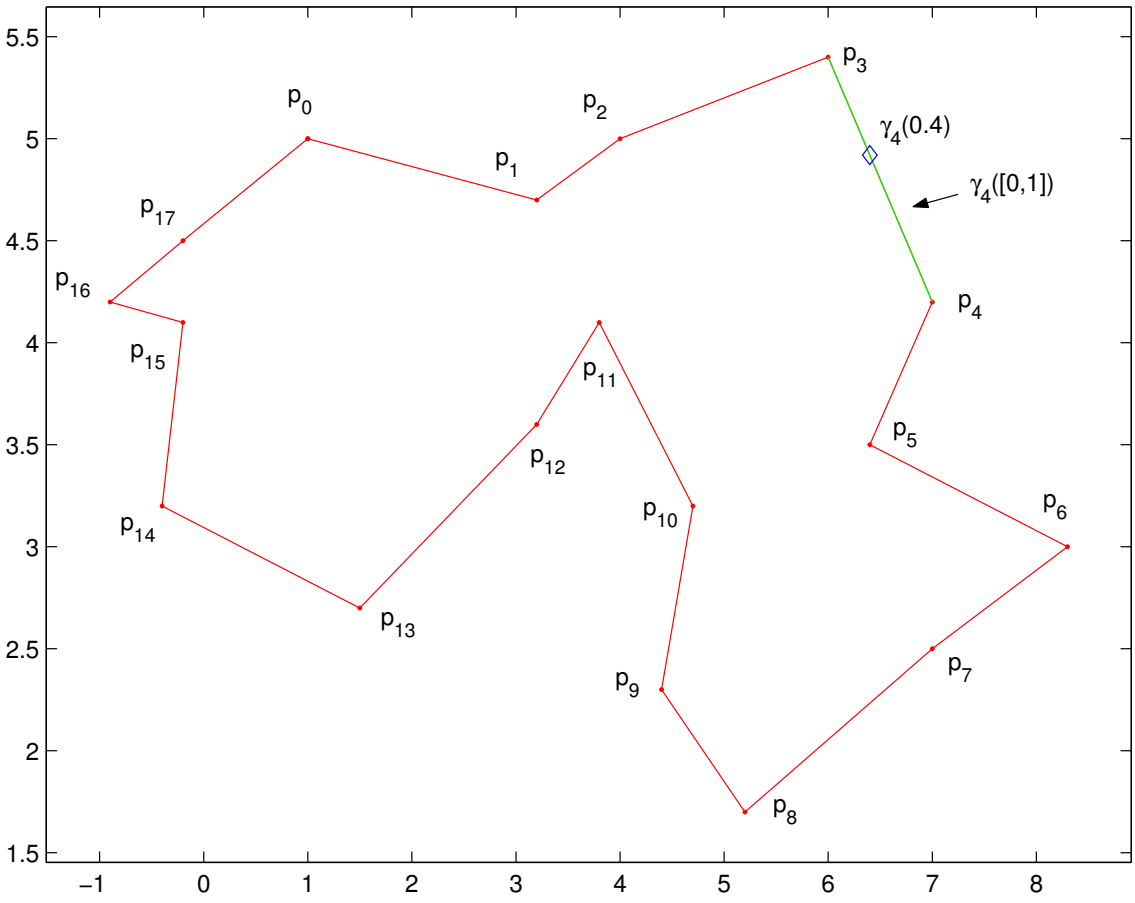
We consider the following points:



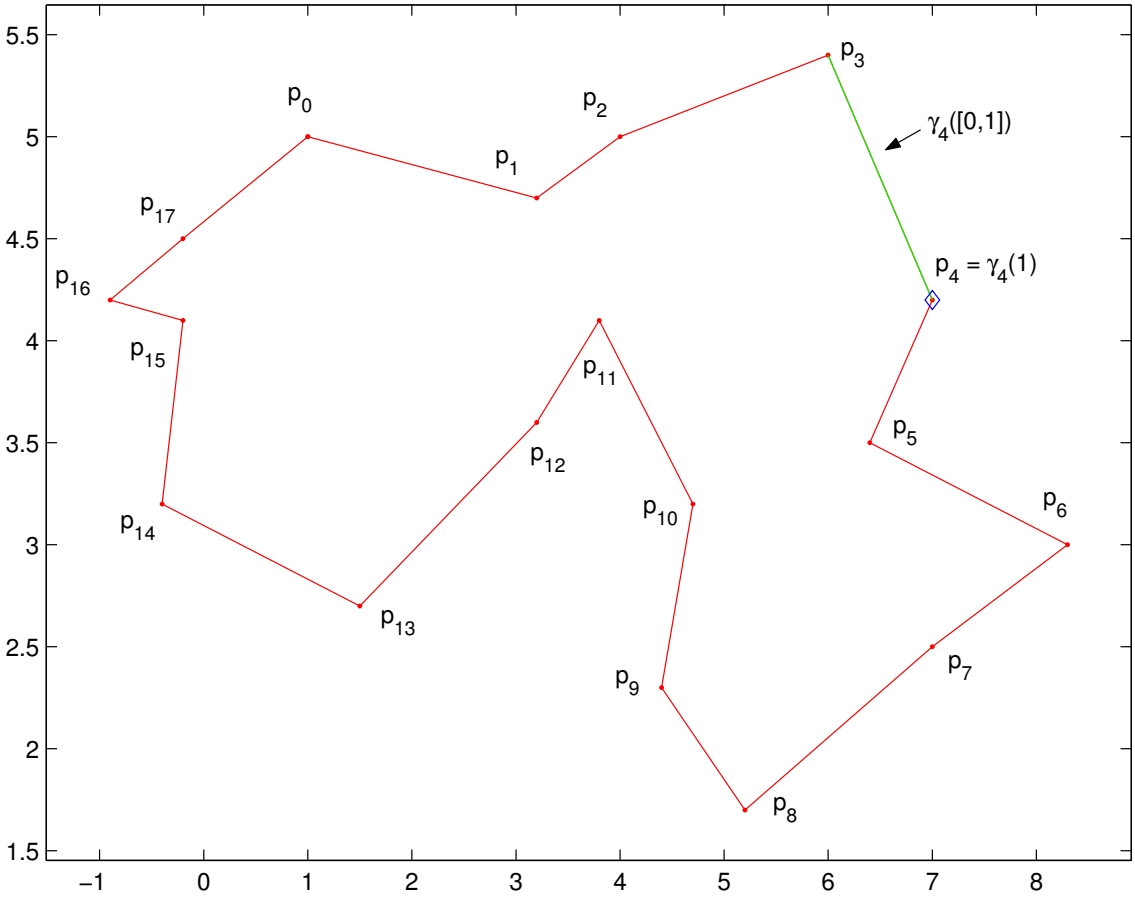
We observe γ_4 and its evaluation in t=0:



t=0.4:


p0
p1
p2
p3
p4
p5
p6
p7
p8
p9
p10
p11
p12
p13
p14
p15
p16
p17
γ4(0.4)
γ4([0,1])

and t=1:


p0
p1
p2
p3
γ4([0,1])
p4 = γ4(1)
p5
p6
p7
p8
p9
p10
p11
p12
p13
p14
p15
p16
p17
5.5
5
4.5
4
3.5
3
2.5
2
1.5
−1
0
1
2
3
4
5
6
7
8

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We discretize $a_{slp}(\cdot,\cdot)$ using piecewise $c^{\underline{st}}$ functions $(\varphi_i)_{i=1}^n$ defined through

$$\varphi_i \circ \gamma_j \equiv \delta_{ij}$$

for $i,j \in \mathcal{I} := 1,...,n$. The coeff. of the corresponding matrix are given by

$$\begin{aligned} G_{ij} &= a_{slp}(\varphi_i,\varphi_j) = \int_\Gamma \varphi_i(x) \int_\Gamma \log(\|x-y\|)\varphi_j(y)dydx \\ &= \|p_i - p_{i-1}\|\|p_j - p_{j-1}\| \int_0^1 \int_0^1 \log(\|\gamma_i(x)-\gamma_j(y)\|)dydx. \end{aligned}$$

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From $p_i \neq p_j$ it follows that this matrix is full. As long as $\gamma_i([0,1])$ doesn't intersect $\gamma_j([0,1])\forall i,j \in$ 1,...,n, we don't have singularities and can replace the kernel by degenerate approximations.

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We'll now focus on computing the entries of the hierarchical matrix, in particular the low-rank blocks, so we suppose that we have a function that initializes the full matrices and now consider the treatment of the low-rank blocks.

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They correspond to admissible pairs (t,s) of clusters and require the evaluation of a degenerate approximation of the kernel function. Let's assume that $\mathrm{diam}(Q^t) \leq \mathrm{diam}(Q^s)$, it follows

$$\tilde{(}g)(x,y) = \sum_{\nu \in K} \log(\|x_\nu^t - y\|)(L)_\nu^t(x)$$

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γ_i is affine, so $A_{i\nu}^{t,s}$ are polynomials of degree m. We can thus use an exact quadrature rule for its evaluation.

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- An array `l` of dim. `p` containing the transformed interpolation pts.

Recall(Gauss-quadrature):
The idea is to approximate an integral as

$$\int_{\Omega} f(\xi)d\xi \approx \sum_{K\in\mathcal{M}} |K| \sum_{l=1}^{P_K} w_l^K f(\pi_l^K).$$

The w_l^K are called local weights and the points π_l^K local nodes. The Gauss-quadrature is exact for polynomials of degree up to 2P - 1 using only P nodes.
The quadrature is numerically stable if all the weights are positive.

Example(Gauss-quadrature, 1-dimensional):

We consider the integral $\int_{-1}^{1} f(x)dx$ and want to approximate it with a Gauss-quadrature of degree P=2:

$$\int_{-1}^{1} f(\xi)d\xi = w_1 f(x_1) + w_2 f(x_2)$$

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And so we have

$$\begin{aligned} w_1 &= w_2 = 1 \\ x_1 &= -\frac{1}{\sqrt{3}} \\ x_2 &= \frac{1}{\sqrt{3}} \end{aligned}$$

The evaluation of $B^{t,s}$ requires us to integrate the kernel function for points x_ν^t on intervals given by p_{i-1} and p_i. Here, it can be done analytically. In more general cases, we can use the same quadrature rule as for polynomials, but the result will no longer necessarily be exact.

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and if we change γ to

$$\gamma(t) := \begin{pmatrix} 1+2t \\ 1+4t \end{pmatrix}$$

(and thus doubling its"speed") we obtain b=5.4231.

The `supermatrix` structure can be initialized by a simple recursion: If the `supermatrix` contains an `rkmatrix`, we compare the diameters of the clusters involved and use the procedure described above to initialize the fields `a` and `b` of the `rkmatrix`.

If the `supermatrix` contains a `fullmatrix`, we evaluate singular integrals and fill its field `e`.

Otherwise, we proceed recursively with the subblocks that are given by the array `s`.