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Fast integration of rapidly oscillatory functions

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Dedicated to the memory of Ivor M. Longman

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Abstract

A collocation method for approximating integrals of rapidly oscillatory functions is presented. The method is efficient for integrals involving Bessel functions $J_\nu(rx)$ with large oscillation parameter r, as well as for many other one- and multi-dimensional integrals of functions with rapid irregular oscillations.

Keywords: Oscillatory integrals; Collocation

1. Introduction

In many areas of applied mathematics one encounters the problem of computing rapidly oscillatory integrals of the type

$$I = \int_a^b g(x)S(rx)\,\mathrm{d}x, \tag{1.1}$$

where S is an oscillatory function, r is a large parameter and a and b are real and finite. Approximating I by usual numerical integration algorithms requires many function evaluations of g and S. Sometimes the evaluation of g is very expensive, and the computation becomes highly time consuming for very large values of r. The methods of Filon [1,2] and of Longman [4] are best appropriate if S is a trigonometric function. An efficient method for integrals of the type

$$I = \int_a^b g(x)\mathrm{e}^{\mathrm{i}q(x)}\,\mathrm{d}x, \tag{1.2}$$

with $\max_{x\in[a,b]}\{|q'(x)|\} \gg (b-a)^{-1}$ is presented in [3]. The present work is a direct extension of the method in [3] for the evaluation of integrals with more general oscillatory weights S. The problem of computing the integral is replaced by a problem of finding a solution of a system of linear ordinary differential equation, with no boundary conditions, and this last problem is efficiently

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solved by collocation. As in [3], it is shown here that the efficiency of the method does not deteriorate as r increases. The particular case of $S(x) = J_\nu(rx)$ is used as an example, and a Mathematica program for testing the method for $\nu = 0, 1$ is appended. We point out that for this class of integrals, of the form $\int_a^b g(x) J_\nu(rx)\,dx$, there exist other efficient methods [4–8]. The collocation method presented here is applicable to a wide class oscillatory integrals with weight functions S satisfying certain differential conditions. For example, it is appropriate for computing integrals of the form

$$\int_a^b g(x) \cos(r_1 x) J_\nu(r_2 x)\,dx \tag{1.3}$$

for large r_1 and r_2 and integrals involving $S(x) = J_\nu^2(rx)$ (Examples 2 and 3 in Section 3). Simple classification rules for a large class of oscillatory functions S which satisfy the required conditions are presented. The real power of the method is for multidimensional integration of rapidly oscillatory functions. The extension of the present work to the multidimensional case is straightforward, and is a direct extension of two-dimensional method in [3].

2. The general scheme

We consider a general class of rapidly oscillatory integrals of the form

$$I = \int_a^b f^t(x) w(x)\,dx \equiv \int_a^b \langle f, w\rangle (x)\,dx, \tag{2.1}$$

where $f(x) = (f_1(x), \ldots, f_m(x))^t$ is an m-vector of non-rapidly oscillatory functions and $w(x) = (w_1(x), \ldots, w_m(x))^t$ is an m-vector of linearly independent rapidly oscillatory functions. We further assume that$\{w_i\}_{i=1}^m$ satisfy a system of ordinary differential equation of the form

$$w'(x) = A(x) w(x), \tag{2.2}$$

where $A(x)$ is an $m \times m$ matrix of non-rapidly oscillatory functions. The integrals considered in [3] satisfy (2.1)–(2.2) with $m = 1$, while integrals involving Bessel functions, as $\int_a^b g(x) J_\nu(rx)\,dx$ with large r, satisfy (2.1)–(2.2) with $m = 2$. Following [3], the idea is to approximate the integrand in (2.1) by a derivative of a known function. In particular, here we would like to find an m-vector function $p(x) = (p_1(x), \ldots, p_m(x))^t$ such that

$$\langle p, w\rangle' \approx \langle f, w\rangle. \tag{2.3}$$

Then we would approximate the integral I as

$$I \approx \int_a^b \langle p, w\rangle'(x)\,dx = p^t(b) w(b) - p^t(a) w(a). \tag{2.4}$$

Expanding (2.3), using (2.2), we find out that $p(x)$ should satisfy

$$\langle p, w\rangle' = \langle p', w\rangle + \langle p, w'\rangle = \langle p', w\rangle + \langle p, Aw\rangle = \langle p' + A^t p, w\rangle \approx \langle f, w\rangle. \tag{2.5}$$

By the linear independence of the functions $\{w_i\}_{i=1}^m$ it then follows that p should be an approximation to a solution of the system of ordinary differential equations

$$Lq \equiv q' + A^t q = f. \tag{2.6}$$

By the above assumptions, f and A are not rapidly oscillatory. Therefore, as argued in [3], the system (2.6) has a particular solution which is not rapidly oscillatory, and we shall look for an approximation to this particular solution by collocation with 'nice' functions, e.g. polynomials.

For $i = 1, \ldots, m$ let $\{u_k^{(i)}\}_{k=1}^n$ be some linearly independent basis functions on $[a,b]$. An n-point collocation approximation to the solution of (2.6) is defined as $p^{(n)}(x) = (p_1^{(n)}(x), \ldots, p_m^{(n)}(x))^t$, where

$$p_i^{(n)}(x) = \sum_{k=1}^{n} c_k^{(i)} u_k^{(i)}(x), \quad i = 1, \ldots, m, \tag{2.7}$$

where the coefficients $\{c_k^{(i)}\}_{i=1,k=1}^{m,n}$ are determined by the collocation conditions

$$Lp^{(n)}(x_j) = f(x_j), \quad j = 1, \ldots, n, \tag{2.8}$$

where $\{x_j\}_{j=1}^n$ are regularly distributed in $[a,b]$. Following (2.4) the corresponding n-point approximation to the integral (2.1) is given by

$$I_n \equiv (p^{(n)}(b))^t w(b) - (p^{(n)}(a))^t w(a). \tag{2.9}$$

Some guidelines and examples for choosing the basis functions are presented in [3]. In the next section we describe in detail the case of integrals involving Bessel functions, including some numerical examples. We conclude this section by the following two simple lemmas which are helpful for identifying oscillatory weight functions satisfying (2.2).

Lemma 1. *If $w(x) = (w_1(x), \ldots, w_m(x))^t$ satisfies (2.2) and $q(x)$ is a monotone function on $[a,b]$, then $u(x) = w(q(x))$ satisfies*

$$u'(x) = B(x)u(x), \tag{2.10}$$

where $B(x)$ is an $m \times m$ matrix of non-rapidly oscillatory functions.

Lemma 2. *Let $u(x) = (u_1(x), \ldots, u_k(x))^t$ satisfy $u'(x) = B(x)u(x)$ and let $v(x) = (v_1(x), \ldots, v_l(x))^t$ satisfy $v'(x) = C(x)v(x)$ with $B(x)$ a $k \times k$ matrix and $C(x)$ an $l \times l$ matrix of non-rapidly oscillatory functions. Then*

$$w = \{u_i v_j \mid i = 1, \ldots, k,\ j = 1, \ldots, l\}$$

satisfies $w'(x) = A(x)w(x)$ with $A(x)$ an $m \times m$ matrix of non-rapidly oscillatory functions, $m = kl$.

3. Numerical examples

In this section we demonstrate the application of the collocation method to three type of oscillatory functions. The first is the well-studied case $S(x) = J_\nu(rx)$, also treated in [4–8]. The other two examples are $S(x) = J_\nu^2(rx)$ and $S(x) = \cos(r_1 x)J_\nu(r_2 x)$ which seem more complicated to handle. However, as shown below, the collocation method is easily applicable for all these cases, and is very efficient.

Example 1. The computation of $\int_a^b g(x)J_\nu(rx)\,dx$.

The basis to the treatment of integrals involving Bessel functions are the following two differential recurrence relations:

$$J'_{\nu-1}(x) = \frac{\nu-1}{x} J_{\nu-1}(x) - J_\nu(x), \tag{3.1}$$

$$J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x), \tag{3.2}$$

It follows that the 2-vector function $w(x) = (J_{\nu-1}(rx), J_\nu(rx))^t$ satisfies (2.2) with

$$A = \begin{pmatrix} (\nu-1)/x & -r \\ r & -\nu/x \end{pmatrix}. \tag{3.3}$$

For $\nu = 1$ the collocation method is applied for the approximation of integrals of the form $\int_a^b (f_1(x)J_0(rx) + f_2(x)J_1(x))\,dx$. An n-point approximation was computed by solving the collocation equations (2.8) with L defined by (2.6) and A given by (3.3) with $\nu = 1$. As collocation points we chose equidistant points in $[a,b]$, $x_j = a + [(j-1)/(n-1)](b-a)$, $j = 1,\dots,n$, and as basis functions we take the polynomials

$$u_k^{(i)}(x) = \left(x - \frac{a+b}{2}\right)^{k-1}, \quad k = 1,\dots,n, \; i = 1,2.$$

Table 1 shows the relative errors in n-point approximation to the integral $\int_1^2 (x^2+1)^{-1} J_0(rx)\,dx$. The results were obtained by the Mathematica program given in the Appendix. The table exhibits the fast convergence of the approximation as n increases. It also shows that with the same amount of work we can obtain similar relative errors for small and for large oscillation parameter r.

We remark that for $r = 1$ similar accuracy can be obtained by using n-point Gauss-quadrature rules. For $r = 1000$ however, to achieve the accuracy obtained here with only 9 points, classical quadrature rules require $\sim 10^6$ points.

Table 1
Relative errors in n-point approximation to $\int_1^2 (x^2+1)^{-1} J_0(rx)\,dx$

r	$n = 3$	$n = 5$	$n = 9$
1	2.7E − 3	1.7E − 5	2.8E − 9
10	1.8E − 2	6.4E − 4	7.7E − 8
100	2.6E − 4	9.1E − 6	4.2E − 9
1000	6.0E − 4	5.4E − 5	2.2E − 9

Example 2. $\int_a^b g(x)\cos(r_1 x)J_\nu(r_2 x)\,dx$.

$u(x) = e^{ir_1x}$ satisfies (2.2) with $m = 1$ and $v(x) = (J_{\nu-1}(r_2x), J_\nu(r_2x))^t$ satisfies (2.2) with A given in (3.3). Then, by Lemma 2, an integral of the form (1.3) can be handled by the collocation method with $m = 2$. Explicitly $w(x) = e^{ir_1x}(J_{\nu-1}(r_2x), J_\nu(r_2x))^t$ satisfies (2.2) with $m = 2$ and

$$A = \begin{pmatrix} ir_1 + (\nu-1)/x & -r_2 \\ r_2 & ir_1 - \nu/x \end{pmatrix}. \tag{3.4}$$

We have applied the collocation method to the integral

$$I[r_1, r_2] = \int_1^2 (x^2+1)^{-1}\cos(r_1x)J_0(r_2x)\,dx$$

for several values of r_1 and r_2, using the same basis functions and sets of collocation points as in Example 1. In Fig. 1, we plot the function for $r_1 = 100$ and $r_2 = 170$. In Table 2 we present the relative errors in the collocation approximation.

Example 3. $\int_a^b g(x)J_\nu^2(rx)\,dx$.

Applying Lemma 2 for $u(x) = v(x) = (J_{\nu-1}(rx), J_\nu(rx))^t$, it follows that $w(x) = (J_{\nu-1}^2(rx), J_{\nu-1}(rx)J_\nu(rx), J_\nu^2(rx))^t$ satisfies (2.2) with $A(x)$ being a 3×3 matrix. Explicitly here

$$A = \begin{pmatrix} 2(\nu-1)/x & -2r & 0 \\ r & -1/x & -r \\ 0 & 2r & -2\nu/x \end{pmatrix}. \tag{3.5}$$

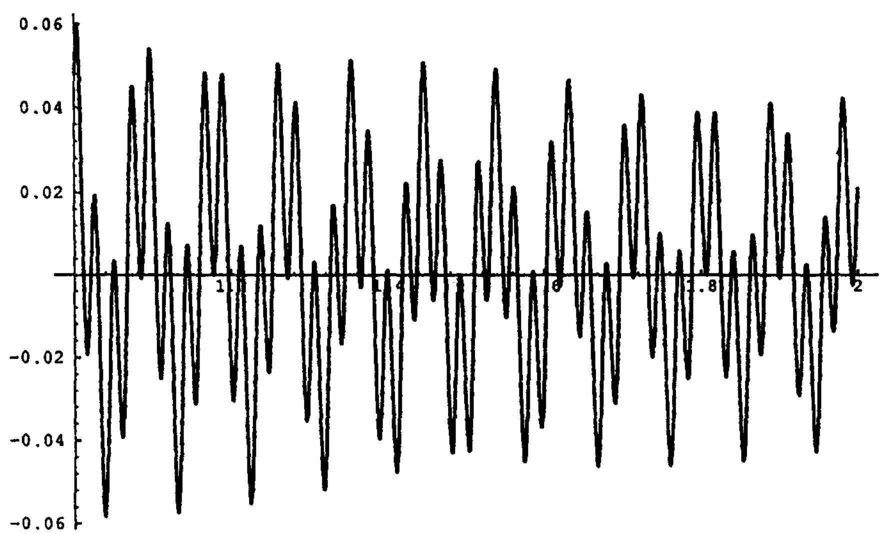

Fig. 1. Plot of the function $(x^2+1)^{-1}\cos(100x)J_0(170x)$ in [1, 2].

Table 2
Relative errors in the n-point approximation to $I[r_1, r_2]$

r_1, r_2	$n = 5$	$n = 9$	$n = 17$
10,17	2.1E − 4	6.5E − 8	2.9E − 13
100,170	2.0E − 4	1.1E − 7	2.6E − 12
1000,1700	9.8E − 7	7.7E − 10	1.9E − 14

Table 3
Relative errors in the n-point approximation to $\int_1^2 J_0^2(rx)\,\mathrm{d}x$

r	$n = 3$	$n = 5$	$n = 9$
1	3.7E − 3	2.4E − 5	7.2E − 9
10	9.0E − 3	2.4E − 4	4.2E − 8
100	2.3E − 3	2.3E − 5	7.8E − 8
1000	1.8E − 3	1.6E − 4	7.9E − 8

We applied the collocation method to the computation of $\int_1^2 J_0^2(rx)\,\mathrm{d}x$ using the same basis functions and sets of collocation points as in Example 1. Table 3 depicts the relative errors in the n-point collocation approximation.

4. Summary

A simple general scheme for computing rapidly oscillatory integrals is presented and tested. Although an error analysis and error estimation are still missing, it is shown that the method handles efficiently quite complicated oscillatory integrals.

Appendix

A Mathematica testing program for approximating $\int_a^b (f(x)J_0(rx) + g(x)J_1(x))\,\mathrm{d}x$

```
a = 1; b = 2; r = 100; n = 5; n2 = 2n
j0[x_] := BesselJ[0, x]; j1[x_] := BesselJ[1, x]
d = (a + b)/2 + 0.0000000000001 (* to avoid 0^0 in u[x, k] *)
u[x_, k_] := (x − d)^(k − 1); uprime[x_, k_] := (k − 1)(x − d)^(k − 2)
f[x_] := 1/(x^2 + 1); g[x_] := 0
point = Table[a + (j − 1)(b − a)/(n − 1), {j, 1, n}]
rhs = Table[0, {j, 1, n2}]
Do[rhs[[i]] = f[point[[i]]], {i, 1, n}]
```

```
Do[rhs[[n + i]] = g[point[[i]]], {i, 1, n}]
mat = Table[0, {j, 1, n2}, {k, 1, n2}]
Do[mat[[j, k]] = uprime[point[[j]], k], {j, 1, n}, {k, 1, n}]
Do[mat[[j, k]] = r u[point[[j]], k − n], {j, 1, n}, {k, n + 1, n2}]
Do[mat[[j + n, k]] = −r u[point[[j]], k], {j, 1, n}, {k, 1, n}]
Do[mat[[j + n, k]] = uprime[point[[j]], k − n] − u[point[[j]], k − n]/point[[j]], {j, 1, n},
{k, n + 1, n2}]
c = LinearSolve[mat, rhs]
approx = N[Sum[c[[k]]u[b, k], {k, 1, n}] j0[r b] − Sum[c[[k]]u[a, k], {k, 1, n}] j0[r a] +
Sum[c[[n + k]]u[b, k], {k, 1, n}] j1[r b] − Sum[c[[n + k]]u[a, k], {k, 1, n}] j1[r a]]
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