1 Linear Multistep Methods

1.1 Cauchy Problem

Consider the autonomous Cauchy problem:

$$\begin{cases} y' = f(y) \\ y(0) = y_0 \end{cases} \tag{1}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ satisfies a local Lipschitz conditio, in order to guarantee existence and uniqueness of solutions.
Given a **step-size** $h > 0$ and a **time-step** $k \in \mathbb{N}_0$, we define a **time-grid** as $\mathcal{G} = \{t_k := kh\}$. We aim to construct a grid function $\mathcal{G} \to \mathcal{R}^d$.

As a notational convention, we use $f_n := f(y_n)$.

Definition 1. *A **k-step Linear Multistep Method** is a numerical method for the approximation of* (1) *of the form*

$$\sum_{l=0}^{k} \alpha_l y_{j+l} = h \sum_{l=0}^{k} \beta_l f_{j+l}, \tag{2}$$

where $\alpha_l, \beta_l \in \mathbb{R}$*, and* $\alpha_k = 1$.

If $\beta_k \neq 0$, we call this an implicit method, otherwise it is an explicit method. For the implicit case, a unique solution for y_k exists for sufficiently small h.

1.2 Consistency

Suppose that y is a solution of (1). We are interested in the residual generated by using exact values of the solution in our numerical method. We consider

$$\tau(h; y_0) = \sum_{l=0}^{k} \alpha_l y(t_l) - h \sum_{l=0}^{k} \beta_l y(t_l)$$

Definition 2. *We say a linear multistep method for* (1) *is **consistent of order p** provided*

$$\tau(h; y_0) = o(h^{p+1}).$$

1.2.1 Necessary conditions for consistency

Definition 3. *Given a LMSM* (2)*, we define the **characteristic polynomials** of the method to be:*

$$\rho(\xi) = \sum_{l=0}^{k} \alpha_l \xi^l; \quad \sigma(\xi) = \sum_{l=0}^{k} \beta_l \xi^l.$$

Letting $f \equiv 0$ on the right-hand side of (1), it follows that $\sum_{l=0}^{k} \alpha_l = 0$, is a necessary condition for consistency of (2). Or equivalently, $\rho(1) = 0$. If $f(y) = y$, then we see that consistency implies $\rho'(1) - \sigma(1) = 0$.

1.3 Stability

Suppose $f = 0$. We are naturally led to the homogeneous linear difference equation

$$\sum_{j=0}^{k} \alpha_j y_j = 0. \tag{3}$$

If ζ is a root of ρ, then $y_j = \zeta^j$ is a solution of the homogeneous linear difference equation. In particular, if the polynomial $\rho(\zeta)$ splits with roots $\{\zeta_0, \dots, \zeta_{k-1}\}$, then

$$y_j = \gamma_0 \zeta_0^j + \gamma_1 \zeta_1^j + \cdots + \gamma_{k-1} \zeta_{k-1}^j$$

is again a solution to the difference equation, by linearity.

Remark 1. *If $\rho(\zeta) = 0$ with $|\zeta| > 1$, then slight perturbations of the initial conditions of the problem are amplified exponentially by the method. If $|\zeta| = 1$, and $\rho'(\zeta) = 0$ as well, then slight perturbations induce polynomial growth in the outputs of the method. Both of these cases are undesirable in a linear multistep method, especially in the case where exact solutions of* (1) *decay at infinity.*

Definition 4. *A Linear Multistep Method is **stable** provided:*

(i) $\rho(\zeta) = 0 \quad \Rightarrow \quad |\zeta| \leq 1$;

(ii) $\rho(\zeta) = 0 \wedge |\zeta| = 1 \quad \Rightarrow \quad \zeta$ *is a simple zero.*

1.4 Convergence

"*Consistency + Stability $\implies$ Convergence*"

More precisely, let $h > 0$ and define $\epsilon_k = y_k - y(kh)$, for $0 \leq k \leq T/h$. Then for a stable LMSM of order p we have:

$$\max_{0 \leq k \leq T/h} \|e_k\| = C(\max_{0 < l < k-1} \|y_l - y(t_l)\| + h^p),$$

where $C > 0$ is a constant independent of h.

Theorem 1. ***(Dahlquist barrier)****The order of a stable k-step LMSM can be at most*

$$\begin{cases} k+2, & k \text{ even} \\ k+1, & k \text{ odd} \\ k, & \text{if the method is explicit, i.e. } \beta_k = 0 \end{cases}$$

1.5 Stability Preservation

When will multistep methods be stiff integrators? That is, if the Cauchy problem is asymptotically stable, when can we expect that multistep approximations of the solutions will also decay at infinity?
Consider the linear model problem:

$$y' = \lambda y. \tag{4}$$

For $\mathrm{Re}\lambda < 0$, solutions of the model problem decay at infinity. A linear k-step method for this problem has the form:

$$\sum_{l=0}^{k} \alpha_l y_{j+l} = \lambda h \sum_{l=0}^{k} \beta_l y_{j+l}$$

Or equivalently,

$$\sum_{l=0}^{k} (\alpha_l - \lambda h \beta_l) y_{j+l} = 0.$$

This is a homogeneous linear difference equation. As before, we are naturally led to investigate the zeros of the polynomial $\rho_z(\zeta) := \rho(\zeta) - z\sigma(\zeta) \in \mathrm{Poly}_k$. We define the **stability domain** to be the set

$$\mathcal{A} := \{\lambda h : \ |y_n| \to 0, \text{ as } t_n \to \infty\}.$$

Desirable:

(i) $\mathcal{A} = \mathbb{C}^-$ **A-stability**

(ii) $\mathrm{Re}(z) \to -\infty \quad \Rightarrow \quad y_k \to 0$ *

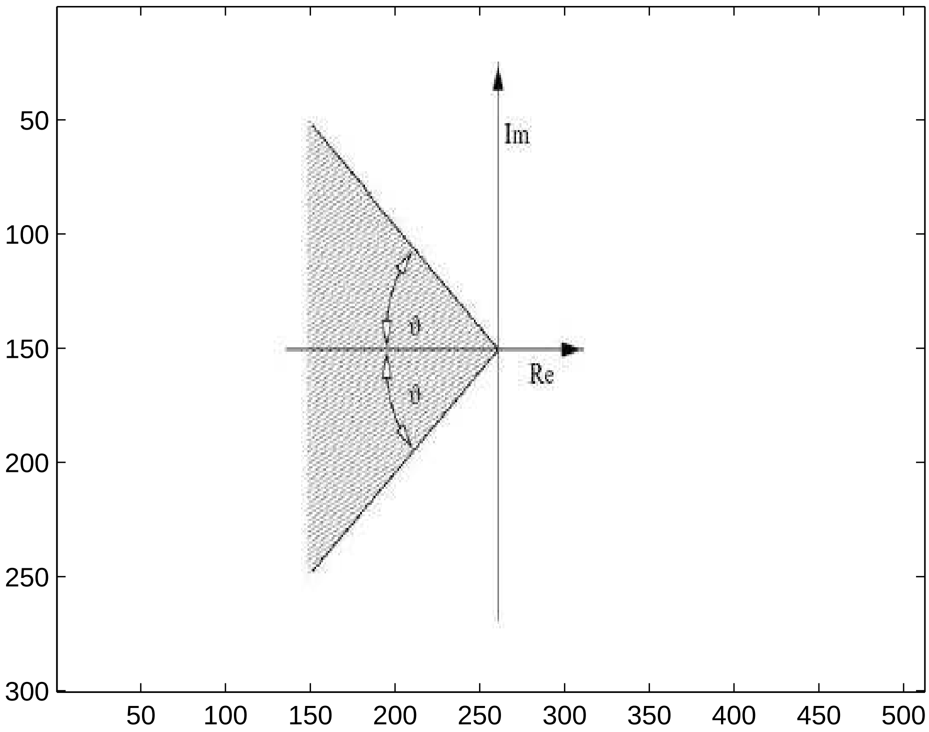
Remark 2. *Explicit linear multistep methods cannot be A-stable!*

Theorem 2. ***Dahlquist*** *Any A-stable linear multistep method has order at most 2.*

In fact, the only linear multistep method that is A-stable and achieves order 2 is the one-step method

$$y_{k+1} - y_k = \frac{1}{2}(f(y_k) + f(y_{k+1})),$$

also known as the implicit trapezoidal rule.
Big Disappointment!
In general, linear multistep methods have to settle for $\mathbf{A}(\vartheta)$-stability, i.e. the stability region contains a wedge



(amateur)

1.6 Backwards Differentiation Formulae Methods

Given $y_0, \dots, y_{k-1}$, fix y_k by

$$y_k = p(t_k),$$

where $p \in \mathrm{Poly}_k$, with $p(t_j) = y_j$, for $j = 0, \dots, k-1$ and $p'(t_k) = f(p(t_k))$. To show this polynomial exists, one has to invoke a fixed point argument. However, k+1 conditions uniquely determine a polynomial p. Let l_j be a Lagrange basis polynomial for the point t_j, then

$$p(t) = \sum_{j=0}^{k} y_j l_j(t)$$

and

$$p'(t_k) = \sum_{j=0}^{k} y_j l_j'(t_k) = f(y_k).$$

This is a linear multistep method with $\sigma(\xi) = \xi^k$ which is stable for $k \leq 6$ and A(ϑ)-stable (The larger the k, the smaller the ϑ)