1

z-TRANSFORM DEFINITION AND THEOREMS

The techniques of the z-transform method are not new, for they can be actually traced back as early as 1730 when DeMoivre[1] introduced the concept of the "generating function" (which is actually identical to the z-transform) to probability theory. The concept of the generating function was later extensively used in 1812 by Laplace[2] and others in probability theory. In a much later article by H. L. Seal,[3] a historical survey of the use of the generating function in probability theory was presented. Recently, the development and extensive applications of the z-transform[31-35] are much enhanced as a result of the use of digital computers in systems. These systems are referred to as discrete, because of the discrete nature of the signals or information flowing in them. Thus a new discipline of system theory is being developed, to be known as discrete system theory. The material here is devoted for the most part to discussing the various facets of this discrete theory.

The z-transform method constitutes one of the transform methods that can be applied to the solution of linear difference equations. It reduces the solutions of such equations into those of algebraic equations. The Laplace transform method, which is well developed for the solution of differential equations and extensively used in the literature, can be modified to extend its applicability to discrete systems. Such modifications have resulted in introducing the various associated transform techniques which are briefly discussed in the last section of this chapter.

This chapter is mainly devoted to the development of the theory of z-transform and the modified z-transform. Many useful theorems related to these transforms are either derived or stated. In addition, other theorems are introduced in the problem section related to this chapter.

1.1 Discrete time function and z-transform definitions[14–16,24–26]

In many discrete systems, the signals flowing are considered at discrete values of t, usually at nT, $n = 0, 1, 2, \ldots$, where T is a fixed positive number usually referred to as the sampling period. In Fig. 1.1, a continuous function of time $f(t)$ is shown where its values at $t = nT$ are indicated. The study of such discrete systems may be carried through by using the z-transform method. This method will be extensively developed in this and other chapters with its modifications, extensions, and applications.

Definition

Let T be a fixed positive number (it could be taken as unity). Let $f(t)$ be defined for this discussion for $t \geq 0$. This case will be extended in Chapter 4 to cover values of t which are also negative. The z-transform of $f(t)$ is the function

$$\mathfrak{z}[f] = \mathscr{F}(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}, \qquad \text{for } |z| > R = \frac{1}{\rho} \tag{1.1}$$

ρ = radius of convergence of the series

of the complex variable z. We use the symbol $\mathfrak{z}$ to denote the z-transform of f.

Since only the values $f_n = f(nT)$ of f at nT are used, the z-transform is actually defined for the sequence $\{f_n\}$.

$$\mathfrak{z}[\{f_n\}] = \sum_{n=0}^{\infty} f_n z^{-n} = \mathscr{F}(z) \tag{1.2}$$

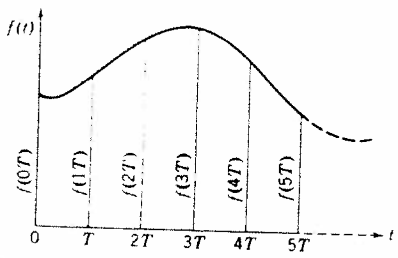

FIGURE 1.1 Discrete and continuous functions.

The series in equation (1.1) can always be considered as a formal series to be manipulated in certain ways and not necessarily to be summed.

If $f(t)$ has a jump discontinuity at a value nT, we shall always interpret $f(nT)$ as the limit of $f(t)$ as $t \to nT^+$, and we shall assume the existence of this limit, for $n = 0, 1, 2, \ldots$ for all $f(t)$ considered.

EXAMPLE

To obtain the z-transform of $f(t) = t$, we use equation (1.1) as follows:

$$\mathscr{F}(z) = \mathfrak{z}[f] = \sum_{n=0}^{\infty} f(nT)z^{-n} = Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + \ldots$$

$$= Tz^{-1}[1 + 2z^{-1} + 3z^{-2} + \ldots] = \frac{Tz}{(z-1)^2}, \qquad \text{for } |z| > 1 \tag{1.3}$$

1.2 Properties of z-transforms[5,14,25,26]

In the following we shall show a few properties and theorems related to the z-transform. Some theorems will be presented whose proofs could be easily obtained as an exercise in the problem section. Their use will enable us to develop the z-transform method and indicate its applications in the following chapters.

Linearity of the z-transform

For all constants c_1 and c_2, the following property holds:

$$\mathfrak{z}(c_1 f_1 + c_2 f_2) = \sum_{n=0}^{\infty} [c_1 f_1(nT) + c_2 f_2(nT)]z^{-n}$$

$$= c_1 \sum_{n=0}^{\infty} f_1(nT)z^{-n} + c_2 \sum_{n=0}^{\infty} f_2(nT)z^{-n}$$

$$= c_1\mathfrak{z}[f_1] + c_2\mathfrak{z}[f_2] \tag{1.4}$$

Thus $\mathfrak{z}$ is a linear operator on the linear space of all z-transformable functions $f(t)$, $(t \geq 0)$.

Shifting theorem

If $\mathfrak{z}[f] = \mathscr{F}(z)$,

$$\mathfrak{z}[f(t + T)] = z[\mathscr{F}(z) - f(0^+)] \tag{1.5}$$

Proof: By definition

$$\mathscr{Z}[f(t+T)] = \sum_{n=0}^{\infty} f[(n+1)T]z^{-n}$$

$$= z\sum_{n=0}^{\infty} f[(n+1)T]z^{-(n+1)} = z\sum_{k=1}^{\infty} f(kT)z^{-k} \qquad (1.6)$$

where $k = n + 1$. By adding and substracting $f(0^+)$ term under the summation sign of equation (1.6), we can write the summation over the range from $k = 0$ to $k = \infty$. Thus

$$\mathscr{Z}[f(t+T)] = z\left[\sum_{k=0}^{\infty} f(kT)z^{-k} - f(0^+)\right] = z[\mathscr{F}(z) - f(0^+)] \qquad (1.7)$$

Extending these procedures, we can readily obtain for any positive integer m the following results:

$$\mathscr{Z}[f(t+mT)] = z^m\left[\mathscr{F}(z) - \sum_{k=0}^{m-1} f(kT)z^{-k}\right] \qquad (1.8)$$

COROLLARY: If $\mathscr{Z}[f(t)] = \mathscr{F}(z)$, then

$$\mathscr{Z}[f(t-nT)u(t-nT)] = z^{-n}\mathscr{F}(z) \text{ where } u(t)$$
$$= \text{unit step function} \qquad \text{for } n = 0, 1, 2, \ldots \qquad (1.9)$$

Proof: By definition

$$\mathscr{Z}[f(t-nT)u(t-nT)] = \sum_{m=0}^{\infty} [f(m-n)Tu(m-n)T]z^{-m}$$

$$= z^{-n}\sum_{m=0}^{\infty} \{f[(m-n)T]u[(m-n)T]\}z^{-(m-n)} \qquad (1.10)$$

Letting $m - n = k$, we obtain

$$\mathscr{Z}[f(t-nT)u(t-nT)] = z^{-n}\sum_{k=-n}^{\infty} f(kT)u(kT)z^{-k}$$

$$= z^{-n}\sum_{k=0}^{\infty} f(kT)z^{-k} = z^{-n}\mathscr{F}(z) \qquad (1.11)$$

The use of the shifting theorem is important in the solution of difference equations as indicated in Chapter 2. Following a similar procedure, we can easily obtain the z-transform of the forward difference as well as the backward difference as follows:

$$\mathscr{Z}[\Delta^k f(nT)] = (z-1)^k\mathscr{F}(z) - z\sum_{j=0}^{k-1}(z-1)^{k-j-1}\Delta^j f(0T) \qquad (1.12)$$

where

$$\Delta f(nT) = f(n+1)T - f(nT) \qquad (1.13)$$

Furthermore,

$$\mathscr{Z}[\nabla^k f(nT)] = (1 - z^{-1})^k\mathscr{F}(z) \qquad (1.14)$$

where

$$\nabla f(nT) = f(nT) - f(n-1)T \qquad (1.15)$$

Complex scale change

If the z-transform of $f(t)$ is $\mathscr{F}(z)$,

$$\mathscr{Z}[e^{-at}f(t)] = \mathscr{F}(e^{aT}z). \qquad (1.16)$$

Proof: From the definition of the z-transform we have

$$\mathscr{Z}[e^{-at}f(t)] = \sum_{n=0}^{\infty} e^{-anT}f(nT)z^{-n} = \sum_{n=0}^{\infty} f(nT)(e^{aT}z)^{-n} \qquad (1.17)$$
$$= \mathscr{F}(e^{aT}z)$$

Finite summation[14,25,28]

To obtain the z-transform of

$$\mathscr{Z}\left[\sum_{k=0}^{n} f(kT)\right] \qquad (1.18)$$

first we define

$$\sum_{k=0}^{n} f(kT) \triangleq g[(nT)], \quad \text{or} \quad g[(n-1)T] = \sum_{k=0}^{n-1} f(kT) \qquad (1.19)$$

We can write a relation between successive values of the sum by noting this definition and equation (1.19).

$$g(nT) = g(n-1)Tu(n-1)T + f(nT) \qquad (1.20)$$

Applying the z-transform to this equation, we have

$$\mathscr{G}(z) = z^{-1}\mathscr{G}(z) + \mathscr{F}(z) \qquad (1.21)$$

Solving for $\mathscr{G}(z)$, we finally obtain

$$\mathscr{G}(z) = \mathscr{Z}\left[\sum_{k=0}^{n} f(kT)\right] = \frac{z}{z-1}\mathscr{F}(z). \qquad \text{for } |z| > 1 \qquad (1.22)$$

Initial and final values[14,24]

From the definition of the z-transform,

$$\mathscr{F}(z) = \sum_{n=0}^{\infty} f(nT)z^{-n} = f(0T) + \frac{f(1T)}{z} + \frac{f(2T)}{z^2} + \ldots \qquad (1.23)$$

We readily notice that the initial value is obtained as:

$$f(0) = \lim_{z\to\infty} \mathscr{F}(z) \tag{1.24}$$

If $f(0) = 0$, we can obtain $f(T)$ as the $\lim_{z\to\infty} z\mathscr{F}(z)$.

For the final value, let us write

$$\mathfrak{z}[f(t+T) - f(t)] = \lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1)T - f(kT)]z^{-k} \tag{1.25}$$

The transform of the left-hand side is obtained from equation (1.4) and (1.5) thus

$$z\mathscr{F}(z) - zf(0T) - \mathscr{F}(z) = \lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1)T - f(kT)]z^{-k} \tag{1.26}$$

We now let $z \to 1$ for both sides of equation (1.26), assuming the order of taking the limits may be interchanged.

$$\begin{aligned}
\lim_{z\to1} (z-1)\mathscr{F}(z) - f(0T) &= \lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1)T - f(kT)] \\
&= \lim_{n\to\infty} \{[f(1T) - f(0T)] \\
&\quad + [f(2T) - f(1T)] \\
&\quad + \ldots + [f(nT) - f(n-1)T] \\
&\quad + [f(n+1)T - f(nT)]\} \\
&= \lim_{n\to\infty} [-f(0T) + f(n+1)T] \\
&= -f(0T) + f(\infty)
\end{aligned} \tag{1.27}$$

so that we finally obtain

$$\lim_{n\to\infty} f(nT) = \lim_{z\to1} (z-1)\mathscr{F}(z) \tag{1.28}$$

if the limit exists.

Complex multiplication (real convolution)

If f_1 and f_2 have the z-transform $\mathscr{F}_1(z)$ and $\mathscr{F}_2(z)$, then

$$\mathscr{F}_1(z)\mathscr{F}_2(z) = \mathfrak{z}\left[\sum_{k=0}^{n} f_1(kT)f_2(n-k)T\right] \tag{1.29}$$

Proof: By definition

$$\mathscr{F}_1(z)\mathscr{F}_2(z) = \sum_{k=0}^{\infty} f_1(kT)z^{-k}\mathscr{F}_2(z) \tag{1.30}$$

but from equation (1.11)

$$z^{-k}\mathscr{F}_2(z) = \mathfrak{z}[f_2(t-kT)] \tag{1.31}$$

if $f_2\,(n-k)\,T = 0$, for $n < k$

Hence

$$\begin{aligned}
\mathscr{F}_1(z)\mathscr{F}_2(z) &= \sum_{k=0}^{\infty} f_1(kT)\mathfrak{z}[f_2(t-kT)] \\
&= \sum_{k=0}^{\infty} f_1(kT) \sum_{n=0}^{\infty} f_2[(n-k)T]z^{-n} \\
&= \sum_{n=0}^{\infty} \left\{\sum_{k=0}^{\infty} f_1(kT)f_2[(n-k)T]\right\} z^{-n}
\end{aligned} \tag{1.32}$$

but $f_2\,(n-k)\,T = 0$, for $n < k$ and therefore

$$\mathscr{F}_1(z)\mathscr{F}_2(z) = \mathfrak{z}\left[\sum_{k=0}^{n} f_1(kT)f_2(n-k)T\right] \tag{1.33}$$

Complex differentiation or multiplication by t

If $\mathscr{F}(z)$ is the z-transform of f, then

$$\mathfrak{z}[tf] = -Tz\frac{d}{dz}\mathscr{F}(z) \tag{1.34}$$

Proof: By definition

$$\mathfrak{z}[tf] = \sum_{n=0}^{\infty} (nT)f(nT)z^{-n} = -Tz\sum_{n=0}^{\infty} f(nT)[-nz^{-n-1}] \tag{1.35}$$

The term in the brackets is a derivative of z^{-n} with respect to z.

$$\begin{aligned}
\mathfrak{z}[tf] &= -Tz\sum_{n=0}^{\infty} f(nT)\frac{d}{dz}z^{-n} = -Tz\frac{d}{dz}\sum_{n=0}^{\infty} f(nT)z^{-n} \\
&= -Tz\frac{d}{dz}\mathscr{F}(z)
\end{aligned} \tag{1.36}$$

Similarly, we can write

$$\mathfrak{z}[t^k f] = -Tz\frac{d}{dz}\mathscr{F}_1(z) \tag{1.37}$$

where

$$\mathscr{F}_1(z) = \mathfrak{z}[t^{k-1}f], \qquad k > 0 \quad \text{and integer} \tag{1.38}$$

As a corollary to this theorem we can deduce

$$\mathfrak{z}[n^{(k)}f(n)] = z^{-k}\frac{d^k\mathscr{F}(z)}{d(z^{-1})^k} \tag{1.39}$$

where the function $n^{(k)}$ is given by

$$n^{(k)} = n(n-1)(n-2)\ldots(n-k+1) \tag{1.40}$$

As special cases of the preceding,

$$\mathfrak{z}[(-1)^k n^{(k)} f(n-k+1)] = z\frac{d^k\mathscr{F}(z)}{dz^k} \tag{1.41}$$

and

$$\mathfrak{z}[n(n+1)(n+2)\ldots(n+k-1)f(n)] = (-1)^k z^k \frac{d^k\mathscr{F}(z)}{dz^k} \tag{1.42}$$

Special summation theorem

If $\mathscr{F}(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}$, then

$$\mathscr{F}(z^k) = \mathfrak{z}[\nabla f_1(nT)] \tag{1.43}$$

where

$$f_1(nT) = \sum_{m=0}^{[n/k]} f(mT), \qquad \nabla f_1(nT) = f_1(nT) - f_1(n-1)T \tag{1.44}$$

and $[n/k]$ denotes the largest integer in n/k.

Proof: From the definition of the z-transform we can write

$$\mathfrak{z}\left[\sum_{m=0}^{[n/k]} f(mT)\right] \triangleq \sum_{n=0}^{\infty} z^{-n} \sum_{m=0}^{[n/k]} f(mT) = \sum_{n=0}^{\infty}\sum_{m=0}^{[n/k]} z^{-n} f(mT) \tag{1.45}$$

Since n varies in integer values from zero to infinity and m also varies in integer values, we can write for the right-hand side of this equation,

$$\mathfrak{z}\left[\sum_{m=0}^{[n/k]} f(mT)\right] = \sum_{m=0}^{\infty}\sum_{n=mk}^{\infty} f(mT)z^{-n} = \sum_{m=0}^{\infty} f(mT) \sum_{n=mk}^{\infty} z^{-n}$$

$$= \sum_{m=0}^{\infty} f(mT)z^{-mk} \sum_{n=0}^{\infty} z^{-n} \tag{1.46}$$

However,

$$\sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1}, \qquad \text{for } |z| > 1$$

Therefore equation (1.46) can be written

$$\mathfrak{z}\left[\sum_{m=0}^{[n/k]} f(mT)\right] = \frac{z}{z-1}\sum_{m=0}^{\infty} f(mT)z^{-mk} = \frac{z}{z-1}\mathscr{F}(z^k) \tag{1.47}$$

From equation (1.14), we have for $k = 1$

$$\mathfrak{z}[\nabla f(nT)] = \frac{z-1}{z}\mathscr{F}(z) \tag{1.48}$$

Using this equation in (1.47) we finally obtain

$$\mathscr{F}(z^k) = \mathfrak{z}[\nabla f_1(nT)] \tag{1.49}$$

This theorem is very important in obtaining the inverse z-transform of **special functions of $\mathscr{F}(z)$ containing essential singularities. These functions** are discussed in detail in the next section.

We introduce the following additional theorems whose proofs are left as an exercise to the reader.

Differentiation with respect to second independent variable

$$\mathfrak{z}\left[\frac{\partial}{\partial a} f(t, a)\right] = \frac{\partial}{\partial a}\mathscr{F}(z, a) \tag{1.50}$$

Second independent variable limit value

$$\mathfrak{z}\left[\lim_{a\to a_0} f(t, a)\right] = \lim_{a\to a_0}\mathscr{F}(z, a) \tag{1.51}$$

Integration with respect to second independent variable

$$\mathfrak{z}\left[\int_{a_0}^{a_1} f(t, a)\,da\right] = \int_{a_0}^{a_1}\mathscr{F}(z, a)\,da \tag{1.52}$$

if the integral is finite.

1.3 Inverse z-transform and branch points[14,27]

The discrete function $f(t)$ at $t = nT$ or $f(nT)$ can be obtained from $\mathscr{F}(z)$ by a process called the inverse z-transform. This process is symbolically denoted as

$$f(nT) = \mathfrak{z}^{-1}[\mathscr{F}(z)] \tag{1.53}$$

where $\mathscr{F}(z)$ is the z-transform of $f(t)$† or $f(nT)$.

In the following, we discuss the several methods from which we can obtain $f(nT)$ from $\mathscr{F}(z)$ or the inverse z-transform.

The power series method

When $\mathscr{F}(z)$ is given as a function analytic for $|z| > R$ (and at $z = \infty$), the value of $f(nT)$ can be readily obtained as the coefficient of z^{-n} in the power series expansion (Taylor's series) of $\mathscr{F}(z)$ as a function of z^{-1}.

† It may also be written as f.

From equation (1.1) it is observed that

$$\mathscr{F}(z) = f(0T) + f(T)z^{-1} + \dots f(kT)z^{-k} + \dots f(nT)z^{-n} + \dots \tag{1.54}$$

Thus it is noticed that $f(nT)$ can be read off as the coefficient of z^{-n} and so can values of f at other instants of time.

If $\mathscr{F}(z)$ is given as a ratio of two polynomials in z^{-1}, the coefficients $f(0T), \dots, f(nT)$ are obtained as follows:

$$\begin{aligned}\mathscr{F}(z) &= \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + \dots + p_n z^{-n}}{q_0 + q_1 z^{-1} + q_2 z^{-2} + \dots + q_n z^{-n}} \\ &= f(0T) + f(1T)z^{-1} + f(2T)z^{-2} + \dots \end{aligned} \tag{1.55}$$

where

$$\begin{aligned} p_0 &= f(0T)q_0 \\ p_1 &= f(1T)q_0 + f(0T)q_1 \\ &\vdots \\ p_n &= f(nT)q_0 + f(n-1)Tq_1 + f(n-2)Tq_2 + \dots + f(0T)q_n \end{aligned} \tag{1.56}$$

It is also observed that $f(nT)$ can be obtained by a synthetic division of the numerator by the denominator.

Partial fraction expansion

If $\mathscr{F}(z)$ is a rational function of z, analytic at ∞, it can be expressed by a partial fraction expansion,

$$\mathscr{F}(z) = \mathscr{F}_1(z) + \mathscr{F}_2(z) + \mathscr{F}_3(z) + \dots \tag{1.57}$$

The inverse of this equation $f(nT)$ can be obtained as the sum of the individual inverses obtained from the expansion, that is,

$$f(nT) = \mathscr{Z}^{-1}\mathscr{F}(z) = \mathscr{Z}^{-1}[\mathscr{F}_1(z)] + \mathscr{Z}^{-1}[\mathscr{F}_2(z)] + \dots \tag{1.58}$$

We can easily identify the inverse of a typical $\mathscr{F}_k(z)$, from tables† or power series and thus obtain $f(nT)$.‡

† See Appendix, Table 1.

‡ A determinant method for obtaining $f(nT)$ from equation (1.55) is in the Appendix of this chapter.

Complex integral formula

We can also represent the coefficient $f(nT)$ as a complex integral. Since $\mathscr{F}(z)$ can be regarded as a Laurent series, we can multiply $\mathscr{F}(z)$ in equation (1.54) by z^{n-1} **and integrate around a circle** Γ **on which** $|z| = R_0$, $R_0 > R$, **or any simple closed path on or outside of which** $\mathscr{F}(z)$ **is analytic. This integral yields** $2\pi j$ **times the residue of the integrand, which is, in this case,** $f(nT)$, **the coefficient of** z^{-1}. **Hence**

$$\oint_\Gamma \mathscr{F}(z)z^{n-1}\,dz = f(nT)\cdot 2\pi j \tag{1.59}$$

or

$$f(nT) = \frac{1}{2\pi j}\oint_\Gamma \mathscr{F}(z)z^{n-1}\,dz, \qquad (n = 0, 1, 2, \dots) \tag{1.60}$$

The contour Γ encloses all singularities of $\mathscr{F}(z)$ as shown in Fig. 1.2.

The contour integral in equation (1.60) can be evaluated when $\mathscr{F}(z)$ has only isolated singularities by using Cauchy's integral formula,

$$f(nT) = \frac{1}{2\pi j}\oint_\Gamma \mathscr{F}(z)z^{n-1}\,dz = \text{sum of the residues of } \mathscr{F}(z)z^{n-1} \tag{1.61}$$

The following cases are discussed for the inversion formula.

THE POLES OF $\mathscr{F}(z)$ ARE SIMPLE. Assume that

$$\mathscr{F}(z) = \frac{\mathscr{H}(z)}{\mathscr{G}(z)} \tag{1.62}$$

When $\mathscr{G}(z)$ **has simple zeros only, the residue at a simple singularity** a **is given by**

$$\lim_{z\to a}(z-a)\mathscr{F}(z)z^{n-1} = \lim_{z\to a}\left[(z-a)\frac{\mathscr{H}(z)}{\mathscr{G}(z)}z^{n-1}\right] \tag{1.63}$$

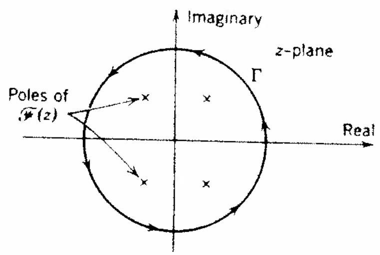

FIGURE 1.2 Contour integration in the z-plane.

POLES OF $\mathscr{F}(z)$ ARE NOT GIVEN IN A FACTORED FORM, BUT ARE SIMPLE. The residue at the singularity a_m is

$$\left[\frac{\mathscr{H}(z)}{\mathscr{G}'(z)} z^{n-1}\right]_{z=a_m} \tag{1.64}$$

where

$$\mathscr{G}'(z) = \frac{d\mathscr{G}(z)}{dz} \tag{1.65}$$

$\mathscr{F}(z)$ HAS MULTIPLE POLES. The residue at a kth-order pole of $\mathscr{F}(z)$ is given by the following expression: residue at kth-order pole at a,

$$h_1 = \frac{1}{k-1!} \frac{d^{k-1}}{dz^{k-1}} [\mathscr{F}(z)(z-a)^k z^{n-1}]\bigg|_{z=a} \tag{1.66}$$

$\mathscr{F}(z)$ has essential singularities[37]

In some cases when $\mathscr{F}(z)$ has an essential singularity at z other than infinity, we can utilize the preceding theorems of the z-transform to obtain the inverse without power series expansion. This is best illustrated by the following examples.

If $\mathscr{F}(z)$ is given as

$$\mathscr{F}(z) = e^{-(x/z)-1/2z^2} \tag{1.67}$$

to obtain its inverse, that is, $f(nT) = \mathscr{Z}^{-1}[\mathscr{F}(z)]$.

We let $\mathscr{G}(z) = e^{-x/z}$ and $\mathscr{H}(z) = e^{-1/2z^2}$; then by using the real convolution theorem on p. 6,

$$f(nT) = \mathscr{Z}^{-1}[\mathscr{G}(z)\mathscr{H}(z)] = \sum_{k=0}^{n} h(kT)g[(n-k)T] \tag{1.68}$$

The inverse of $e^{-x/z}$ can be readily obtained from the series expansion

$$g(nT) = \mathscr{Z}^{-1}[\mathscr{G}(z)] = \frac{(-x)^n}{n!} \tag{1.69}$$

Furthermore, from the theorem on p. 8, we have for $k = 2$,

$$\mathscr{Z}^{-1}[e^{-1/2z^2}] = h(nT) = \nabla \sum_{m=0}^{[n/2]} \frac{(-1)^m}{2^m m!} \tag{1.70}$$

Using equations (1.69) and (1.70) in equation (1.68), we obtain (replacing k by s)

$$f(nT) = \mathscr{Z}^{-1}[e^{-x/z}e^{-1/2z^2}] = \sum_{s=0}^{n} \left[\nabla \sum_{m=0}^{[s/2]} \frac{(-1)^m}{2^m m!}\right] \frac{(-x)^{n-s}}{(n-s)!} \tag{1.71}$$

The expression in the bracketed term of equation (1.71) is different from zero only for s even, and then it is equal to

$$\frac{(-1)^{s/2}}{2^{s/2}(s/2)!} \tag{1.72}$$

Putting $s = 2k$ in equation (1.71) and using (1.72), we finally obtain for the inverse z-transform

$$f(nT) = \mathscr{Z}^{-1}[e^{-x/z}e^{-1/2z^2}] = \sum_{k=0}^{[n/2]} \frac{(-1)^{n-k}}{k!} \frac{x^{n-2k}}{(n-2k)!\,2^k} = \frac{H_n(x)}{n!} \tag{1.73}$$

where H_n is the Hermite polynomial of degree n.

$\mathscr{F}(z)$ is irrational function of "z"[37]

The inverse z-transform can be obtained either by power series expansion or the integral formula. If we use the latter form, we must be specially careful in the integration because of branch points in the integrand. As in the previous case, we shall illustrate the procedure by using the following example.

Let $\mathscr{F}(z)$ be given as

$$\mathscr{F}(z) = \left(\frac{z+b}{z}\right)^{\alpha} \tag{1.74}$$

where α can be any real value and assumed to be noninteger.

To obtain $f(nT)$ we use the two procedures of contour integration and power series expansion. To simplify the integration process we introduce a change of variable to normalize the constant b to be unity.

Let

$$z = by \tag{1.75}$$

Then $\mathscr{F}(z)$ becomes

$$\mathscr{F}(by) = \left(\frac{y+1}{y}\right)^{\alpha} \tag{1.76}$$

The function $\mathscr{F}(by)$ has a branch cut in the y-plane that extends from zero to minus unity as shown in Fig. 1.3. By using equation (1.60), the inverse is given

$$f(nT) = \mathscr{Z}^{-1}[\mathscr{F}(by)] = \frac{b^n}{2\pi j} \int_{\Gamma} \left(\frac{y+1}{y}\right)^{\alpha} y^{n-1}\, dy \tag{1.77}$$

where the closed contour Γ is as shown in Fig. 1.3.

We can easily show that at the limit $y \to 0$ the integral around the small circle BCD is zero. Furthermore, the integral along EA is also zero.

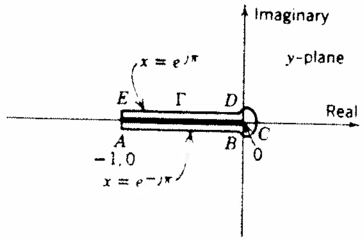

FIGURE 1.3 Contour integration enclosing branch point of $\mathscr{F}(by)$.

Therefore $f(nT)$ can be obtained by integrating around the barrier.

$$f(nT) = \frac{b^n}{2\pi j}\left[\int_1^0 \left(\frac{xe^{-j\pi}+1}{xe^{-j\pi}}\right)^\alpha x^{n-1}e^{-j\pi n}\,dx + \int_0^1 \left(\frac{xe^{j\pi}+1}{xe^{j\pi}}\right)^\alpha x^{n-1}e^{j\pi n}\,dx\right] \tag{1.78}$$

This equation can also be written as

$$f(nT) = \frac{b^n}{2\pi j}\left[\int_0^1 x^{n-1-\alpha}(1-x)^\alpha e^{j\pi(n-\alpha)}\,dx - \int_0^1 x^{n-1-\alpha}(1-x)^\alpha e^{-j\pi(n-\alpha)}\,dx\right]$$

$$= b^n \frac{\sin(n-\alpha)\pi}{\pi}\int_0^1 x^{n-1-\alpha}(1-x)^\alpha\,dx \tag{1.79}$$

By utilizing the following identity,

$$B(m,k) = \frac{\Gamma(m)\Gamma(k)}{\Gamma(m+k)} = \int_0^1 x^{m-1}(1-x)^{k-1}\,dx \tag{1.80}$$

we obtain for $f(nT)$

$$f(nT) = b^n \frac{\sin(n-\alpha)\pi}{\pi}\,\frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(n+1)} \tag{1.81}$$

By using the identity

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin \pi m} \tag{1.82}$$

we finally obtain

$$f(nT) = b^n \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)} = \binom{\alpha}{n} b^n \tag{1.83}$$

and for α negative integer

$$\binom{\alpha}{n} = \frac{\Gamma(n-\alpha)}{\Gamma(n+1)\Gamma(-\alpha)}(-1)^n \tag{1.84}$$

We can also obtain $f(nT)$ from the Taylor's series expansion of $\mathscr{F}(z)$ as follows:

$$\mathscr{F}(z) = \left(\frac{z+b}{z}\right)^\alpha = (1+bz^{-1})^\alpha$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n(1+bz^{-1})^\alpha}{(dz^{-1})^n}\bigg|_{z^{-1}=0} z^{-n} \tag{1.85}$$

Equation (1.85) yields

$$\mathscr{F}(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)b^n z^{-n} \tag{1.86}$$

We know that

$$\Gamma(\alpha+1) = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)\cdot\Gamma(\alpha-n+1) \tag{1.87}$$

and

$$\Gamma(n+1) = n! \tag{1.88}$$

By using equations (1.87) and (1.88) in equation (1.86),

$$\mathscr{F}(z) = \left(\frac{z+b}{z}\right)^\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)b^n}{\Gamma(n+1)\Gamma(\alpha-n+1)} z^{-n}$$

$$= \sum_{n=0}^{\infty}\binom{\alpha}{n} b^n z^{-n} \tag{1.89}$$

From the definition of the z-transform we readily ascertain

$$f(nT) = \binom{\alpha}{n} b^n \tag{1.90}$$

The entries of Table I in the Appendix readily yield the inverse z-transforms of $f(nT)$. They have been obtained by using the theorems and properties of the z-transforms discussed in the preceding sections.

It should be noted from the inverse theorem that $f(nT)$ can be obtained from the power series expansion without having to evaluate the poles of $\mathscr{F}(z)$. This feature offers a decisive advantage over the continuous case using the Laplace transform. Thus the z-transform is used for approximating a continuous function, which will be discussed in Chapter 7.

1.4 The modified z-transform[14,28,31,32]

In many applications of discrete systems and particularly in the use of digital computers in control systems, the output between the sampling instants is very important. In studying hybrid systems (mixed digital and analog systems) the output is a continuous function of time, and thus the z-transform method is not quite adequate for a critical study of such

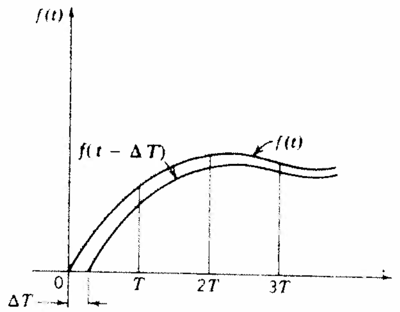

FIGURE 1.4 A fictitiously delayed output to scan values of $f(t)$ other than at $t = nT$.

systems. However, the z-transform can be easily modified to cover the system behavior at all instants of time; The extension of this method is called the modified z-transform method.

The modified z-transform is also important in the study of linear systems for periodic inputs, in sampled-data systems with pure delay, in limit cycle analysis of discrete systems, in the solution of difference equations whose coefficients are periodic functions (with period equal to unity), in the solution of mixed difference-differential equations, in approximation techniques for continuous systems, and in summing up of infinite convergent series. The introduction of the modified z-transform as well as its applications will be discussed both in the remainder of this chapter and in the following chapters.

Modified z-transform definition[14,28]

To obtain the values of $f(t)$ other than at $t = nT$, $(n = 0, 1, 2, \ldots)$ we can delay the function $f(t)$ by a fictitious negative delay ΔT as shown in Fig. 1.4. By letting Δ vary between zero and unity we obtain all the points of $f(t)$ for $t = (n - \Delta)T$, $n = 0, 1, 2, \ldots$ and $0 \leq \Delta \leq 1$. As will be shown in Section 1.5 in order to avoid any convergence problem in the integral evaluation of the modified z-transform from the Laplace transform of $f(t)$, and also in order to utilize the existing extensive tables of the modified z-transform, we make the following change of variable:

$$\Delta = 1 - m, \qquad 0 \leq m \leq 1$$

With this change of variable, t becomes

$$t = (n - 1 + m)T, \qquad n = 0, 1, 2, \ldots, \qquad 0 \leq m \leq 1 \quad (1.91)$$

We can also scan the values of the continuous function between the sampling instants by fictitiously advancing the function $f(t)$ by the amount ηT, such that

$$t = (n + \eta)T, \qquad 0 \leq \eta \leq 1, \qquad n = 0, 1, 2, \ldots \quad (1.92)$$

The preceding description of the time is utilized in some other works, thus avoiding any convergence difficulties in the integration process as will be explained in Section 1.5.

The modified z-transform of f is defined as follows:

$$\mathscr{F}(z, m) \triangleq \mathscr{Z}_m(f) = \sum_{n=0}^{\infty} [f(n - 1 + m)T]z^{-n}, \qquad 0 \leq m \leq 1 \quad (1.93)$$

This definition also relates to the modified z-transform of the function $f[(n - 1 + m)T]$.

By using equation (1.9), the preceding equation can be written as

$$\mathscr{F}(z, m) = z^{-1}\sum_{n=0}^{\infty} [f(n + m)T]z^{-n}, \qquad 0 \leq m \leq 1 \quad (1.94)$$

From this equation by letting $m = 0$, we readily deduce

$$z\mathscr{F}(z, m)\big|_{m=0} = \sum_{n=0}^{\infty} f(nT)z^{-n} = \mathscr{F}(z) \quad (1.95)$$

Therefore the z-transform is obtained as a special case from the modified z-transform. Furthermore, if $f(t)$ has no discontinuity at the sampling instants (or jumps), the z-transformation can also be obtained as follows:

$$\mathscr{F}(z) = \mathscr{F}(z, m)\big|_{m=1} \quad (1.96)$$

When $f(t)$ has a discontinuity at the sampling instants, the time function related to this equation yields the value at the left side of the discontinuity, that is, at $t = nT^-$, for $n = 1, 2, \ldots$ and the value zero at $n = 0$. This can be readily ascertained by noting equation (1.7).

Similar to the z-transform we can derive several properties of the modified z-transform. These properties and important theorems only are stated in the following discussions and the proofs can be easily derived by the reader. The steps for the derivation follows exactly as for the z-transform.

Theorems related to the modified z-transform[14,27]

INITIAL VALUE THEOREM

$$\lim_{\substack{n \to 0 \\ m \to 0}} f(n + m)T = \lim_{\substack{z \to \infty \\ m \to 0}} z\mathscr{F}(z, m) \quad (1.97)$$

Special case, the response over the first interval, $n = 0$, $0 \leq m \leq 1$, is

$$\lim_{\substack{n \to 0 \\ 0 \leq m \leq 1}} f(n+m)T = \lim_{\substack{z \to \infty \\ 0 \leq m \leq 1}} z\mathscr{F}(z, m) \tag{1.98}$$

FINAL VALUE

$$\lim_{n \to \infty} f(n, m)T = \lim_{z \to 1} (z-1)\mathscr{F}(z, m) \tag{1.99}$$

if the limit exists.

REAL TRANSLATION

$$\mathfrak{z}_m[f(t-kT)] = z^{-k}\mathscr{F}(z, m), \qquad k = 0, 1, 2, \ldots \tag{1.100}$$

$$\begin{aligned} \mathfrak{z}_m[f(t-\Delta T)] &= z^{-1}\mathscr{F}[z, m+1-\Delta], \qquad 0 \leq m \leq \Delta \\ &= \mathscr{F}(z, m-\Delta], \qquad \Delta \leq m \leq 1 \end{aligned} \tag{1.101}$$

where

$0 < \Delta < 1$, and zero initial conditions of f.

LINEARITY

$$\mathfrak{z}_m\left[\sum_{l=0}^{k} a_l f_l(t)\right] = \sum_{l=0}^{k} a_l \mathscr{F}_l(z, m), \quad \text{where } a_l \text{ are constants independent of } t \tag{1.102}$$

COMPLEX SCALE CHANGE

$$\mathfrak{z}_m[e^{\pm bT}f(t)] = e^{\pm bT(m-1)}\mathscr{F}(ze^{\pm bT}, m) \tag{1.103}$$

DIFFERENTIATION WITH RESPECT TO m

$$\mathfrak{z}_m\left[\frac{\partial}{\partial m} f(n, m)T\right] = \frac{\partial}{\partial m}\mathscr{F}(z, m) \tag{1.104}$$

MULTIPLICATION BY t^k

$$\mathfrak{z}_m[t^k f(t)] = T\left[(m-1)\mathscr{F}_1(z, m) - z\frac{\partial}{\partial z}\mathscr{F}_1(z, m)\right] \tag{1.105}$$

where

$$\mathscr{F}_1(z, m) = \mathfrak{z}_m[t^{k-1}f(t)] \tag{1.106}$$

and k integer larger than zero.

DIVISION BY t

$$\mathfrak{z}_m\left[\frac{f(t)}{t}\right] = \frac{1}{T} z^{m-1}\int_z^{\infty} z^{-m}\mathscr{F}(z, m)\, dz + \lim_{t \to 0}\frac{f(t)}{t}, \qquad 0 \leq m \leq 1 \tag{1.107}$$

INTEGRATION WITH RESPECT TO t

$$\mathfrak{z}_m\left[\int_0^t f(t)\, dt\right] = \frac{T}{z-1}\int_0^1 \mathscr{F}(z, m)\, dm + T\int_0^m \mathscr{F}(z, m)\, dm \tag{1.108}$$

SUMMATION OF SERIES

$$\sum_{n=0}^{\infty} f(n+m)T = \lim_{z \to 1} z\mathscr{F}(z, m), \qquad 0 \leq m \leq 1 \tag{1.109}$$

if the sum exists.

INVERSE MODIFIED z-TRANSFORM. The continuous time function $f(t)|_{t=(n-1+m)T}$ can be obtained from the modified z-transform by a process called inverse modified z-transformation, that is,

$$f(t)|_{t=(n-1+m)T} = f(n, m)T = \mathfrak{z}_m^{-1}[\mathscr{F}(z, m)] \tag{1.110}$$

Methods similar to those for the inverse z-transform exist for the inverse modified z-transform, namely the integral formula and the power series. The integral formula yields the time function in a closed form,

$$f(t)|_{t=(n-1+m)T} = \frac{1}{2\pi j}\oint_{\Gamma} \mathscr{F}(z, m)z^{n-1}\, dz, \qquad 0 \leq m \leq 1$$

$$t = (n-1+m)T \tag{1.111}$$

where the closed contour Γ encloses in a counterclockwise direction the poles of $\mathscr{F}(z, m)$.

The power series method yields the continuous function in a piecewise form,

$$\begin{aligned} z\mathscr{F}(z, m) &= f_0(m) + f_1(m)z^{-1} + f_2(m)z^{-2} + \ldots \\ &\quad + f_n(m)z^{-n} + \ldots, |z| > R \qquad 0 \leq m \leq 1 \end{aligned} \tag{1.112†}$$

where $f_0(m)$ with $0 \leq m \leq 1$ represent the continuous time function in the first sampling interval, $f_1(m)$ represents the same function in the second interval, and so forth. Actually, the time function is related to $f_n(m)$ as follows:

$$f(t) = f(n+m)T = f_n(m) \tag{1.113}$$

It should be noted that since m can be considered as a constant in the contour integration of equation (1.111), the same tables for the inverse z-transform are readily applicable to the inverse modified z-transform.

MAXIMUM OR MINIMUM POINTS OF $f(t) = f_n(m)$. To obtain the maxima or minima points we can differentiate the modified z-transform with respect to m to obtain, using the power series form,

$$\begin{aligned} \frac{\partial z\mathscr{F}(z, m)}{\partial m} &= f_0'(m) + f_1'(m)z^{-1} + f_2'(m)z^{-2} \\ &\quad + \ldots + f_n'(m)z^{-n} + \ldots \end{aligned} \tag{1.114}$$

† The general form of $f_n(m)$ is obtained using the determinant form discussed in the Appendix to this chapter.

If we let

$$f_0'(m) = 0,\ f_1'(m) = 0, \ldots, f_n'(m) = 0 \tag{1.115}$$

the solution of this equation for $0 < m < 1$ yields the maxima or minima points. The sign of the second derivative determines which points are maxima and which are minima. The preceding theorem is very important in determining the quality of response in discrete systems.

MODIFIED z-TRANSFORM OF A kth DERIVATIVE

$$\mathfrak{Z}_m[f^{(k)}(t)] = \frac{1}{T^k}\frac{\partial^k}{\partial m^k}\mathscr{F}(z, m), \quad \text{provided that } \lim_{t\to 0} f^{(n)}(t) = 0 \text{ for } 0 \le n \le k-1 \tag{1.116}$$

and

$$\mathfrak{Z}_m[f^{(k)}(t + hT)] = \frac{1}{T^k}\frac{\partial^k}{\partial m^k} \times \left\{ z^{h-1}\left[z\mathscr{F}(z, m) - \sum_{p=0}^{h-1} f_p(m) z^{-p} \right] \right\} \tag{1.117}$$

where h is a positive integer and $\lim_{t\to 0} f^{(n)}(t) = 0$ for $0 \le n \le k-1$.

1.5 Relationship between Laplace and z-transforms[14,29,34]

The one-sided Laplace transform of a function "f" is defined as follows:

$$F(s) \triangleq \mathscr{L}[f] = \int_0^\infty f(t)e^{-st}\,dt, \qquad \operatorname{Re}[s] > \sigma_a \tag{1.118}$$

where σ_a is the abscissa of absolute convergence associated with $f(t)$.

The inverse Laplace transform is represented by the following Bromwitch integral, that is,

$$f(t) = \mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{ts}\,ds, \qquad c > \sigma_a, \quad \text{for } t > 0 \tag{1.119}$$

We can obtain both $\mathscr{F}(z, m)$ and $\mathscr{F}(z)$ directly from $F(s)$ by an integral transformation symbolically denoted as follows:

$$\mathscr{F}(z, m) \triangleq \mathcal{Z}_m[F(s)] \tag{1.120}$$

and

$$\mathscr{F}(z) \triangleq \mathcal{Z}[F(s)] \tag{1.121}$$

The purpose of this section is to show the preceding relationship and develop the required transformation to enable us to obtain the z-transform directly from the Laplace transform.

The function $f(nT)$ shown in Fig. 1.1 can be considered as a sum of rectangular pulses of area $hf(nT)$, where h is the width of the rectangle (or pulse). Such a sum approximates $\sum hf(nT)\,\delta(t - nT)$, since $\delta(t - nT)$ should be considered as the limit of a pulse of unit area. Thus to convert $f(nT)$ into a train of pulses

$$f(0)\,\delta(t), f(T)\,\delta(t - T), f(2T)\,\delta(t - 2T), \ldots, f(nT)\,\delta(t - nT) \tag{1.122}$$

a scale factor $1/h$ is needed for such a conversion. Therefore we can replace the sampled function $f(nT)$ by the impulse function $f^*(t)$ provided the scale factor (or pulse width) is accounted for. The definition of such an impulse function is given (assuming a full impulse occurring at $t = 0^+$).

$$f^*(t) = f(t)\sum_{n=0}^{\infty}\delta(t - nT) = \sum_{n=0}^{\infty} f(nT)\,\delta(t - nT) \tag{1.123}$$

Taking the Laplace transform of this equation and noting that

$$\delta(t) = 0, \qquad t \neq 0, \quad \text{and} \quad \int_a^b \delta(t)\,dt = 1, \qquad \text{for } a < 0 < b \tag{1.124}$$

and

$$\mathscr{L}[\delta(t - kT)] = \int_0^\infty \delta(t - kT)e^{-st}\,dt = e^{-kTs} \tag{1.125}$$

we obtain

$$F^*(s) \triangleq \mathscr{L}[f^*(t)] = \sum_{n=0}^{\infty} f(nT)e^{-nTs} \tag{1.126}$$

This equation is readily recognized as the z-transform of f, if we replace e^{Ts} by z. Hence we establish

$$\mathscr{F}(z) = F^*(s)\big|_{s = T^{-1}\ln z} \tag{1.127}$$

If we denote $\sum_{n=0}^{\infty}\delta(t - nT) = \delta_T(t)$, we readily establish the connection (using Eqs. 1.123 and 1.127) between the z-transform and the Laplace transform

$$\mathscr{F}(z)\big|_{z = e^{Ts}} = F^*(s) \triangleq \mathscr{L}[f^*(t)] = \mathscr{L}[f(t)\,\delta_T(t)] \tag{1.128}$$

By using the convolution theorem for the Laplace transform and **assuming that $f(t)$ contains no impulses and is initially zero, we obtain**

$$F^*(s) = \mathscr{F}(z)\big|_{z=e^{Ts}} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)\,\frac{1}{1-e^{-T(s-p)}}\,dp \qquad (1.129)$$

where

$$F(p) = F(s)\big|_{s=p} = \mathscr{L}[f(t)]\big|_{s=p} \qquad (1.130)$$

$$\frac{1}{1-e^{-Ts}} = 1 + e^{-Ts} + e^{-2Ts} + \ldots e^{-nTs} + \ldots$$

$$= \mathscr{L}[\delta_T(t)], \qquad \text{for } |e^{-Ts}| < 1 \qquad (1.131)$$

and

$$\sigma_{a_2} < c < \sigma - \sigma_{a_1}, \qquad \max(\sigma_{a_1}, \sigma_{a_2}, \sigma_{a_1} + \sigma_{a_2}) < \sigma \qquad (1.131a)\dagger$$

For the case $F(s)$ has only one degree higher denominator than numerator, Eq. 1.129 should be modified,[40]

$$F^*(s) = \mathscr{F}(z)\big|_{z=e^{Ts}} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)\,\frac{1}{1-e^{-T(s-p)}}\,dp + \tfrac{1}{2}f(0^+) \qquad (1.132)$$

The addition of $\frac{1}{2}f(0^+)$ is required in view of our definition that a full impulse occurs at $t = 0^+$.

Both equations (1.129) and (1.132) yield a relationship between $\mathscr{F}(z)$ and $F(s)$, hence we define the following transformation:[40]

$$\mathscr{F}(z) \triangleq \mathscr{Z}[F(s)]$$

$$= \left[\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)\,\frac{1}{1-e^{-T(s-p)}}\,dp + \tfrac{1}{2}f(0^+)\right]_{e^{Ts}=z} \qquad (1.133)$$

Evaluation of $\mathscr{Z}[F(s)]$

To evaluate $\mathscr{F}(z)$, we assume first that $F(s)$ has two degrees in s higher denominator than numerator; thus we can use equation (1.129). The path of integration in equation (1.129) should lie in an analytic strip which does not enclose or pass through the poles of the integrand. This is assured in view of the restriction imposed on c.

In effecting the line integral, we can readily enclose in a negative sense (clockwise direction) the poles of $1/1 - e^{-T(s-p)}$ in the right half of the p-plane, or alternatively we may enclose in a positive sense the left half of

† Where $\sigma = \mathrm{Re}\,[s]$, σ_{a_2} = abscissa of absolute convergence of $f(t)$, σ_{a_1} = abscissa of absolute convergence of $\delta_T(t)$. Here $\sigma_{a_1} = 0$, $c = \mathrm{Re}\,[p]$.

the plane. Because of the assumed form of $F(s)$ the integrals on both the infinite semicircles are zero.

If we integrate along the left half p-plane as shown in Fig. 1.5 equation (1.129) becomes

$$\mathscr{F}(z)\big|_{z=e^{Ts}} = \frac{1}{2\pi j}\oint F(p)\,\frac{1}{1-e^{-T(s-p)}}\,dp \qquad (1.134)$$

If $F(s)$ has only simple poles, the integral using Cauchy's formula yields the sum of the residue of the function in the closed path, that is,

$$\mathscr{F}(z) = \sum_{\substack{\text{roots of}\\ B(p)}} \text{residue of } \frac{A(p)}{B(p)}\,\frac{1}{1-e^{Tp}e^{-Ts}}\bigg|_{z=e^{Ts}} \qquad (1.135)$$

where

$$F(p) = F(s)\big|_{s=p} = \frac{A(p)}{B(p)} \qquad (1.136)$$

When $B(s)|_{s=p} = 0$ has simple roots only, this equation becomes

$$\mathscr{F}(z) = \sum_{n=1}^{n} \frac{A(s_n)}{B'(s_n)}\,\frac{1}{1-e^{Ts_n}z^{-1}} \qquad (1.137)$$

where $s_1, s_2, s_3, \ldots, s_n$ are the simple roots of $B(s) = 0$, and

$$B'(s_n) = \frac{dB}{ds}\bigg|_{s=s_n} \qquad (1.138)$$

Where $F(s)$ has branch points in addition to regular singularities, the z-transform can also be obtained using equation (1.129); however, here

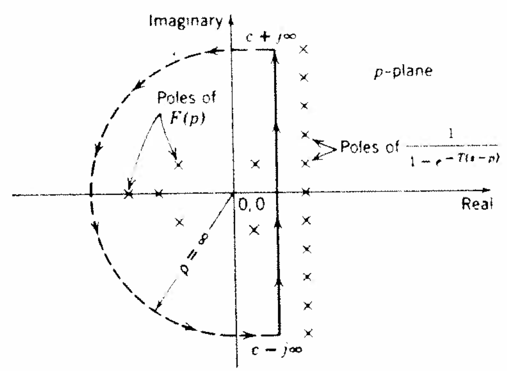

FIGURE 1.5 Path of integration in the left half of the p-plane.

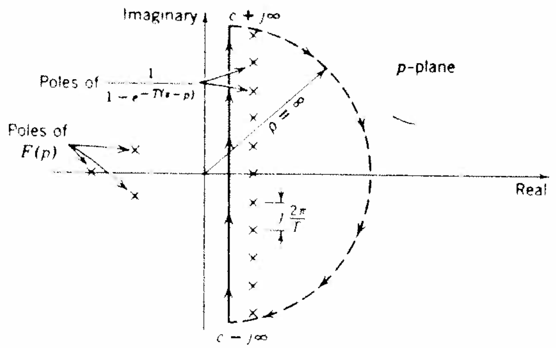

FIGURE 1.6 Path of integration in the right half of the p-plane.

special care is required to evaluate the line integral. An example will be presented later.

The integral in equation (1.129) can also be evaluated by the contour integration which encloses only the poles of $1/1 - e^{-T(s-p)}$ in the right half of the p-plane as shown in Fig. 1.6.

Thus equation (1.129) can be represented by

$$F^*(s) = \mathscr{L}[f^*(t)] = \frac{1}{2\pi j}\oint F(p)\frac{1}{1-e^{-T(s-p)}}\,dp \tag{1.139}$$

The integral of equation (1.139) is equivalent to the negative sum of the residue according to Cauchy's theorem.

$$F^*(s) = -(\text{sum of the residue of the integrand at the poles enclosed}) \tag{1.140}$$

By evaluating the residues at the infinite poles of $1/1 - e^{-T(s-p)}$, we finally obtain

$$F^*(s) = \frac{1}{T}\sum_{k}^{\infty} F(s + jk\omega_r) \tag{1.141}$$

where

$$\omega_r = \frac{2\pi}{T} \tag{1.142}$$

If we replace in this equation $e^{Ts} = z$, we readily obtain the z-transform $\mathscr{F}(z) = \mathscr{Z}[F(s)]$. The two forms of (1.135 and 1.141) are equivalent can be readily verified for specific examples of $F(s)$.

For $F(s)$ has one degree in s higher denominator than numerator, the integral along the infinite semicircle in the left-half plane is no longer zero, whereas in the right-half plane it is still zero. There equation (1.132) should be used, which yields on the left-half plane the same as equation (1.135), whereas for integration in the right-half plane we obtain[14,30]

$$F^*(s) = \frac{1}{T}\sum_{k=-\infty}^{k=\infty} F(s + jk\omega_r) + \tfrac{1}{2}f(0^+) \tag{1.143}$$

It should be noted that here the infinite summation is not absolutely convergent. However, if the sum is evaluated by taking pairs of terms corresponding to equal positive and negative values of the index k, the sum converges to a definite value.

EXAMPLE

Given $F(s) = \Gamma(1-\beta)s^{\beta-1}e^{\alpha s}$, $\alpha > 0$, $\beta > 0$, and noninteger, we obtain $\mathscr{F}(z) \triangleq \mathscr{Z}[F(s)]$ as follows:

Using the contour integration of equation (1.129), we can write for $T = 1$

$$\mathscr{F}(z) = \mathscr{Z}[F(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{e^{\alpha p}p^{\beta-1}\Gamma(1-\beta)}{1 - z^{-1}e^{p}}\,dp \tag{1.144}$$

The integrand has a branch cut (for β not integer) in the left half of the plane. Using Cauchy's formula the integral of equation (1.144) is equivalent to integration around the branch cut as shown in Fig. 1.7. Hence

$$\mathscr{F}(z) = -\frac{\Gamma(1-\beta)}{2\pi j}\int_{\Gamma_1+\Gamma_2} \frac{e^{\alpha p}p^{\beta-1}}{1 - z^{-1}e^{p}}\,dp \tag{1.145}$$

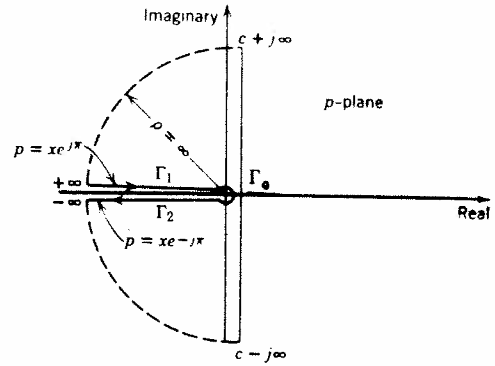

FIGURE 1.7 Path of integration around a branch cut.

Setting $p = xe^{j\pi}$ on Γ_1, and $p = xe^{-j\pi}$ on Γ_2, and noting that the integral around Γ_0 vanishes, we have

$$\mathscr{F}(z) = -\frac{\Gamma(1-\beta)}{2\pi j}\left[\int_{\infty}^{0}\frac{e^{-\alpha x}x^{\beta-1}e^{j\beta\pi}}{1-z^{-1}e^{-x}}\,dx + \int_{0}^{\infty}\frac{e^{-\alpha x}x^{\beta-1}e^{-j\beta\pi}}{1-z^{-1}e^{-x}}\,dx\right]$$

$$= \frac{\Gamma(1-\beta)\sin\beta\pi}{\pi}\int_0^{\infty}\frac{e^{-\alpha x}x^{\beta-1}}{1-z^{-1}e^{-x}}\,dx$$

$$= \frac{1}{\Gamma(\beta)}\int_0^{\infty}\frac{e^{-\alpha x}x^{\beta-1}}{1-z^{-1}e^{-x}}\,dx \tag{1.146}$$

We note from integral tables that

$$\int_0^{\infty}\frac{e^{-\alpha x}x^{\beta-1}}{1-z^{-1}e^{-x}}\,dx = \Gamma(\beta)\Phi(z^{-1},\beta,\alpha), \qquad \beta > 0, \alpha > 0 \tag{1.147}$$

This equation is also defined for β integer. Therefore

$$\mathscr{F}(z) = \Phi(z^{-1},\beta,\alpha), \qquad \text{for } \alpha > 0 \text{ and } \beta > 0 \text{ (and also integer)} \tag{1.148}$$

It is known that the definition of $\Phi(z^{-1},\beta,\alpha)$ is

$$\Phi(z^{-1},\beta,\alpha) = \sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^{\beta}}z^{-n} \tag{1.149}$$

which is the definition of the z-transform of $f(t) = 1/(t+\alpha)^{\beta}$, for $\beta > 0$, $\alpha > 0$, and $T = 1$. **This result is expected because**

$$f(t) = \mathscr{L}^{-1}[F(s)] = \mathscr{L}^{-1}[\Gamma(1-\beta)s^{\beta-1}e^{\alpha s}] = 1/(t+\alpha)^{\beta}$$

for all β's different from 1, 2, 3, ...

Although $\Phi(z^{-1}, \beta, \alpha)$ can be represented only in a summation form, however, if $z^{-1} = 1$, the function $\Phi(1,\beta,\alpha) = \zeta(\beta,\alpha)$ is tabulated for certain α and β. This function is referred to as generalized Riemann-zeta function. Therefore the z-transform does not exist in a closed form; however, the infinite summation $\sum_{n=0}^{\infty} 1/(n+\alpha)^{\beta}$ can be calculated numerically for certain α's and β's.

Evaluation of $\mathscr{F}(z, m) \triangleq \mathcal{Z}_m[F(s)]$

The relationship between the Laplace transform and the modified z-transform can be readily obtained similarly to the relationship of the z-transform. From equations (1.93) and (1.123), we can write

$$\mathscr{F}(z,m)\big|_{z=e^{Ts}} \triangleq F^*(s,m) = \mathscr{L}[f^*(t,m)]$$

$$= \mathscr{L}[f(t-T+mT)\,\delta_T(t)], \quad 0 \le m < 1 \tag{1.150}$$

This equation can also be written as

$$\mathscr{F}(z,m)\big|_{z=e^{Ts}} = \mathscr{L}[f(t-T+mT)\,\delta_T(t-T)]$$

$$= e^{-Ts}\mathscr{L}[f(t+mT)\,\delta_T(t)], \quad 0 \le m < 1 \tag{1.151}$$

From the convolution theorem of Laplace transform this equation is equivalent to

$$\mathscr{F}(z,m)\big|_{z=e^{Ts}} = \frac{1}{2\pi j}z^{-1}\int_{c-j\infty}^{c+j\infty}F(p)e^{mpT}\frac{1}{1-e^{-T(s-p)}}\,dp, \qquad 0 \le m < 1 \tag{1.152}$$

It is noticed from the preceding that to evaluate this integral we require **the change of variable parameter from m through $\Delta = 1 - m$, so as to get the term e^{mpT} such that the integral vanishes on the left half infinite semi-circle. Furthermore, equation (1.152) constitutes the relationship between $\mathscr{F}(z, m)$ and $F(s)$; thus we define**

$$\mathscr{F}(z,m) \triangleq \mathcal{Z}_m[F(s)] = \frac{1}{2\pi j}z^{-1}\int_{c-j\infty}^{c+j\infty}F(p)e^{mpT}\frac{1}{1-e^{-T(s-p)}}\,dp, \qquad 0 \le m < 1 \tag{1.153}$$

Integrating in the left-half plane and assuming $F(s)$ has regular singularities, equation (1.153) becomes

$$\mathscr{F}(z,m) = z^{-1}\sum_{\substack{\text{poles of}\\ F(p)}}\text{residue of }\frac{F(p)e^{mpT}}{1-e^{pT}z^{-1}}, \qquad 0 \le m < 1 \tag{1.154}$$

When $F(s) = A(s)/B(s)$ has simple poles, equation (1.154) can be expressed as

$$\mathscr{F}(z,m) = z^{-1}\left[\sum_{n=1}^{N}\frac{A(s_n)}{B'(s_n)}\frac{e^{ms_nT}}{1-e^{-T(s-s_n)}}\right]_{z=e^{Ts}}, \qquad 0 \le m < 1 \tag{1.155}$$

where $s_1, s_2, \ldots, s_N$ are the simple roots of $B(s) = 0$, and $B'(s_n) = \left.\dfrac{dB}{ds}\right|_{s=s_n}$.

Table II of the Appendix lists extensive forms of the modified z-transform for various forms of $G(s)$ or $F(s)$. This table can also be used to obtain $\mathscr{G}(z) = \mathcal{Z}[G(s)]$ using the relationship in equation (1.157).

Equations (1.154) and (1.155) are also valid for $m = 1$, provided $F(s)$ has two degrees in s higher denominator than numerator. However, if $F(s)$ has only one degree in s higher denominator than numerator, equations (1.154) and (1.155) yield for their inverses, if discontinuities exist, the values at the left side of the discontinuities. The value at $t = 0$ should be taken as at $t = 0^-$, that is, zero.

Evaluating the integral of equation (1.152) in the right half of the p-plane, we (with due care for convergence) obtain similar to equation (1.141) the infinite series form

$$F^*(s, m) \triangleq \mathcal{L}[f^*(t, m)]$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} F\left(s + jk\frac{2\pi}{T}\right)e^{-[s+jk(2\pi/T)](1-m)T},$$
$$0 \leq m < 1\dagger \qquad (1.156)$$

If $F(s)$ is of two degrees higher denominator than numerator, this equation can be extended for $m = 1$.

The z-transform can be obtained as a special case from the preceding by noting

$$\mathscr{F}(z) = z\mathscr{F}(z, m)\big|_{m=0} \qquad (1.157)$$

and for $f(0^+) = 0$

$$\mathscr{F}(z) = \mathscr{F}(z, m)\big|_{m=1} \qquad (1.158)$$

The conditions for obtaining $F(s)$ knowing either $\mathscr{F}(z, m)$ or $\mathscr{F}(z)$, that is, $\mathscr{Z}_m^{-1}$, $\mathscr{Z}^{-1}$, are discussed in Chapter 4.

1.6 Application to sampled-data systems[14-16,20]

One of the basic engineering applications of the z-transform theory is in the field of sampled-data or digital control systems. Such systems are used more and more often in modern technology. We shall briefly obtain the z-transforms of certain systems configurations.

1. Let the sampled data systems be presented as shown in Fig. 1.8. The z-transform of the output is readily obtained by applying the z-transform to $C(s)$ as follows:

$$C(s) = E^*(s)G(s) \qquad (1.159)$$

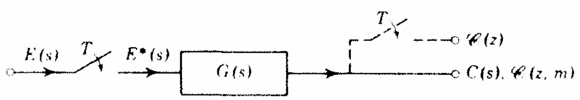

FIGURE 1.8 A sampled-data system.

† It should be noted that if we let $m = 1$ inside the summation and before summing, this equation should be modified as

$$F^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F\left(s + jk\frac{2\pi}{T}\right) + \tfrac{1}{2}f(0^+).$$

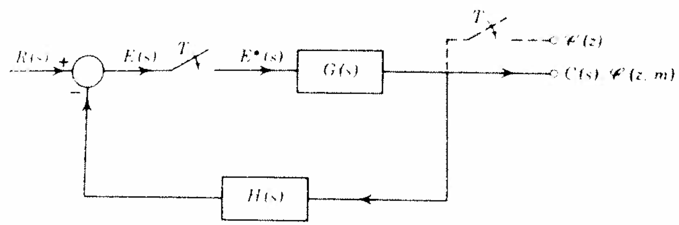

FIGURE 1.9 A sampled-data feedback control system.

or

$$\mathscr{C}(z) \triangleq \mathscr{Z}[C(s)] = \mathscr{E}(z)\mathscr{G}(z), \qquad \text{where } \mathscr{E}(z) = E^*(s)\big|_{s=T^{-1}\ln z} \qquad (1.160)$$

The modified z-transform of equation (1.159) is

$$\mathscr{C}(z, m) \triangleq \mathscr{Z}_m[C(s)] = \mathscr{E}(z)\mathscr{G}(z, m) \qquad (1.161)$$

2. Let the sampled-data feedback system be presented as shown in Fig. 1.9. The z-transform of the output transform $C(s)$ is given as

$$\mathscr{C}(z) \triangleq \mathscr{Z}[C(s)] = \mathscr{Z}[E^*(s)G(s)] = \mathscr{E}(z)\mathscr{G}(z) \qquad (1.162)$$

$$\mathscr{E}(z) = \mathscr{Z}[E(s)] = \mathscr{Z}[R(s) - C(s)H(s)] = \mathscr{Z}[R(s) - E^*(s) \times G(s)H(s)] = \mathscr{R}(z) - \mathscr{E}(z)\mathscr{G}\mathscr{H}(z) \qquad (1.163)$$

Substituting the preceding in equation (1.162), we obtain

$$\mathscr{C}(z) = \mathscr{R}(z)\frac{\mathscr{G}(z)}{1 + \mathscr{H}\mathscr{G}(z)} \qquad (1.164)$$

Similarly, for the modified z-transform of the output we obtain

$$\mathscr{C}(z, m) = \mathscr{Z}_m[C(s)] = \mathscr{R}(z)\frac{\mathscr{G}(z, m)}{1 + \mathscr{H}\mathscr{G}(z)} \qquad (1.165)$$

We can also obtain the system transfer function of any configuration of sampled-data or digital control system. In later chapters some of these systems will be studied in more detail.

1.7 Mean square value theorem[14,37,38]

The following theorem with its extensions is very useful in the study of discrete systems. In particular, the mean square value of the continuous error in sampled-data control systems can be readily obtained.