Chapter 1

THE LAPLACE TRANSFORM

1.1 Introduction

The Laplace Transform is a very neat mathematical method for solving problems which arise in several areas of mathematical analysis. Of particular importance is its ability to solve differential equations, partial differential equations and differential-difference equations which continually arise in engineering problems.

In this chapter we will review the formal mathematical definition of the Laplace Transform and derive some of its fundamental properties. Our interest in the Laplace Transform is quite practical. We wish to solve problems and our experience has shown us that the Laplace Transform is a valuable tool in doing so. We wish to point out also that the use of the Laplace Transform increases in power when it is used in somewhat unconventional ways.

With the availability of modern large scale computers, its applications have become increasingly important as a tool in the numerical solution of mathematical problems. With this clear understanding the analyst and engineer must, in dealing with today's broad spectrum of technical problems, consider both analytical and numerical properties of the Laplace Transform.

Because of our interest in examining unconventional applications of the Laplace Transform, we shall consider further the concept of the Stieltjes integral which allows us to extend the ideas of the transform to problems involving discontinuities. To begin, let $u(t)$ be a continuous real function defined for all $t \geq 0$. The function $L(u)$, given by the expression,

$$L(u) = \int_0^{\infty} e^{-st} u(t)dt = F(s) \tag{1.1.1}$$

is defined as the Laplace Transform of u. Traditionally the function is written either as $L(u)$, to show its dependence on the function $u(t)$, or as $F(s)$, indicating that (1.1.1) is a function of the complex variable $s = (x + iy)$.

Before we examine some of the properties of the Laplace Transform of $u(t)$, we must pay careful attention to convincing ourselves that the integral (1.1.1) does, in fact, exist. To do this, we shall construct a rather general form of the integral, the Stieltjes integral, and show that it includes, as a subcase, the conventional definition of the Laplace Transform.

1.2 Functions of a Bounded Variation

The function $F(s)$ exists whenever the integral,

$$\int_0^{\infty} e^{-st} u(t)dt = F(s) \tag{1.2.1}$$

exist over a range of the complex parameter s. Clearly this puts some restrictions on the behavior of $u(t)$, particularly as t gets very large.

To broaden our vision as much as possible, let us consider some of the generalizations which can be made under which (1.2.1) exists. To this end we introduce the concept of a function of a bounded variation. Such a function $f(t)$ is said to be of a bounded variation in the interval (a, b) if it can be expressed in the form $g(t) - h(t)$ where both functions $g(t)$ and $h(t)$ are

nondecreasing bounded functions.

1.3 The Stieltjes Integral

The integral defined in (1.1.1) is the usual Riemann integral used for the conventional definition of the Laplace Transform. Yet to make sure we will be working in a mathematical frame which is broad enough to include a large variety of applications, we shall introduce the Stieltjes integral as a generalization of the more widely used Riemann integral. Our aim in constructing this integral is to demonstrate the general conditions under which the Laplace Transform exists.

To construct the Stieltjes integral, let $\alpha(x)$ and $f(x)$ be real bounded functions of a real variable x for $a \leq x \leq b$. Define a subdivision A of the interval (a, b) by the points,

$$a = x_0 < x_1 < x_2 \dots < x_n = b \quad .$$

Let δ be the largest $|x_{i+1} - x_i|$, $i = 0,1,\dots,n-1$. If

$$\lim_{\delta \to 0} \sum_{i=0}^{n-1} f(\xi_i) \left(\alpha(x_{i+1}) - \alpha(x_i)\right)$$

$$x_i < \xi_i < x_{i+1}$$

exists independently of the manner of the subdivision and the choice of ξ_i, then the limit is called the Stieltjes integral of $f(x)$ with respect to $\alpha(x)$ and is denoted,

$$\int_a^b f(t)\, d\alpha(t) \tag{1.3.1}$$

Using the definitions we have just introduced, it can be shown to be true that if $f(x)$ is continuous and $\alpha(x)$ is of bounded variation in the interval (a, b), then the Stieltjes integral of $f(x)$ with respect to $\alpha(x)$ space from a to b exists.

A further characterization of the existence of the Stieltjes integral is given by the following result. If $f(x)$ and $\alpha(x)$ are real bounded functions in $a<x<b$ and in addition $\alpha(x)$ is nondecreasing, then the necessary and sufficient condition that,

$$\int_a^b f(x)\, d\alpha(x)$$

exists, is that

$$\lim_{\delta \to 0} (S_A - s_A) = 0 \quad ,$$

independent of the manner of subdivision, where

$$S_A = \sum_{k=0}^{n-1} M_k(\alpha_{k+1}(x) - \alpha_k(x))$$

$$s_A = \sum_{k=0}^{n-1} m_k(\alpha_{k+1}(x) - \alpha_k(x))$$

$$M_k = \text{l.u.b. } f(x)$$

$$m_k = \text{g.l.b. } f(x)$$

$$x_k \le x \le x_{k+1}$$

By using these results, we have a means of establishing the existence of the Stieltjes integral when the need arises.

1.4 Improper Stieltjes Integral

If we recall that the Laplace Transform $L(u)$ defined in (1.1.1) has a range of integration from 0 to infinity, we inquire what the counterpart in the framework of the Stieltjes integral would be. The improper Stieltjes integral can be defined as a limiting process in the following way. If $f(x)$ is continuous in $a<x<R$, for every R, then

$$(1.4.1) \qquad \int_a^\infty f(x)\, d\alpha(x) = \lim_{R\to\infty} \int_a^R f(x)\, d\alpha(x) \quad ,$$

when the limit exists. If the integral does, in fact, exist, the improper integral (1.4.1) is said to converge. Furthermore, we say (1.4.1) converges absolutely if and only if,

$$(1.4.2) \qquad \int_a^\infty |f(x)|\, dV(x) < \infty$$

where $V(x)$ is associated with the bounded variation property of $\alpha(x)$ and is defined as,

$$(1.4.3) \qquad V(x) = g(x) - h(x) \quad .$$

where both $g(x)$ and $h(x)$ are nondecreasing bounded functions.

1.5 The Laplace Transform

To demonstrate the conditions under which the Laplace Transform exists, we shall consider the results we have found for the Stieltjes integral. Recall from the last section, that if $f(x)$ is continuous and $\alpha(x)$ is of bounded variation, and,

$$(1.5.1) \qquad \int_0^\infty |f(x)|\, dV(x) < \infty \quad ,$$

then the integral

$$(1.5.2) \qquad \int_0^R f(x)\, d\alpha(x)$$

exists and in the limit as $R\to\infty$ converges absolutely.

Now, if $\alpha(x)$ is a continuous function of a nondecreasing parameter t, then we can write $\alpha(x)=t$ so that $d\alpha(x)=dt$, and

$$(1.5.3) \qquad f(x) = f(t) = e^{-st}\, u(t) \quad ,$$

where s is a complex parameter.

Then, the Stieltjes integral (1.4.1) becomes the Laplace Transform, (1.5.4) $$\int_0^\infty e^{-st}\, u(t)dt \quad .$$

Now if we impose the bounded condition on the function $u(t)$,

(1.5.5) $$|u(t)| < ae^{bt} \quad ,$$

for some constants a and b as $t \to \infty$, and that

(1.5.6) $$\int_0^T |u(t)|dt < \infty$$

for any finite T, then the conditions are satisfied to insure that the integral will converge absolutely and uniformly for $Re(s) > b$, since

$$\int_0^\infty |e^{-st}\, u(t)|dt \le a\int_0^\infty |e^{(b-s)t}|dt < \infty$$

for $Re(s) > b$.

Therefore, depending on the bounded behavior of $u(t)$ for large t, a convergent Laplace Transform exists only for those complex parameters s whose real parts are greater than b.

1.6 Existence and Convergence

The explicit results of the last section may be restated in the following way. If $u(t)$ is continuous and satisfies a bound of the form,

(1.6.1) $$|u(t)| \le ae^{bt}$$

for some constants a and b as $t \to \infty$ and if

(1.6.2) $$\int_0^T |u(t)|dt < \infty$$

for every finite T, then the combination of these two reasonable assumptions permits us to conclude that the integral,

(1.6.3) $$\int_0^\infty e^{-st}\, u(t)dt$$

exists and converges absolutely and uniformly for $Re(s) > b$.

1.7 Properties of the Laplace Transform

Some of the elementary properties of the Laplace Transform are of particular interest to us as we proceed through the book. We shall begin with the most important of the elementary attributes of the Laplace Transform, namely the relative invariance under translations in both the t- and s-spaces.

We first observe that from its definition, the Laplace Transform can be written,

(1.7.1) $$\int_0^\infty e^{-st}\, e^{-bt}\, u(t)dt = \int_0^\infty e^{-(s+b)t}\, u(t)dt \quad .$$

Or in other words,

(1.7.2) $$L(e^{-bt}u(t)) = F(s+b) \quad .$$

Also we have, using the same technique of examination,

(1.7.3) $$\int_1^\infty e^{-st}\, u(t-1)dt$$

$$= \int_0^\infty e^{-s(t+1)}\, u(t)dt = e^{-s}\, L(u) \quad .$$

Another property of the Laplace Transform can be illustrated by considering,

(1.7.4) $$L(du/dt) = \int_0^\infty e^{-st}\, (du/dt)dt$$

$$= (e^{-st}\, u(t)) \Big|_0^\infty + s\int_0^\infty e^{-st}\, u(t)dt \quad ,$$

by integrating by parts.

From the convergence requirements of the Laplace Transform, we assume that,

(1.7.5) $$\lim_{t\to\infty} e^{-st} u(t) = 0 \quad .$$

Using these results we obtain the property that,

(1.7.6) $$L(du/dt) = sL(u) - u(0) \quad .$$

Similarly,

(1.7.7) $$L(d^2u/dt^2) = sL(du/dt) - u'(0)$$
$$= s^2L(u) - su(0) - u'(0) \quad ,$$

and inductively,

(1.7.8) $$L(d^nu/dt^n) = s^n L(u) - s^{n-1}u(0) - \ldots - u^{(n-1)}(0) \quad ,$$

where,

$$u^{(n)}(x) = d^nu/dx^n \quad .$$

This is a remarkable property of the Laplace Transform for it transforms derivatives into simple algebraic expressions together with the inclusion of initial conditions in a very natural way.

The Laplace Transform has several other far reaching properties. For example, if $f(t)$ is Laplace Transformable, that is, its Laplace Transform exists, then we can show that

(1.7.9) $$L\left(\int_0^t f(t)dt\right) = F(s)/s \quad .$$

Hence, we follow the formal definition of the Laplace Transform,

(1.7.10) $$L\left(\int_0^t f(t)dt\right) = \int_0^\infty e^{-st} \int_0^t f(p)dpdt \quad .$$

Integrating the right side of (1.7.10) by parts, we have,

(1.7.11) $$L(\int_0^t f(t)dt) = -\frac{1}{s} e^{-st} \int_0^t f(p)dp \Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f(t)dt$$
$$= F(s)/s.$$

Using the same technique, we can obtain two interesting similar results.

(1.7.12) $$L(t\ f(t)) = \int_0^\infty e^{-st} t\ f(t)dt$$
$$= -\int_0^\infty \frac{d}{ds} (e^{-st} f(t))dt$$
$$= -\frac{dF(s)}{ds} \quad .$$

We also take note of the following fact,

(1.7.13) $$L(f(t)\ /\ t)) = \int_0^\infty e^{-st} f(t)/t\ dt$$
$$= \int_0^\infty \int_0^s e^{-pt} f(t)dpdt$$
$$= \int_0^s \int_0^\infty e^{-pt} f(t)dtdp$$
$$= \int_0^s F(p)dp \quad .$$

Finally, we wish to examine the Laplace Transform as it defines a complex function over a complex domain. By definition, if $u(t)$ is continuous for $t > 0$, then we have defined,

(1.7.14) $$L(u) = \int_0^\infty e^{-st} u(t)dt \quad ,$$

where now we wish to emphasize that s is a complex number, i.e.,

(1.7.15) $$s = x + iy$$

where the symbol i is the imaginary number $i = \sqrt{-1}$.

Recalling that,

$$(1.7.16) \qquad e^{-st} = e^{-(x+iy)t} = e^{-xt}\,(\cos(yt) - i\,\sin(yt)) \quad ,$$

we see that the transform (1.7.14) can be written,

$$(1.7.17) \qquad L(u) = \int_0^{\infty} e^{-xt}\,(\cos(yt) - i\,\sin(yt))\,u(t)dt \quad .$$

By inspection we can write,

$$(1.7.18) \qquad L(u) = F(x,y) + i\,G(x,y) \quad ,$$

demonstrating clearly that $L(u)$ is a complex function of a complex variable. Quite often problems are encountered in which the unit step function plays a prominent part. This function is usually denoted as $U(t)$ and its definition is

$$(1.7.19) \qquad U(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad .$$

We now have the following results, sometimes called the shifting theorem. The theorem states that if

$$L(f) = F(s) \quad ,$$

then

$$L(f(t-T)\,U(t-T)) = e^{-sT}\,F(s) \quad .$$

We demonstrate the proof of this result by noting that,

$$(1.7.20) \qquad L(f(t-T)\,U(t-T)) = \int_0^{\infty} f(t-T)\,U(t-T)\,e^{-st}\,dt$$

$$= \int_T^{\infty} f(t-T)\,e^{-st}\,dt$$

by letting $p = t-T$,

$$= \int_0^{\infty} f(p)\,e^{-s(p+T)}\,dp$$

$$= e^{-sT}\int_0^{\infty} f(p)\,e^{-sp}\,dp$$

$$= e^{-sT}\,F(s) \quad .$$

1.8 The Inversion of the Laplace Transform

The Laplace Transform of a function $u(t)$ is of little value to the analyst if there is no assurance that the inverse exists and is unique. Therefore we shall be concerned in this section with demonstrating that such an inverse exists uniquely and in so doing we will determine the conditions under which the inversion can be successfully performed.

Since we are not interested in the most general case, but only in the class of functions which arise naturally in the course of our investigations, we shall restrict ourselves to proving the following: If $F(s)$ satisfies the following conditions,

1. $F(s)$ is analytic for $\mathrm{Re}(s) > a$
2. $F(s) = \frac{c_i}{s} + O(\frac{1}{|s|^2})$ as $|s| \to \infty$ along $s = b + it$, $b > a$,

then

$$(1.8.1) \qquad f(t) = \frac{1}{2\pi i}\int_{(c)} F(s)\,e^{st}\,ds \quad , \qquad t > 0 \quad ,$$

where (c) is a contour in the region of analyticity, exists and

$$(1.8.2) \qquad F(s) = L(f) \quad .$$

To demonstrate the result is true, let us write the Laplace Transform of a function $f(t)$, $t \geq 0$, as follows,

$$(1.8.3) \qquad L(f) = \lim_{T\to\infty}\int_0^{T} f(t)\,e^{-pt}\,dt \quad ,$$

where p is a complex parameter.

Using (1.8.1) we have

$$(1.8.4) \qquad L(f) = \lim_{T\to\infty} \int_0^T \left[\frac{1}{2\pi i}\int_{(c)} F(s)e^{st}ds\right] e^{-pt}dt \quad .$$

Because of the absolute convergence of the double integral, the order of integration can be reversed, and we have

$$(1.8.5) \qquad L(f) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{(c)} F(s) \int_0^T e^{(s-p)t}\, dtds \quad .$$

If $Re(p) > Re(s)$, the inner integral exists in the limit as $T\to\infty$ and may be explicitly evaluated.
Therefore,

$$(1.8.6) \qquad L(f) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{(c)} F(s)\left[\frac{e^{(s-p)T} - 1}{(s-p)}\right] ds \quad ,$$

or in the limit as $T\to\infty$

$$(1.8.7) \qquad L(f) = \frac{-1}{2\pi i}\int_{(c)} \frac{F(s)\, ds}{(s-p)} = F(p)$$

because $F(s)$ is analytic for $Re(s) > a$.

Alternatively by direct manipulation, we can show the existence of the inverse Laplace Transform by the following analysis. If $f(t)$ is defined for $t > 0$ and the correspond transform,

$$(1.8.8) \qquad F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

is absolutely convergent for $Re(s) > a$, as in Fig. 1.1. Then for $x > a$, we can write,

$$F(x+iy) = \int_0^\infty f(t)\, e^{-(x+iy)t}\, dt \quad ,$$

where the integral is absolutely convergent.

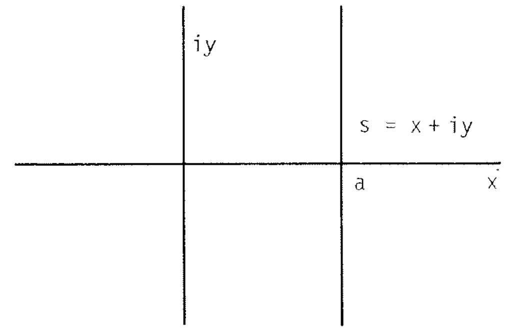

Fig. 1.1 Region of absolute convergence for the Laplace Transform

Now, if we multiply both sides of (1.8.8) by $e^{u(x+iy)}$, u being a real parameter, and integrate between $-T$ and T, along the imaginary y-axis, we see,

$$(1.8.9) \qquad \int_{-T}^{T} e^{u(x+iy)}\, F(x+iy)dy$$

$$= \int_{-T}^{T} e^{u(x+iy)} \int_0^\infty e^{-(x+iy)t}\, f(t)dtdy$$

But because the double integral converges absolutely, the order of integration can be interchanged,

$$(1.8.10) \qquad \int_{-T}^{T} e^{u(x+iy)}\, F(x+iy)dy$$

$$= e^{ux}\int_0^\infty f(t)e^{-xt}\int_{-T}^{T} e^{(iuy-iyt)}dydt \quad .$$

We can now note that

$$(1.8.11) \qquad e^{i(u-t)y} = \cos(u-t)y + i\,\sin(u-t)y \quad ,$$

and therefore

$$(1.8.12) \qquad \int_{-T}^{T} e^{i(u-t)y}\, dy = \frac{2\sin T(u-t)}{(u-t)} \quad .$$

Substituting (1.8.12) into (1.8.10), we see

$$\text{(1.8.13)}\qquad \int_{-T}^{T} e^{u(x+iy)} F(x+iy)dy$$

$$= 2e^{ux} \int_0^{\infty} f(t)\, e^{-xt}\, \frac{\sin T(u-t)}{(u-t)}\, dt \quad .$$

To investigate the behavior of the right-hand integral for $u > 0$, we can break the interval $(0, \infty)$ into $(0, u\text{-}d)$, $(u\text{-}d, u\text{+}d)$, $(u\text{+}d, \infty)$, so (1.8.13) can be written as,

$$\text{(1.8.14)}\qquad \int_{-T}^{T} e^{u(x+iy)} F(x+iy)dy$$

$$= 2e^{ux} \left[\int_0^{(u-d)} (.)dt + \int_{(u-d)}^{(u+d)} (.)dt + \int_{(u+d)}^{\infty} (.)dt\right] \quad .$$

Each integral of (1.8.14) is of the form,

$$\int_a^b g(t) \left\{ \begin{array}{c} \sin(tT) \\ \cos(tT) \end{array} \right\} dt$$

where $g(t)$ is continuous over the intervals defined on the first and third terms. If we assume that $g(t)$ has a derivative, then integrating by parts gives, for example,

$$\int_a^b g(t) \sin(tT)dt$$

$$= \frac{g(t)\, \cos(tT)}{T} \Bigg|_a^b - \frac{1}{T} \int_a^b g'(t) \cos(tT)dt$$

and we have the result,

$$\lim_{T\to\infty} \int_a^b g(t) \left\{ \begin{array}{c} \sin(tT) \\ \cos(tT) \end{array} \right\} dt = 0 \quad ,$$

therefore in the limit as $T\to\infty$, the first and third integrals of (1.8.14) vanish.

Therefore, at this point we can say, in the limit,

$$\text{(1.8.15)}\qquad \int_{-T}^{T} e^{u(x+iy)} F(x+iy)dy$$

$$= 2e^{ux} \int_{(u-d)}^{(u+d)} f(t)e^{-xt}\, \frac{\sin T(u-t)}{(u-t)}\, dt \quad .$$

Since d is small, we assume $f(t)$ is sufficiently smooth in the neighborhood of u that,

$$f(t)e^{-xt} = f(u)e^{-xu} + h(u, t)(t-t) \quad ,$$

where $|h(u, t)| < k$, $(u\text{-}d) < t < (u\text{+}d)$.
Therefore, we can immediately write,

$$\text{(1.8.16)}\qquad \int_{-T}^{T} e^{u(x+iy)} F(x+iy)dy$$

$$= 2f(u)\int_{(u-d)}^{(u+d)} \frac{\sin T(u-t)}{(u-t)}\, dt$$

$$+ 2e^{xu} \int_{(u-d)}^{(u+d)} h(u, t) \sin T(u-t)dt \quad .$$

Since $|\sin T(u\text{-}d)| < 1$ and $|h(u, t)| < k$, the second integral is $O(d)$. Now set $v = T(u-t)$, $dv = -Tdt$, we have then,

$$\text{(1.8.17)}\qquad \int_{-T}^{T} e^{u(x+iy)} F(x+iy)dy$$

$$= -\, 2f(u) \int_{-Td}^{Td} \frac{\sin v}{v}\, dv + O(d)$$

We can use the fact that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi \quad .$$

If we let $T\to\infty$ and $d\to 0$ such that $Td\to\infty$, then we have

$$\text{(1.8.18)} \qquad \lim_{T\to\infty} \int_{-T}^{T} e^{u(x+iy)} F(x+iy)dy = -2\pi f(u) + O(d) \quad ,$$

and therefore,

$$\text{(1.8.19)} \qquad f(u) = \lim_{T\to\infty} -\frac{1}{2\pi} \int_{-T}^{T} e^{u(x+iy)} F(x+iy)dy \quad .$$

Since $F(s)$ is analytic for $Re(s) > a$, the integral (1.8.19) exists in the right half plane, $Re(s) > a$. Since $s = x + iy$, we can choose a path c in the complex plane parallel to the imaginary axis so that $ds = -idy$. The limits of integration of (1.8.19) become $a \pm iT$.
The final expression can then be written,

$$\text{(1.8.20)} \qquad f(u) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} e^{us} F(s)ds \quad ,$$

which is the expression for the inverse Laplace Transform.

1.9 The Convolution Theorem

A fundamental property of the Laplace Transform is associated with the expression

$$\text{(1.9.1)} \qquad h(t) = \int_0^t f(r)\, g(t-r)dr \quad .$$

This mathematical operation creates a function $h(t)$ as a composite of two functions $f(t)$ and $g(t)$ and plays important roles in analysis in mathematical physics and probability theory. The notation,

$$\text{(1.9.2)} \qquad h = f \star g$$

is frequently used to symbolize (1.9.1) and the integral itself is the convolution of f and g. If we consider the Laplace Transform of $h(t)$ as a limiting process, we can write,

$$\text{(1.9.3)} \qquad \int_0^T h(t)\, e^{-st}\, dt = \int_0^T e^{-st} \int_0^t f(r)\, g(t-r)\, drdt \quad ,$$

as $T\to\infty$.

Now consider the repeated integral as a double integral over a region S in Fig. 1.2,

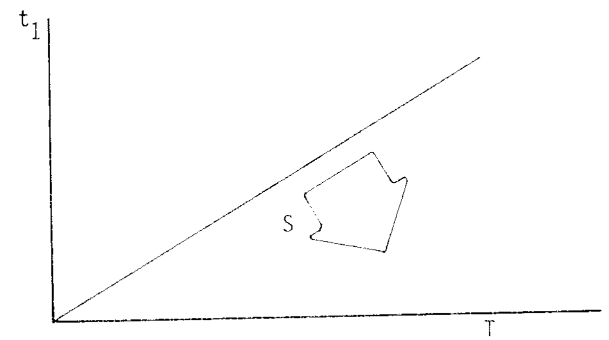

Fig. 1.2

Inverting the order of integration, we have,

$$\text{(1.9.4)} \qquad \iint_S e^{-st} f(t_1)\, g(t-t_1)dt_1 dt$$
$$= \int_0^T f(t_1) \left[\int_{t_1}^T e^{-st} g(t-t_1)dt\right]dt_1$$
$$= \int_0^T e^{-st_1} f(t_1) \left[\int_0^{T-t_1} e^{-su} g(u)du\right]dt_1 \quad .$$

As $T\to\infty$, we obtain formally $L(f\star g) = L(f)L(g)$. To put this heuristic result on a rigorous foundation, we shall prove the following theorem.
If

$$\text{(a)} \qquad \int_0^\infty e^{-at_1} |f(t_1)|dt_1 < \infty$$

(b) $\int_0^\infty e^{-(a+it)t_1} g(t_1)dt_1$ converges for $t \geq 0$,

then

$$\int_0^\infty h(t)e^{-st}\,dt = \left[\int_0^\infty e^{-st} f(t)dt\right]\left[\int_0^\infty e^{-st} g(t)dt\right]$$

for $s = s + ib$, and generally for $\mathrm{Re}(s) > a$.

To begin the proof, we have, referring to (1.9.4),

$$\int_0^T h(t)e^{-st}\,dt = \int_0^T e^{-st_1} f(t_1)\left[\int_0^{T-t_1} e^{-su} g(u)dt\right]dt_1 \tag{1.9.5}$$

$$= \int_0^T e^{-st_1} f(t_1)\left[\int_0^\infty e^{-su} g(u)du\right]dt_1$$

$$- \int_0^T e^{-st_1} f(t_1)\left[\int_{T-t_1}^\infty e^{-su} g(u)du\right]dt_1 ,$$

The second integral in (1.9.5) can be shown to be bounded. We break up the range of t_1 integration into two parts, (0, R) and (R, T). Since, by assumption the integrals $\int_0^\infty e^{-su} g(u)du$ and $\int_0^\infty e^{-at}|f(t)|dt$ converge, then for any $\varepsilon > 0$, there is an R, depending on ε for which,

(a) $\left|\int_{R(\varepsilon)}^\infty e^{-su} g(u)du\right| \leq \varepsilon$

(b) $\left|\int_{R(\varepsilon)}^\infty e^{-st} f(t)dt\right| \leq \varepsilon c_2$

(c) $\left|\int_0^\infty e^{-st} g(u)du\right| < c_1$.

Therefore for $R(\varepsilon)$ selected to satisfy (a), we can say

$$\left|\int_0^{R(\varepsilon)} e^{-st_1} f(t_1)\left[\int_{R(\varepsilon)-c_1}^\infty e^{-su} g(u)du\right]dt_1\right| \tag{1.9.6}$$

$$\leq \varepsilon\int_0^{R(\varepsilon)} e^{-at_1} |f(t_1)|dt_1$$

$$\leq \varepsilon\int_0^\infty e^{-at_1} f(t_1)\,dt_1$$

$$\leq \varepsilon c_2 \quad .$$

The remaining integral has the bound,

$$\left|\int_{R(\varepsilon)}^T f(t_1)\left[\int_{R(\varepsilon)-t_1}^\infty e^{-su} g(u)du\right]dt_1\right| \tag{1.9.7}$$

$$\leq c_1\int_{R(\varepsilon)}^T e^{-at_1} |f(t_1)|dt_1$$

$$< c_1\varepsilon \quad .$$

By letting $T \to \infty$, the second integral in (1.9.5) goes to zero and we are left with the result,

$$\int_0^\infty h(t)\,e^{-st}\,dt = \int_0^\infty e^{-st} f(t)dt \int_0^\infty e^{-su} g(u)du \tag{1.9.8}$$

i.e.,

$$L(f * g) = L(f)L(g) \quad . \tag{1.9.9}$$

Result (1.9.9) informs us that rather than constructing the Laplace Transform for a complicated convolution involving both $f(t)$ and $g(t)$, we need only construct the transform of each separately and multiply. This device will be used later with the renewal equation.

1.10 Instability of the Inverse of the Laplace Transform

Let $v(t)$ be a function for which

$$(1.10.1) \qquad \int_0^\infty |v(t)|\ e^{-kt}\ dt < \varepsilon \quad .$$

Then, for $\mathrm{Re}(s) \ge k$,

$$(1.10.2) \qquad |L(u+v) - L(u)| = \left|\int_0^\infty v(t)\ e^{-st}\ dt\right|$$

$$\le \int_0^\infty |v(t)| e^{-\mathrm{Re}(s)t} < \varepsilon \quad .$$

In other words, a "small" change in u produces an equally small change in $L(u)$. In mathematical terms, $L(u)$ is stable under perturbations of this type.

The impossibility of usable universal algorithms for inverting the Laplace Transform is a consequence of the fact that the inverse of the Laplace Transform is not stable under reasonable perturbations. Two simple examples illustrate this. Consider first the well-known formula,

$$(1.10.3) \qquad L(\sin at) = a/(s^2 + a^2) \quad .$$

As a increases, the function $\sin at$ oscillates more and more rapidly, but remains of constant amplitude. The Laplace Transform, however, is uniformly bounded by $1/a$ for $s \ge 0$ and thus approaches 0 uniformly as $a \to \infty$.

As a second example we consider the formula,

$$(1.10.4) \qquad L(u) = L\left(\frac{a}{2\sqrt{\pi}}\ \frac{e^{-a^2/4t}}{t^{3/2}}\right) = e^{-a\sqrt{s}} \quad .$$

As $a \to 0$, the function $e^{-a\sqrt{s}}$ remains uniformly bounded by 1 for $s \ge 0$. Observe how $u(t)$ behaves as a function of t. At $t = a^2/4$, we see that it has the value c_1/a^2 where c_1 is a positive constant. Nonetheless, at $t = 0$, for all $a > 0$, $u(t)$ assumes the value 0. Hence $u(t)$ has a "spike" form (see Fig. 1.3), one which is sharper and sharper as $a \to 0$. We see then that $u(t)$ is

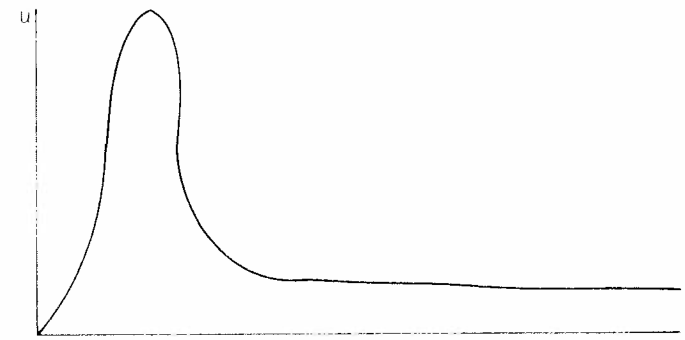

Fig. 1.3

is highly localized in the vicinity of $t = 0$ for a small, and thus that $u(t)$ is an excellent approximation to the delta function $\delta(t)$.

These examples make evident some of the difficulties we face in finding $u(t)$ given $F(s)$. Let $F(s)$ be calculated to an accuracy of 1 in 10^{10}, say to ten significant figures, then if $u_0(t)$ is the function giving rise to $F(s)$ via $F(s) = L(u_0)$, we see that

$$(1.10.5) \qquad u_0 + \sin 10^{20}(t-t_1) + 10^{-20}\ \frac{a_1 e^{-a_1^2/4(t-s_1)}}{a\sqrt{\pi}\ (t-s_1)^{3/2}}$$

will have, to ten significant figures, the same Laplace Transform for any a_1, $s_1 > 0$.

We cannot therefore "filter out" extremely rapid oscillations or spike behavior of $u(t)$ on the basis of numerical values of $F(s)$ alone. What we can do to escape from this simultaneous nightmare and quagmire of pathological behavior is to agree to restrict our attention from the beginning to functions $u(t)$ which are essentially smooth. In other words, we can use knowledge of the structural behavior of $u(t)$ to obtain numerical values. In many cases, as we shall show, the inverse transform must be accomplished by numerical means and we shall be continually keeping the analytical and numerical behavior of each problem clearly in mind.

1.11 The Laplace Transform and Differential Equations

An important application of Laplace Transform occur in the solution of ordinary differential equations which are cast in the form of initial value problems. Properties of the Laplace Transform make the transform very appealing as a means of finding solutions provided the inverse transform can be easily found.

Let us now turn to the solution of linear equations with constant coefficients by means of the Laplace Transform. To begin with, consider the first-order scalar equation,

$$(1.11.1) \qquad du/dt = au + v \;, \qquad u(0) = c \quad .$$

Taking the Laplace Transform of both sides, we have

$$(1.11.2) \qquad \int_0^\infty e^{-st}(du/dt)dt = a\int_0^\infty e^{-st}u(t)dt + \int_0^\infty e^{-st}v(t)dt \quad .$$

Hence by integrating by parts,

$$(1.11.3) \qquad e^{-st}u\Big|_0^\infty + s\int_0^\infty e^{-st}udt = a\int_0^\infty e^{-st}udt + \int_0^\infty e^{-st}vdt \quad .$$

Writing,

$$(1.11.4) \qquad L(u) = \int_0^\infty e^{-st}udt \;, \qquad L(v) = \int_0^\infty e^{-st}vdt \quad ,$$

we have, by solving (1.11.3) for $L(u)$,

$$(1.11.5) \qquad L(u) = \frac{c}{s-a} + \frac{L(v)}{s-a} \quad .$$

The inverse of the first term is known to be ce^{at}. To obtain the inverse of the second term, we apply the convolution theorem. The result is

$$(1.11.6) \qquad u(t) = ce^{at} + \int_0^t e^{a(t-s)}v(s)\,ds \quad .$$

Turning to the vector-matrix case,

$$(1.11.7) \qquad dx/dt = Ax + y \;, \qquad x(0) = c \quad ,$$

we have

$$(1.11.8) \qquad L(x) = (sI-A)^{-1}c + (sI-A)^{-1}L(y) \quad .$$

Since the inverse transform of $(sI-A)^{-1}$ is e^{At}, we obtain the expression,

$$(1.11.9) \qquad x = e^{At}c + \int_0^t e^{A(t-s)}y(s)\,ds \quad .$$

1.12 Transient Solutions

A byproduct of the Laplace Transform technique when it is applied to differential equations is the fact that one may be able to get explicit expressions for the transient solutions which reflect the initial conditions. Referring to the last section, we see that the differential equation itself (1.11.1) carries the initial condition, $u(0)=c$, as an auxilary condition. The form of the solution given by the Laplace Transform (1.11.5) shows that the initial condition, c, is incorporated in the explicit analytic expression.

If the transform can be inverted analytically, then the results shows explicitly how the initial conditions are propagated in time. If the transform must be inverted numerically, then one cannot tell for sure how the initial conditions propagate and the numerical solution has to be constructed very carefully.

1.13 Generating Functions

The generating function is an example of a transform on functions of a discrete variable, or index, and is formally quite similar to the Laplace Transform.

If $\{u_n\}$, $n=1,2,\ldots$ is a sequence, the generating function associated with set $\{u_n\}$ is defined as,

(1.13.1) $$G(z) = \sum_{n=0}^{\infty} u_n z^n$$

where z is a complex number. Several examples come immediately to mind,

(a) $u_n = 1, \quad n \geq 0$ $\qquad G(z) = \sum_{n=0}^{\infty} z^n$,

(b) $u_n = \begin{Bmatrix} 0 & 0 \leq n \leq m \\ 1 & n \geq m \end{Bmatrix}$ $\qquad G(z) = \sum_{n=m}^{\infty} z^n$,

(c) $u_n = \begin{Bmatrix} 0 & 0 \leq n \leq m \\ u_{n-m} & n \geq m \end{Bmatrix}$ $\qquad G(z) = \sum_{n=m}^{\infty} u_{m-n} z^n$.

The inversion of the transform defined by (1.13.1) can quickly be obtained by two different methods. Noting that $G(z)$ is analytic in z, we obtain by differentiation,

(1.13.2) $$u_n = 1/n! \left. \frac{d^n G(z)}{dz^n} \right|_{z=0} .$$

Relating to the analyticity of $G(z)$ we multiply both sides of (1.13.1) by z^{-m-1}, for a given m. Then

(1.13.3) $$z^{-m-1}\, G(z) = \sum_{n=0}^{\infty} u_n\, z^{n-m-1} .$$

If we integrate (1.13.3) by a contour integral about a simple closed contour including the origin and with the region of analyticity,

(1.13.4) $$\int_C z^{-m-1}\, G(z)dz = 2\pi i u_m ,$$

or,

(1.13.5) $$u_m = \frac{1}{2\pi i} \int_C z^{-m-1}\, G(z)dz .$$

The convolution of two sequences can be defined in a way analogous to the convolution for the Laplace Transform. Let $\{u_n\}$ and $\{v_n\}$ be two sequences and defined a third sequence $\{w_n\} = \sum_{i=0}^{n} u_i v_{n-i}$. We then write $\{w_n\} = \{u_n\} * \{v_n\}$ and say that $\{w_n\}$ is the convolution of $\{u_n\}$ and $\{v_n\}$. Under these definitions it is easy to see that the generating function of the convolution of the two sequences is the product of the generating functions of the two sequences.

Problems

1. The Laplace Transform is defined as,

$$F(s) = \int_0^{\infty} e^{-st}\, f(t)dt .$$

If s is a complex number $(s = x + iy)$:

(a) Show that $F(s)$ is a complex function of a complex variable.

(b) If $f(t)$ is bounded on $0 < t < \infty$, determine the region of convergence of $F(s)$ in the complex plane. Show $F(s)$ is analytic in this region.

2. Find the Laplace Transforms for the following functions.

(a) $f(t) = 1$

(b) $f(t) = t$

(c) $f(t) = t^n$

(d) $f(t) = \sin at$

(e) $f(t) = \cosh at$

(f) $f(t) = e^{-at}$

(g) $f(t) = t \sin at$

(h) $f(t) = U(t)$ where $U(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$.

3. Show that the Laplace Transform of $f(t) = e^{-at} \sin bt$ is

$$F(s) = b/((s+a)^2 + b^2) \quad .$$

(a) Where in the complex plane is $F(s)$ analytic?

(b) If $a > 0$, is $f(t)$ stable, (i.e., is it bounded at infinity?) and where are the poles of $F(s)$?

(c) If $a < 0$, $f(t)$ is unstable. How are the poles of $F(s)$ shifted?

(d) Given $F(s)$, find $f(t)$ by the inversion formulae.

4. What is the Laplace Transform of $f'(t)$?
Under what conditions on $f(t)$ does this transform exist?
What must be specified to make the transform unique?
Does the transform have a zero or pole in the complex plane and if so where are they located?

5. Repeat (4) for $f''(t)$.

6. Repeat (4) for $\int_0^t f(s)ds$.

7. If $F(s) = F(s-a)$, then what can be said for the function $f(t)$?
State in words how a linear transformation in the transform will effect the function $f(t)$.
What happens if a is complex?

8. Given a function of two variables $f(x, y)$, the Laplace Transform is defined to be,

$$F(u, v) = \int_0^\infty \int_0^\infty e^{-ux - vy} f(x, y)dxdy \quad .$$

Under what conditions does $F(u, v)$ exist?

9. Show $L(f(t) + g(t)) = L(f(t)) + L(g(t))$.

10. Using (9) and (10), explain why $L(f)$ is a linear operator.
Why is this an important observation?