Application: Volterra integral equations

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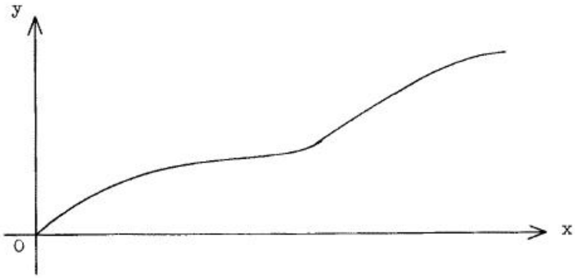
First appearance

Abel's Mechanical Problem

In the (x,y)-plane find a curve C which is the graph of an increasing function $x = \varphi(y), y \in [0, H]$, along which under constant downward acceleration g a particle must be constrained to fall, in order that its falling time equals a prescribed function t(y) of the initial height y.



Abel's Mechanical Problem

In the absence of friction, the problem can be reduced to solving for φ in the equation

$$\int_0^y (y-z)^{-1/2} \sqrt{1+\varphi'(z)^2}\, \mathrm{d}z = \sqrt{2g}\, t(y).$$

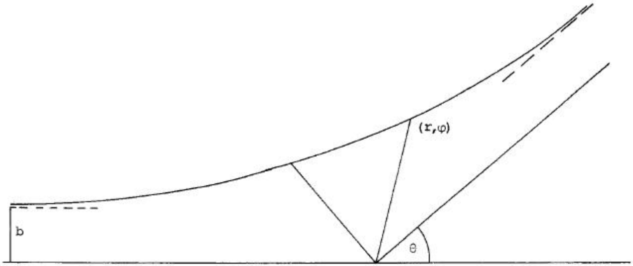
Remark

The $-1/2$ exponent for the singular kernel often occurs for Abel Integral Equations arising from physics

Inverse scattering problem for a repelling potential

Shooting a particle at an atom nucleus

We are interested in determining the potential $V(r)$ of the repelling field of an Atom nucleus. We can do so by measuring the angle of deflection θ a particle with impact parameter $b > 0$ experiences.



By impact parameter, we mean the closest distance the particle would approach the atom if it were to travel in a straight line.

Inverse scattering problem for a repelling potential

One can define a function for the angle of deflection θ by varying the impact parameter b, i.e. $\theta = \theta(b)$.

Inverse scattering problem for a repelling potential

One can define a function for the angle of deflection θ by varying the impact parameter b, i.e. $\theta = \theta(b)$.

$$\theta(b) = \pi - 2\int_{r_0}^{\infty} \frac{\mathrm{dr}}{r^2(b^{-2} - r^{-2} - E^{-1}b^{-2}V(r))^{1/2}}$$

where r_0 is the solution to

$$E - b^2 E r_0^{-2} - V(r_0) = 0$$

Inverse scattering problem for a repelling potential

A few change of variables later:

One can obtain an integral equation of the form

$$\beta(x) = \int_0^x \frac{g(w)\,\mathrm{d}w}{(x-w)^{1/2}},\ 0 \le x \le \frac{1}{b_{\text{min}}^2}.$$

Remark: The bound on the integral arises from the fact energy has been fixed.

Previously in the Seminar

We have previously discussed the approximation of

$$(f \star g)(x) = \int_0^x f(s)g(x-s)\,\mathrm{d}s$$

- g is given explicitly. Only "scant" information about f is given (i.e. conditions suitable for the inversion of its Laplace transform)
- Obtained convolution quadratures from an appropriate linear multistep method(LMSM)
- Error analysis performed justified the use of convolution quadratures. In particular, the case $g(t) = t^{\alpha-1}$ was discussed

Convention

Unless otherwise stated, all LMSMs satisfy

- The method is stable and consistent of order p
- The method is implicit
- All zeros of $\sigma(\zeta)$ have absolute value ≤ 1.

Purpose of this talk

We discuss a method for the numerical solution of a weakly singular Abel-Volterra integral equation

$$y(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, y(s))\, \mathrm{d}s, \quad 0 < \alpha < 1 \text{ fixed}$$

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How is this problem different?

- We are approximating y! More precisely, we are *solving* the (nonlinear) integral equation numerically.
- Known quantities g and f (the latter not appearing in the integral).
- No information about the Laplace transform of y is assumed. Instead, we rely on a priori regularity and asymptotic properties.

Definition

A suitable numerical scheme can be developed for a larger class of integral equations

Abel-Volterra Integral Equation of the Second Kind

$$y(t) = f(t) + \int_0^t K(t, s, y(s))\, \mathrm{d}s$$

Remarks:

- Special case: $K(t, s, y(s)) = (t - s)^{\alpha-1} g(s, y(s))$.
- If K, f are independent of t, the equation reduces to the initial value problem
$$y' = K(t, y), \quad y(0) = y_0.$$

Convolution Quadratures for the Abel-Volterra Integral Equation

We shall show a natural scheme to consider is of the form

$$y_n = f(x_n) + h \sum_{j=-1}^{-k} w_{nj} K(x_n, x_j, y_j) + h \sum_{j=0}^{n} \omega_{n-j} K(x_n, x_j, y_j) \ (n \geq 0)$$

where w_{nj} and ω_{n-j} are weights of (potentially) different methods

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Why is there an extra sum?!

Applying LMSM to Abel-Volterra Integral Equation

Let (ρ, σ) be a LMSM.

Rewrite the Abel-Volterra equation

$$y(x) = J(x, x_n) + \int_{x_n}^{x} K(x, s, y(s))\, \mathrm{d}s, \ (x \geq x_n)$$

where $J(x, x_n) = f(x) + \int_0^{x_n} K(x, s, y(s))\, \mathrm{d}s$

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Note: $y(x_n) = J(x_n, x_n)$

ODE Problem:

$$J(x, \xi) = f(x) + \int_0^{\xi} K(x, s, y(s))\, \mathrm{d}s$$

Taking a partial derivative yields

$$\frac{\partial}{\partial \xi} J(x, \xi) = K(x, \xi, y(\xi)) \tag{1}$$

Approximation of $J_n = J(\cdot, x_n)$

Applying a LMSM to the ODE problem

Recall: k-step LMSMs require k initial values to "take-off"

Starting values $y_{-k}, \ldots, y_{-1}$ are given. We assume the starting functions $\tilde{J}_{-k}, \ldots, \tilde{J}_{-1}$ are given by *some* quadrature (not necessarily related to (ρ, σ))

$$\tilde{J}_n(x) = f(x) + h \sum_{j=-1}^{-k} w_{nj} K(x, x_j, y_j) \; n \in \{-k, \ldots, -1\}$$

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Apply (ρ, σ) to (1) and obtain an approximation $\tilde{J}_n$ of J_n

$$\sum_{j=0}^{k} \alpha_j \tilde{J}_{n+j-k}(x) = h \sum_{j=0}^{k} \beta_j K(x, x_{n+j-k}, y_{n+j-k}) \tag{2}$$

Set $y_n = \tilde{J}_n(x_n)$.

y_n is an approximation of $y(x_n)$

Convolution Quadrature

Lemma

The LMSM (2) with starting functions(as above) can be *rewritten* as a quadrature method

$$y_n = f(x_n) + h \sum_{j=-1}^{-k} w_{nj} K(x_n, x_j, y_j) + h \sum_{j=0}^{n} \omega_{n-j} K(x_n, x_j, y_j) \; (n \geq 0)$$

where the weights ω_n and w_{nj} are bounded. The weights ω_n are the coefficients of the power series

$$\omega(\zeta) = \frac{\sigma(\zeta^{-1})}{\rho(\zeta^{-1})}.$$

Converse: If $\omega(\zeta)$ is a rational function, one can recover the LMSM

Adapted Numerical Method for Convolution Type Kernels

We wish to apply a convolution quadrature to approximate

Abel-Volterra Integral Equation of the 2nd kind with a Convolution Kernel

$$y(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, y(s))\, \mathrm{d}s,$$

for $t \in [0, T]$ and $0 < \alpha < 1$ fixed.

Can we furnish a method adapted to this particular type of Abel-Volterra Integral Equation?

Idea: Consider *fractional* step-sizes and weights!

Fractional Linear Multistep Method

We seek to construct

Fractional Linear Multistep Method

$$y_n = f(t_n) + h^\alpha \sum_{j=0}^{m} w_{nj} g(t_j, y_j) + h^\alpha \sum_{j=0}^{n} \omega_{n-j}^{(\alpha)} g(t_j, y_j)$$

- $h > 0$ denotes step-size
- $t_n = nh$
- starting quadrature weights w_{nj} (independent of h), $j = 0, \ldots, m$, with m fixed
- convolution quadrature weights $\omega_n^{(\alpha)}$

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- starting quadrature weights w_{nj} (independent of h), $j = 0, \ldots, m$, with m fixed
- convolution quadrature weights $\omega_n^{(\alpha)}$

Given (ρ, σ), convolution quadrature weights $\omega_n^{(\alpha)}$ are obtained from the coefficients of the power series

$$\omega^{(\alpha)}(\zeta) = \left(\frac{\sigma(\zeta^{-1})}{\rho(\zeta^{-1})} \right)^\alpha$$

As a (non-trivial) consequence of our convention, $\omega_n^{(\alpha)} = O(n^{\alpha-1})$.

Why Fractional Linear Multistep Methods?

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- Efficient implementation
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The Driving Question

Given a LMSM $\omega = (\rho, \sigma)$, can we construct a Convolution Quadrature satisfying

- **same** convergence properties as (ρ, σ)
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The Driving Question

Given a LMSM $\omega = (\rho, \sigma)$, can we construct a Convolution Quadrature satisfying

- **same** convergence properties as (ρ, σ)
- **same** stability properties as (ρ, σ)

The answer is yes!

Convergence analysis setup

Consider

$$y(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, y(s))\, \mathrm{d}s$$

Assume: g, f are sufficiently smooth

These conditions *guarantee*

- Uniqueness of solution $y(t)$
- Sufficient regularity of $y(t)$
- Our assertions are true

Convergence result

Theorem

Given a LMSM $\omega = (\rho, \sigma)$, there exists a starting quadrature $w_{nj} = O(n^{\alpha-1})$ so that the error of the computed solution satisfies

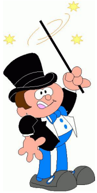
$$|y_n - y(t)| \leq C \cdot t^{\beta-1} \cdot h^p \quad (t = nh \leq T),$$

for all h sufficiently small.

Remarks:

- C independent of n and h
- $\beta > \alpha$

Proof: Choosing the starting quadrature weights



For any sufficiently smooth function $\varphi(t)$, there exists a starting quadrature $w_{nj} = O(n^{\alpha-1})$ satisfying

$$h^\alpha \sum_{j=0}^{n} \omega_{n-j}^{(\alpha)} \varphi(jh) + h^\alpha \sum_{j=0}^{m} w_{nj} \varphi(jh) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)\, \mathrm{d}s + O(t^{\beta-1} \cdot h^p),$$

for some $\beta > \alpha$. In particular,

$$h^\alpha \sum_{j=0}^{n} \omega_{n-j}^{(\alpha)} g(t_j, y(t_j)) + h^\alpha \sum_{j=0}^{m} w_{nj} g(t_j, y(t_j)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, y(s))\, \mathrm{d}s + O(t^{\beta-1} \cdot h^p).$$

Proof: Consistency Error

Define the *consistency error* at $t = nh$ by

$$d_n = \Bigg| h^\alpha \sum_{j=0}^{n} \omega_{n-j}^{(\alpha)} g(t_j, y(t_j)) + h^\alpha \sum_{j=0}^{m} \omega_{nj} g(t_j, y(t_j)) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, y(s))\, \mathrm{d}s \Bigg|$$

It follows

$$d_n \leq C \cdot n^{\beta-1} \cdot h^{p+\beta-1}$$

Proof: Error Propagation

The *global error* at $t = nh$ is defined as

$$e_n = |y_n - y(t)|$$

Claim:

Bound on global error

$$e_n \leq C \cdot h^p \cdot t^{\beta-1}$$

Simplifying Assumption: g is Lipschitz continuous in the second argument.
The triangle inequality yields

$$e_n \leq d_n + h^\alpha L \left(\sum_{j=0}^{n} |\omega_{n-j}^{(\alpha)}| e_j + \sum_{j=0}^{m} |w_{nj}| e_j \right)$$

Recall, the weights are $O(n^{\alpha-1})$

Fact:

$$n^{\alpha-1} \leq (-1)^n \binom{-\alpha}{n},$$

where $(-1)^n \binom{-\alpha}{n}$ is the n-th coefficient of power series expansion of $(1-\zeta)^{-\alpha}$.

Proof: Error Propagation

Therefore, we can bound

$$e_n \le C \cdot h^{p+\beta-1} \cdot (-1)^n \binom{-\beta}{n} + h^\alpha C \sum_{j=0}^{n} (-1)^{n-j} \binom{-\alpha}{n-j} e_j,$$

for some generic constant $C > 0$.

Hint: Try finding a power series $u(\zeta) = \sum_{j=0}^{\infty} u_n \zeta^n$ such that

$$e_n \le C \cdot h^p \cdot u_n$$

with (hopefully) $u_n = O(t^{\beta-1})$.

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with (hopefully) $u_n = O(t^{\beta-1})$.

Fortunately, such a power series **exists**!

$u(\zeta) = \frac{1}{h} V(\frac{1-\zeta}{h})$, where $V(z)$ is the Laplace transform of

$$v(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} + C \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\, ds.$$

works. This completes the proof.

Analytic stability region of the Abel-Volterra Integral Equation

For this discussion, we consider a linearized Abel-Volterra integral equation

$$y(t) = f(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, \mathrm{d}s, \quad t \geq 0, 0 < \alpha < 1$$

Stability Theorem

If $|\arg\lambda - \pi| < (1-\alpha/2)\pi$, the solution $y(t)$ satisfies

- $y(t) \to 0$ as $t \to \infty$ whenever $f(t)$ converges to a finite limit
- $y(t)$ is bounded whenever $f(t)$ is bounded.

Stability region of a Fractional LMSM

Apply fractional LMSM

$$y_n = f_n + h^\alpha \lambda \sum_{j=0}^{n} \omega_{n-j}^{(\alpha)} y_j,$$

where $f_n = f(t_n) + h^\alpha \lambda \sum_{j=0}^{m} w_{nj} y_j$

Stability Region of a Fractional LMSM

$$S = \{\lambda \in \mathbb{C} : \ f_n \to L \text{ implies } y_n \to 0\}$$

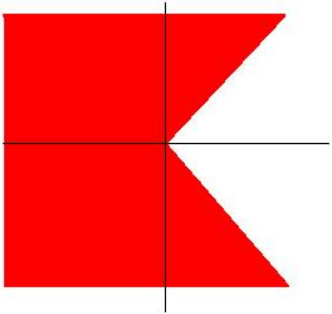
Stability Preservation

Since $w_{nj} = O(n^{\alpha-1})$, it follows that $\lim_{n\to\infty} f_n = L$ whenever $\lim_{t\to\infty} f(t) = L$. If $f_n \to L$ implies $y_n \to 0$, then the stability region should *at least* contain the analytic stability region

- If $\{z \in \mathbb{C} : \ |\arg z - \lambda| < (1 - \alpha/2)\pi\} \subset S$ the method is **A-stable**
- If $\{z \in \mathbb{C} : \ |\arg z - \pi| < \varphi\} \subset S$, the method is $A(\varphi)$-**stable**.

Example 1: Stability region of an A-stable Fractional LMSM

Contains the *red* wedge



Characterization of the Stability Region

Recall: Stability region for a LMSM is given by

$$S = \mathbb{C} \setminus \{\rho(\zeta)/\sigma(\zeta) : \ |\zeta| \geq 1\} = \mathbb{C} \setminus \{1/\omega(\zeta) : \ |\zeta| \leq 1\}$$

A similar characterization for Fractional LMSMs exists

Theorem

The stability region of a fractional LMSM is given by

$$S = \mathbb{C} \setminus \{1/\omega^{(\alpha)}(\zeta) : |\zeta| \leq 1\}.$$

Transformation of Stability Regions

Corollary

Let $\omega = (\rho, \sigma)$ and ω^α its corresponding fractional LMSM. Letting $S_\omega, S_{\omega^\alpha}$ denote their respective stability regions, we have:

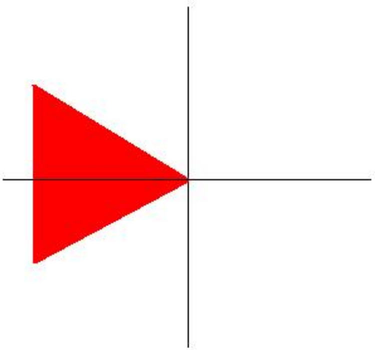
(a) $(\mathbb{C} \backslash S_{\omega^\alpha}) = (\mathbb{C} \backslash S_\omega)^\alpha$

(b) ω^α is A-stable if and only if ω is A-stable

(c) With $\pi - \varphi = \alpha(\pi - \psi)$, ω^α is $A(\varphi)$-stable if and only if ω is $A(\psi)$-stable

Example 2: Stability region of an $A(\psi)$-stable FLMSM

Suppose $\omega = (\rho, \sigma)$ is $A(\varphi)$-stable, i.e. its stability region contains the *red* wedge



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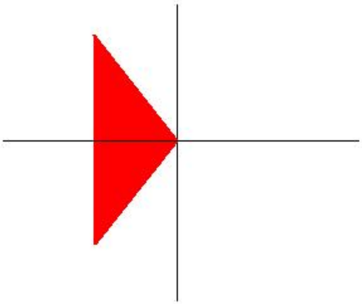
For $0 < \alpha \ll 1$, ω^{α} is $A(\psi)$-stable, so its stability region contains the *red* wedge



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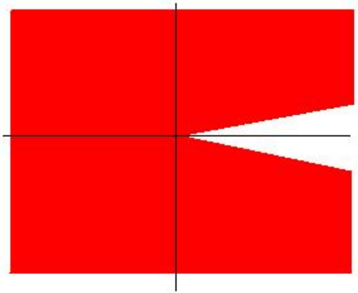
Or $0 \ll \alpha < 1$ ω^{α} is $A(\psi)$-stable, so its stability region contains the *red* wedge



Consistency

Fractional LMSMs consistent of order p satisfy

$$h^\alpha \omega^\alpha(e^{-h}) = 1 + \alpha c^* h^p + O(h^{p+1})$$

c^* is referred to as the *error constant*

Theorem

The order of an A-stable fractional linear multistep method cannot exceed 2.

Example: Fractional Trapezoidal Rule The fractional trapezoidal rule, defined by

$$\omega^\alpha(\zeta) = \left(\frac{1+\zeta}{2-2\zeta}\right)^\alpha$$

is A-stable and has order 2. In particular, it achieves the smallest error constant, $c^* = 1/12$.

A Proof from First Principles

An A-stable fractional method satisfies

$$|\arg \omega^{\alpha}(\zeta)| \leq \frac{\alpha}{2}\pi, \; |\zeta| \leq 1, \; \zeta \neq 1.$$

Complexify $h^{\alpha}\omega^{\alpha}(e^{-h}) - 1$ i.e. consider the function $z^{\alpha}\omega^{\alpha}(e^{-z}) - 1$.

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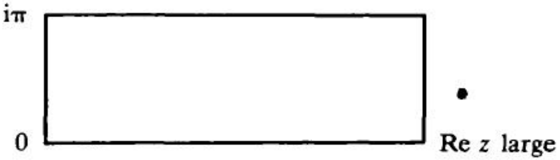
Observe:

- For $z \in (0, i\pi]$, $\mathrm{Im}[z^\alpha \omega^\alpha(e^{-z})] \geq 0$
- For both $z \in \mathbb{R}^+$ and $z \in i\pi + \mathbb{R}^+$, $\mathrm{Im}[z^\alpha \omega^\alpha(e^{-z})] \geq 0$
- $\omega^\alpha(e^{-z}) = \omega^\alpha(0) + O(e^{-\mathrm{Re}z})$, for sufficiently large $\mathrm{Re}z$. and $\omega^\alpha(0) > 0$



A Proof from First Principles

Therefore

$$\mathrm{Im}[z^\alpha \omega^\alpha (e^{-z})] \geq 0 \ \forall z \text{ on the boundary of the rectangle}$$

Maximum Principle implies

$$\mathrm{Im}[z^\alpha \omega^\alpha (e^{-z})] \geq 0 \ \forall z \text{ in the interior of the rectangle}$$

Consistency of order p gives

$$0 \leq \mathrm{Im}[z^\alpha \omega^\alpha (e^{-z}) - 1] = \alpha c^* \mathrm{Im} z^p + O(z^p)$$

which holds for $z \to 0$ and $0 \leq \arg z \leq \pi/2$.

This can only happen if $p \leq 2$.

$(\mathrm{BDF4})^{1/2}$

Consider the integral equation

$$y(t) = -\frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-1/2} (y(s) - \sin s)^3 \, ds.$$

Convolution quadrature weights generated by

$$\omega^{1/2}(\zeta) = \left(\frac{25}{12} - 4\zeta + 4\zeta^2 - \frac{4}{3}\zeta^3 + \frac{1}{4}\zeta^4 \right)^{-\frac{1}{2}}$$

Exact solution $y(8) = 0.3236412904$ is known.

h	**numerical solution**	**error**	**error** $/h^4$
0.1	0.3236520328	1.07×10^{-5}	0.107
0.05	0.3236421096	8.19×10^{-7}	0.1314
0.025	0.3236413206	3.02×10^{-8}	0.0773