Analysis of PML for the Helmholtz equation
What’s the physical situation?
Example

Let $\Omega$ represent an infinite cylinder in $\mathbb{R}^3$. We consider how the magnetic field $\mathbf{H} = \mathbf{H}(\mathbf{x}, t)$ is scattered by the cylinder.
We assume that the magnetic field is parallel to the axis of the cylinder.
With further assumptions, the scattered magnetic field can be written as

\[ \mathbf{H} = u(r, \phi) e^{-iwt} \mathbf{e}_z \]

- where \((r, \phi)\) are the polar coordinates in the plane,
- \(w > 0\) the angular frequency,
- \(\mathbf{e}_z\) the direction of the axis of the cylinder.

**Mathematical analysis of the situation**

- Mathematical description?
- How to describe „the scattering“?
Answer (w.r.t. the Example)

For the amplitude $u = u(r, \phi)$ we have to solve:

1. $(\Delta + k^2)u = 0$ in $\mathbb{R}^2 \setminus \bar{\Omega}$,
2. $\frac{\partial u}{\partial n}\mid_{\partial \Omega} = g$,
3. $\lim_{r \to \infty} r^{1/2}(\frac{\partial u}{\partial r} - iku) = 0$ uniformly in $\hat{x}$,

where $k^2 = w^2 \epsilon_0 \mu_0 > 0$ and $r = |x|$ and $\hat{x} = x/r$, $r \neq 0$.

the equation (3) is called the „Sommerfeld radiation condition at infinity“.
Difficulties

Since $\Omega$ is bounded, $\mathbb{R}^2 \setminus \tilde{\Omega}$ is unbounded.

Solution: The PML-method.

Overview

1. Scattering BVP $\rightarrow u_{sc}$.
2. Full-space Bérenger BVP $\rightarrow u_C$ with $u_C |_D = u_{sc} |_D$.
3. Truncated Bérenger BVP $\rightarrow \tilde{u}_C(\rho)$, bounded domain.
4. Main Theorem: $\tilde{u}_B(\rho) \xrightarrow{\rho \rightarrow \infty} u_{sc}$ near $\Omega$.
5. Outline of the proof.
Part 1: The scattering BVP

1. \((\Delta + k^2)u = 0\) in \(\mathbb{R}^2 \setminus \bar{\Omega}\)

2. \(\frac{\partial u}{\partial n} |_{\partial \Omega} = g \in H^{-1/2}(\partial \Omega)\)

3. \(\lim_{r \to \infty} r^{1/2}(\frac{\partial u}{\partial r} - iku) = 0\) uniformly in \(\hat{x}\)

Note: \((3) \implies u(r, \varphi) \xrightarrow{r \to \infty} C_1 e^{ikr} + C_2 \varphi\)
Existence Theorem

The scattering BVP has a unique (weak) solution $u \in H^1_{rad}(\mathbb{R}^2 \setminus \Omega)$, where

$$H^1_{rad}(\mathbb{R}^2 \setminus \Omega) = \{ u \ ; \ u \in H^1(B_R(0) \setminus \Omega) \text{ for all } R > 0 \}$$

and $u$ satisfies the Sommerfeld radiation condition.

This unique solution of the scattering problem is denoted by $u_{sc}$. 
Part 2: The Construction

We introduce a strictly convex set $D \subset \mathbb{R}^2$, such that

1. $\bar{\Omega} \subset D$,

2. $D$ has a $C^2$-boundary.
Definition of $h$

Let $x \in \mathbb{R}^2 \setminus D$. Then we define

- $h(x) := \text{dist}(x, \partial D) \geq 0$ and
- $p(x) \in \partial D$ such that $h(x) = |x - p(x)|$.

With $x = p(x) + h(x) \cdot n(x)$. 

![Diagram showing the definition of $h$ and $p$](image)
Definition of $\tau$

Further, let $\tau : [0, \infty) \rightarrow [0, \infty)$ be a $C^2$-function with:

- the derivative $\tau'$ is strictly increasing,
- $\lim_{s \to \infty} \tau'(s) = \infty$
- $\tau(0) = \tau'(0+) = \tau''(0+) = 0$
- $\lim_{s \to \infty} e^{-\epsilon \cdot \tau(s)} \tau'(s) = \lim_{s \to \infty} e^{-\epsilon \cdot \tau(s)} \tau''(s) = 0$
  for all $\epsilon > 0$. 

![Graph of \(\tau(x) = x^3\)](image)
Definition of the function $a$

We define a function $a : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by setting

$$a(x) := \begin{cases} 0 & \text{if } x \in D \\ \tau(h(x)) \cdot n(x) & \text{if } x \in \mathbb{R}^2 \setminus D \end{cases}$$
Definition of the stretching function $F$

We define the function $F : \mathbb{R}^2 \rightarrow \mathbb{C}^2$ by setting

$$F(x) := x + i \cdot a(x).$$
Definition of $\Gamma$

$$\Gamma := F(\mathbb{R}^2) = \{z \in \mathbb{C}^2 \mid z = x + i \cdot a(x), x \in \mathbb{R}^2\}$$
The fundamental solution of the Helmholtz equation

\[ \Phi(x, y) = \frac{i}{4} \cdot H_0^{(1)}(k \cdot |x - y|) \]

is the fundamental solution for the Helmholtz equation, i.e.

\[ (\Delta_y + k^2) \Phi(x, y) = \delta(x). \]

The Hankel function of the first kind:

\[ H_0^{(1)} : \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto \frac{1}{i\pi} \int_0^\infty \frac{e^{(z/2)(t^{-1/2})}}{t} \, dt \]

with \( H_0^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{i(x-\pi/4)} \) for \( x \in \mathbb{R} \) and \( x \gg 1 \).
Representation of $u_{sc}$

Definition:

- $S_{\partial \Omega, \mathbb{R}^n \setminus \overline{\Omega}} [\varphi] (x) := \int_{\partial \Omega} \Phi(x, y) \varphi(y) dS(y)$
  - Single layer potential operator

- $K_{\partial \Omega, \mathbb{R}^n \setminus \overline{\Omega}} [\psi] (x) := \int_{\partial \Omega} \frac{\partial \Phi(x,y)}{\partial n(y)} \psi(y) dS(y)$
  - Double layer potential operator

Representation:

\[ u_{sc} = S_{\partial \Omega, \mathbb{R}^n \setminus \overline{\Omega}} [\varphi] + K_{\partial \Omega, \mathbb{R}^n \setminus \overline{\Omega}} [\psi] \]

with some densities $\varphi$ and $\psi$. 
Complexification of the function $|\cdot|$

The function $\rho(x) := |x|$ for $x \in \mathbb{R}^2$ allows an analytic extension to $G \subset \mathbb{C}^2$, where

$$G := \left\{ z = (z_1, z_2) \in \mathbb{C}^2 ; \ z^2 = z_1^2 + z_2^2 \in \mathbb{C}^2 \setminus (-\infty, 0) \right\}$$

This extension is denoted by $\rho$ again and we have:

$$\rho : G \longrightarrow \{ z \in \mathbb{C} ; \ \Re z > 0 \}$$
Neighborhood of $\Gamma$

**Lemma:** The manifold $\Gamma \setminus \bar{\Omega} \subset \mathbb{C}^2$ has a neighborhood $U \subset \mathbb{C}^2$ such that, for all $y \in \partial \Omega$ and $z \in U$ we have $z - y \in G$, i.e.

$$(z_1 - y_1)^2 + (z_2 - y_2)^2 \notin \mathbb{R}^-$$
Extension of the fundamental solution $\Phi$

**Lemma:** The Hankel function $H_0^{(1)}$ is analytic in $\{z \in \mathbb{C} | \Re z > 0\}$.

**Definition:**

$\Phi(z, \zeta) := \frac{i}{4} H_0^{(1)}(k \cdot \rho(z - \zeta)), \quad z - \zeta \in G.$

**Note:**

„Extended $\Phi \rightarrow$ extended $S$ and $K \rightarrow$ extended $u_{sc}“
Extensions

Analytic extensions of the Operators $S$ and $K$:

- $S_{\partial \Omega, U} [\varphi] (z) := \int_{\partial \Omega} \Phi(z, y) \varphi(y) dS(y), \quad z \in U$

- $K_{\partial \Omega, U} [\psi] (z) := \int_{\partial \Omega} \frac{\partial \Phi(z, y)}{\partial n(y)} \psi(y) dS(y), \quad z \in U$

**Definition:** $u : U \longrightarrow \mathbb{R}$, with

$$u(z) := S_{\partial \Omega, U} [\varphi] (z) + K_{\partial \Omega, U} [\psi] (z)$$
Properties of the function $u = u(z)$

1. $z \mapsto u(z)$ is $\mathbb{C}^2$-analytic in $U$.

2. $u(.) \big|_{D\setminus\Omega} = u_{sc} \big|_{D\setminus\Omega}$.

3. $z \mapsto u(z)$ satisfies the complexified Helmholtz equation in $U$,

   $$(\Delta_z + k^2) u(z) = 0,$$

   where $\Delta_z = \partial_{z_1}^2 + \partial_{z_2}^2$.

4. $u \circ F(x) \big|_{x \to \infty} \to 0$ exponentially.
Representation of $\Delta_z u(z)$

Let $u$ be an analytic function defined in a neighborhood of $\Gamma \subset \mathbb{C}^2$. Then, for $z \in \Gamma$,

$$\Delta_z u(z) = (\text{div } H^T H \text{ grad } - m^T H \text{ grad}) [u \circ F] (F^{-1}(z)),$$

where

- $H = (I + i(Da))^{-T} = \begin{pmatrix} 1 + i \frac{\partial a_1}{\partial x_1} & i \frac{\partial a_1}{\partial x_2} \\ i \frac{\partial a_2}{\partial x_1} & 1 + i \frac{\partial a_2}{\partial x_2} \end{pmatrix}^{-T}$

- $m = \begin{pmatrix} \frac{\partial}{\partial x_1} (H)_{1,1} + \frac{\partial}{\partial x_2} (H)_{1,2} \\ \frac{\partial}{\partial x_1} (H)_{2,1} + \frac{\partial}{\partial x_2} (H)_{2,2} \end{pmatrix}$
Corollary

The „Bérenger equation“ \([(\Delta_z + k^2) \cdot u(z)]|_\Gamma = 0\) assumes in \(\mathbb{R}^2\) the form

\[(\text{div } H^T H \text{ grad } - m^T H \text{ grad } + k^2) [u \circ F] = 0.\]

Definition: \(\tilde{\Delta} := \text{div } H^T H \text{ grad } - m^T H \text{ grad}\)

Complexified analogue of the space \(H^1_{rad}(\mathbb{R}^2\setminus\overline{\Omega})\)

\(H^1_{(\delta)}(\mathbb{R}^2\setminus\overline{\Omega}) := \{ u \in H^1(\mathbb{R}^2\setminus\overline{\Omega}) | \)

\(\lim_{h(x)\to\infty} e^{\delta \tau(h(x))} |u(x)| = \lim_{h(x)\to\infty} e^{\delta \tau(h(x))} |\text{grad} u(x)| = 0\)

uniformly in \(\hat{x}\).
The full-space Bérenger problem

We want to find a function $u \in H^1_{(\delta)}(\mathbb{R}^2 \setminus \overline{\Omega})$ such that

1. $(\widetilde{\Delta} + k^2)u = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$
2. $\frac{\partial u}{\partial n} \big|_{\partial \Omega} = g \in H^{-1/2}(\partial \Omega)$

Existence and uniqueness theorem

The full-space Bérenger problem has a unique solution $u_C \in H^1_{k-\epsilon}(\mathbb{R}^2 \setminus \overline{\Omega})$, where $\epsilon > 0$ is arbitrary. Furthermore we have

$$u_C \big|_{D \setminus \Omega} = u_{sc} \big|_{D \setminus \Omega}.$$
Definition: The layer of thickness $\rho > 0$ around $D$ is defined by

$$L(\rho) := \left\{ x \in \mathbb{R}^2 \setminus \overline{D} \mid h(x) < \rho \right\}.$$ 

We define further

$$D(\rho) := D \cup L(\rho).$$
The truncated Bérenger problem

We want to find a function $u_T \in H^1(D(\rho) \setminus \Omega)$ satisfying

1. $(\tilde{\Delta} + k^2)u = 0$ in $D(\rho) \setminus \Omega$
2. $\frac{\partial u}{\partial n} \big|_{\partial \Omega} = g \in H^{-1/2}(\partial \Omega)$
3. $u \big|_{\partial D(\rho)} = 0$
Part 4: Main theorem

For any wavenumber $k > 0$, there exists a positive constant $\rho_0(k)$ such that, for all $\rho \geq \rho_0(k)$, the truncated Bérenger problem (bounded) has a unique solution $u_T = u_T(\rho) \in H^1(D(\rho) \setminus \Omega)$.

Moreover, this solution converges exponentially to the solution $u_{sc}$ of the initial scattering problem (unbounded) near $\Omega$: 

$$\lim_{\rho \to \infty} e^{(k-\epsilon)\tau(\rho)} \|u_{sc} - u_T(\rho)\|_{H^1(D(\rho) \setminus \Omega)} = 0 \text{ for all } \epsilon > 0.$$
It could be so easy...

By linearity of the operator $(\tilde{\Delta} + k^2)$, we have for $\eta := u_C - u_T$

1. $(\tilde{\Delta} + k^2)\eta = 0$ in $D(\rho) \setminus \Omega$

2. $\frac{\partial \eta}{\partial n} \big|_{\partial \Omega} = 0$

3. $\eta \big|_{\partial D(\rho)} = u_C$

If we could show that

$$\|\eta\|_{H^1(D \setminus \Omega)} \leq C \|u_C\|_{H^{1/2}(\partial D(\rho))}, \quad C \text{ indep. of } \rho,$$

the main theorem was proved, since

$$\|u_C\|_{H^{1/2}(\partial D(\rho))} \to 0 \text{ as } \rho \to \infty.$$
... but it isn’t.

Part 5: Outline of the proof of the Main theorem

Three steps:

1. Full-space Bérenger BVP $\iff$ BVP (A) near $\Omega$
2. Truncated Bérenger BVP $\iff$ BVP (B) near $\Omega$
3. BVP $(B) \rightarrow$ BVP (A) if the layer thickness $\rho \rightarrow \infty$

Since we know that the full-space Bérenger BVP $\iff$ scattering BVP near $\Omega$, the Main Theorem is then proved.
The idea behind step 1

Let $0 < \rho_1 < \rho_2$ and $D_j := D(\rho_j)$, $j = 1, 2$. Find $u$ with $(\Delta + k^2)u = 0$ in $D_2 \setminus \Omega$ and

$$\frac{\partial u}{\partial n} \big|_{\partial \Omega} = g, \quad u \big|_{\partial D_2} = P(u \big|_{\partial D_1}),$$

with the double surface operator $P = K_{\partial D_1, \partial D_2} \left( \frac{1}{2} + K_{\partial D_1} \right)^{-1}$,

$$K_{\partial D_1} \varphi(x) = \text{p.v.} \int_{\partial D_1} \frac{\partial \Phi}{\partial n(y)}(x, y) dS(y), \quad x \in \partial D_1.$$
Characterization of $P$

If for a function $u$ we have

1. $(\Delta + k^2)u = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D_1}$

2. $u|_{\partial D_1} = w,$

$\implies \quad Pw = u|_{\partial D_2}.$
The Theorem behind step 1

Assume that $\rho_1$ and $\rho_2$ are so chosen that $k^2$ is not the Dirichlet-eigenvalue of $-\Delta$ in $D_2\setminus D_1$.

The BVP $(\Delta + k^2)u = 0$ in $D_2\setminus \Omega$ with

$$\frac{\partial u}{\partial n}|_{\partial \Omega} = g, \quad u|_{\partial D_2} = P(u|_{\partial D_1}),$$

has a unique solution $u$, and $u_{sc} \equiv u$ in $D_2$.

The task of step 1

- Find $P_C$ analogous to $P$ for the full-space Bérenger problem.
- Prove the „Theorem behind step 1“ with $P$ replaced by $P_C$. 
Definition of the BVP (A)

Let the BVP (A) be defined by

1. \((\tilde{\Delta} + k^2)u = 0\) in \(D_2 \setminus \Omega\)
2. \(\frac{\partial u}{\partial n} \bigg|_{\partial \Omega} = g \in H^{-1/2}(\partial \Omega)\)
3. \(u \big|_{\partial D_2} = P_C(u \big|_{\partial D_1})\),

where \(P_C := \tilde{K}_{(A),\partial D_1,\partial D_2} \left( 1/2 + \tilde{K}_{(A),\partial D_1} \right)^{-1}\),

\(\tilde{K}_{(A),\partial D_1,\partial D_2} [\psi] (x) := \int_{\partial D_1} \frac{\partial \Phi_{(A)}(x,y)}{\partial n(y)} \psi(y) dS(y)\),

\((\tilde{\Delta} + A + k^2)\tilde{\Phi}_{(A)}(x,y) = -\delta(x-y)\) and

\(A = A(\epsilon) : L^2(D_1) \to L^2(D_1), \quad \|A\| < \epsilon\),

\[\lim_{h(x) \to \infty} \sup_{y \in K \subset \mathbb{R}^2} e^{(k-\epsilon)\tau(h(x))}|D_x^\alpha \tilde{\Phi}_{(A)}(x,y)| = 0, \quad |\alpha| \leq 2.\]
**The Theorem of step 1**

The BVP (A) has a unique solution $u$ in $H^1(D_2 \setminus \Omega)$, and $u = u_C$ in $D_2 \setminus \Omega$.

**Lemma**

The BVP

1. $(\tilde{\Delta} + k^2)u = 0$ in $\mathbb{R}^2 \setminus D_1$

2. $u \big|_{\partial D_1} = f \in H^{1/2}(\partial D_1)$

has a unique solution $u \in H_{(1-\epsilon)}^1(\mathbb{R}^2 \setminus D_1)$ and it can be represented as $u = \tilde{K}_{(A), \partial D_1, \mathbb{R}^2 \setminus D_1} [\varphi]$, where $\varphi$ is the unique solution of

$$(\frac{1}{2} + \tilde{K}_{(A), \partial D_1}) [\varphi] = f.$$
The Theorem of step 2

Let $\rho > \rho_2$. There exists an operator $P_\rho \colon H^{1/2}(\partial D_1) \longrightarrow H^{1/2}(\partial D_2)$ such that the truncated Bérenger problem is equivalent to the near-field BVP (B):

1. $(\tilde{\Delta} + k^2)u = 0$ in $D(\rho) \setminus \overline{\Omega}$
2. $\frac{\partial u}{\partial n} |_{\partial \Omega} = g \in H^{1/2}(\partial \Omega)$
3. $u |_{\partial D_2} = P_\rho (u |_{\partial D_1})$.

Moreover, we have

$$\lim_{\rho \to \infty} e^{(k-\epsilon)\tau(\rho)} \| P_\rho - P_C \| = 0 \text{ for all } \epsilon > 0.$$
Lemma

The BVP (C)

1. \((\tilde{\Delta} + k^2)u = 0\) in \(D(\rho)\setminus \overline{D}_1\)
2. \(u\big|_{\partial D_1} = f \in H^{1/2}(\partial D_1)\)
3. \(u\big|_{\partial D_\rho} = 0\)

has a unique solution \(u \in H^1(D(\rho)\setminus \overline{D}_1)\).
Step 3 - The connection between (A) and (B)

Assume that \( \tilde{P} : H^{1/2}(\partial D_1) \to H^{1/2}(\partial D_2) \) is an operator with the property

\[
\| \tilde{P} - P_C \| < \epsilon.
\]

Consider the BVP (A) with \( P_C \) replaced by \( \tilde{P} \). For \( \epsilon > 0 \) small enough, that modified BVP has a unique solution \( \tilde{u} \in H^1(\partial D_2 \setminus \Omega) \), and we have

\[
\| u_C - \tilde{u} \|_{H^1(D_2 \setminus \Omega)} < C \epsilon
\]

for some positive constant \( C > 0 \).
Lemma

This BVP (D) is an "equivalent weak form of" the BVP (A)

1. \((\tilde{\Delta} + k^2)u = Fu \) in \(D_2 \setminus \Omega\)
2. \(\frac{\partial u}{\partial n} \big|_{\partial \Omega} = g \in H^{-1/2}(\partial \Omega)\)
3. \(u \big|_{\partial D_2} = 0\),

where \(Fu = - (\tilde{\Delta} + k^2)R\rho_C(u \big|_{\partial D_1})\) and

\[R : H^{1/2}(\partial D_2) \rightarrow H^1(D_2 \setminus \Omega),\]

\[R(u \big|_{\partial D_2}) = u\]

a right inverse of the trace mapping \(u \mapsto u \big|_{\partial D_2}\).
Thank you for your attention!