Project	:	Numerical simulation of the Einstein-Dirac equations in a spherically symmetric spacetime
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Abstract

In computational relativity, critical behaviour near the black hole threshold and many other phenomena have been studied numerically for several models in the last decade. We develop a spatial Galerkin method, suitable for finding numerical solutions of the EINSTEIN-DIRAC equations in spherically symmetric spacetime (in polar/areal coordinates). The method features exact conservation of the total electric charge and allows for a spatial mesh adaption based on physical arclength. Numerical experiments confirm excellent robustness and convergence properties of our approach. Hence, this new algorithm is well suited for studying critical behaviour, propagating DIRAC waves, stability of static solutions and several other phenomena. Thus we provide a ready to use and tested implementation in C++, which is platform independent and highly optimized for speed on workstations as well as on clusters.

1 The model

We consider a spherically symmetric spacetime in polar/areal coordinates (t, r, θ, ϕ) with metric

$$\mathbf{g} = A(t,r)\,\mathbf{d}t \otimes \mathbf{d}t - B(t,r)\,\mathbf{d}r \otimes \mathbf{d}r - r^2 \mathbf{g}_{S^2}(\theta,\phi) \tag{1}$$

satisfying EINSTEIN's equations driven by the stress-energy of a massive 2-fermion system in singlet state with particle mass $m > 0$. This 2-particle system is modeled by two spinorial wave functions, for which we can make an ansatz similiar to the one used in the theory of the helium atom. In PAULI-DIRAC representation this reads

$$\psi_p = \frac{1}{2\sqrt{\pi}\,r B^{\frac{1}{4}}(t,r)} \begin{bmatrix} \alpha(t,r) \begin{bmatrix} \delta_{p1} \\ \delta_{p2} \end{bmatrix} \\ i\beta(t,r)\,\sigma^r \begin{bmatrix} \delta_{p1} \\ \delta_{p2} \end{bmatrix} \end{bmatrix} \tag{2}$$

for $p \in \{1, 2\}$. By considering the real vector

$$\mathbf{u} := \big(\mathrm{Re}(\alpha), \mathrm{Im}(\alpha), \mathrm{Re}(\beta), \mathrm{Im}(\beta)\big)^T \tag{3}$$

one can give a very useful formulation of the EINSTEIN-DIRAC equations in spherically symmetric spacetime. There is a discrete matrix subgroup

$$\big\{\pm\mathbb{1}, \pm M_{\mathrm{B}}, \pm M_{\mathrm{C}}, \pm M_{\mathrm{D}}, \pm M_{\mathrm{G}}, \pm M_{\mathrm{H}}, \pm M_{\mathrm{I}}, \pm M_{\mathrm{J}}\big\} < \mathrm{SO}(4) \tag{4}$$

such that $\mathrm{D}\textsc{irac}$'s equation reads

$$\dot{\mathbf{u}} = \sqrt{\frac{A}{B}}\, M_{\mathrm{B}}\mathbf{u}' + \frac{1}{2}\left[\sqrt{\frac{A}{B}}\right]' M_{\mathrm{B}}\mathbf{u} + m\,\sqrt{A}\, M_{\mathrm{C}}\mathbf{u} + \frac{\sqrt{A}}{r}\, M_{\mathrm{D}}\mathbf{u} \tag{5}$$

and $\mathrm{H}\textsc{amiltonian}$ constraint and slicing condition then take the form

$$\begin{aligned} \frac{A'}{A} &= \frac{1}{r}\left[\, B-1\,\right] + \frac{4}{r}\,\langle \mathbf{u}', M_{\mathrm{G}}\mathbf{u}\rangle \\ \\ \frac{B'}{B} &= \frac{1}{r}\left[\, 1-B\,\right] + \frac{4}{r}\,\langle \mathbf{u}', M_{\mathrm{G}}\mathbf{u}\rangle \\ \\ &\quad + \frac{4m}{r}\,\sqrt{B}\,\langle \mathbf{u}, M_{\mathrm{H}}\mathbf{u}\rangle + \frac{4}{r^2}\,\sqrt{B}\,\langle \mathbf{u}, M_{\mathrm{I}}\mathbf{u}\rangle. \end{aligned} \tag{6}$$

If each of the particles has electric charge $q \in \mathbb{R}$, then the total electric charge of the system will be given by

$$Q = 2q\int_0^\infty \left[\,|\alpha|^2 + |\beta|^2\,\right]\,dr = 2q\,\|\,\mathbf{u}\,\|_0^{\,2} \overset{!}{=} 2q. \tag{7}$$

2 Conservative discretization

We seek for weak solutions $\mathbf{u} \in V := H_0^1([\,0,\infty), \mathbb{R}^4)$ of $\mathrm{D}\textsc{irac}$'s equation (5) in variational formulation that reads

$$(\dot{\mathbf{u}}, \mathbf{v}) = F\big[\mathbf{u}, \mathbf{v}\big] \quad \text{for all } \mathbf{v} \in V, \tag{8}$$

where $F : V \times V \to \mathbb{R}$ is the skewsymmetric bilinear form

$$\begin{aligned} F\big[\mathbf{u}, \mathbf{v}\big] := & \left(\sqrt{\frac{A}{B}}\, M_{\mathrm{B}}\mathbf{u}', \mathbf{v}\right) + \frac{1}{2}\left(\left[\sqrt{\frac{A}{B}}\right]' M_{\mathrm{B}}\mathbf{u}, \mathbf{v}\right) \\ & + m\left(\sqrt{A}\, M_{\mathrm{C}}\mathbf{u}, \mathbf{v}\right) + \left(\frac{\sqrt{A}}{r}\, M_{\mathrm{D}}\mathbf{u}, \mathbf{v}\right) \end{aligned} \tag{9}$$

and $(.,.)$ the L^2–product on $[\,0,\infty)$. Let $V_h < V$ be some finite element subspace. We seek for semidiscrete solutions $\mathbf{u}_h \in V_h$, such that

$$(\dot{\mathbf{u}}_h, \mathbf{v}_h) = F\big[\mathbf{u}_h, \mathbf{v}_h\big] \quad \text{for all} \quad \mathbf{v}_h \in V_h. \tag{10}$$

Skewsymmetry of F imediately provides conservation of the weak and also the semidiscrete electric charge, i.e.

$$\frac{d}{dt}\,\|\,\mathbf{u}\,\|_0 = \frac{d}{dt}\,\|\,\mathbf{u}_h\,\|_0 = 0. \tag{11}$$

For discretization in time we use the implicit midpoint rule

$$\mathbf{u}_h^{k+1} - \mathbf{u}_h^k = \frac{\tau_{k+1}}{2}\, F_h\left[\,\mathbf{u}_h^{k+1} + \mathbf{u}_h^k\,\right], \tag{12}$$

that preserves quadratic first integrals and formally is second order accurate w.r.t. timestep size τ_{k+1}. Thus we get the fully discrete conservation law

$$\|\,\mathbf{u}_h^{k+1}\,\|_0 - \|\,\mathbf{u}_h^k\,\|_0 = 0. \tag{13}$$

3 Error analysis

We performed detailed error analysis for the discretization described above. In particular we extended the GEVECI method derived in [7] for the wave equation to DIRAC's equation in spherically symmetric spacetime. In our case we used classical LAGRANGIAN finite element spaces $V_{(p,h)}$ containing compactly supported piecewise polynomials of degree $p \in \mathbb{N}$ on a spatial meshgrid with maximal cell-width $h > 0$. This yields the L^2–estimate

$$\left\| \mathbf{u} - \mathbf{u}_{(p,h)}(t) \right\|_0 \leq \left\| \mathbf{u}(0) - \mathbf{u}_{(p,h)}(0) \right\|_0 \tag{14}$$

$$+ C[A,B](m,p,k)\, h^k \left[\|\mathbf{u}(0)\|_{k+1} + \int_0^t \|\dot{\mathbf{u}}(s)\|_{k+1}\, ds \right]$$

for all $1 \leq k \leq p+1$, where $\mathbf{u}$ is the exact solution to (8).

4 Computer implementation

We implemented a simulation software with the following features:

- All code is based on standard C++ only. It doesn't need any non-standard package, and can be compiled on all platforms recently available, including unix based clusters.
- If supported by hardware and operating system, the binary makes use of 64–bit addressing and multi threading. Some complex algorithms have been parallelized.
- The binary has a commandline user interface or can be controlled by shell scripts alternatively. Further it's highly customizable in the sense, that all relevant parameters can be edited.
- Data can be saved in binary format and exported as tab delimited text files, which can directly be used in visualization tools as Gnuplot or Matlab.
- The core authority of our code is the computation of the time evolutions of initial data due to the discretized EINSTEIN-DIRAC equations. Some standard initial data can be specified by their defining parameters. Arbitrary initial data, approximated by piecewise linear functions, can be imported in a tab delimited text file.
- Time evolution takes place on a fixed finte spatial interval $r \in [0, R)$ where reflecting or approximately absorbing boundary conditions can be applied at $r = R$.
- Another cornerstone is automatic mesh grid adaption, which is based on a manually choosen cell densitiy function $\rho(r)$, such that

  $$\text{Number of cells in } [a,b] \geq \int_a^b \rho(r)\, \sqrt{B(t,r)}\, dr. \tag{15}$$

  That means the number of cells per unit physical arclenght in radial direction is bounded from below, a condition imposed for good reason from the principle of equivalence.
- The static solutions of the spherically symmetric EINSTEIN-DIRAC system found by FINSTER from ODE methods in [4] can also be computed and represented in the finite element discretization with our software.

- Our algorithm passed convergence tests due to RICHARDSON's extrapolation that yield a mean convergence rate of approximately 2 for $h \to 0$.

5 Current results and research aims

Results and publications

- The current version of our software is fully functional and runs stable.
- Convergence tests due to Richardson's extrapolation method show satisfying convergence rates for several initial conditions.
- Using our software, we were able to find the black hole tresholds in one parameter families of initial data for low and moderate particle masses. We can confirm some results published in [12] and extend them to a complete survey of critical collapse in the particle mass range $0 < m \leq 0.6$.
- With help of our software, we reproduced static solutions of the spherically symmetric EINSTEIN-DIRAC system and studied its properties in detail. So we can confirm and extend results of [4]. In particular, we found that initial conditions close to unstable static solutions, up to some exceptions, decay to form a black hole.
- We published detailed information about our algorithms in [15].

Ongoing steps

- Writing further publications.
- Simulation of other phenomena of Dirac's equations, e.g. simulation of the zitterbeweung.

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