Lecture IX 14-11-2008

Review: Interpolation estimates for Whitney l-forms

$$\| u - I^l u \|_{L^2(\Omega)} \le C h_{\mathcal{T}_h} \| u \|_{H^1(\Omega)}$$

$C = C(\rho_{\mathcal{T}_h})$ squash tetrahedra

Summary: $\mathcal{T}_h \triangleq$ tetrahedral mesh of $\Omega \subset \mathbb{R}^3$.

with $C = C(\rho_{\mathcal{T}_h}) > 0$, we have

$l = 0$. $\| u - I^0 u \|_{L^2(\Omega)} \le C h^2_{\mathcal{T}_h} | u |^2_{H^2(\Omega)}$ $\quad W^0(T) = \underset{\text{full set of linear poly}}{P_1(T)}$

$l = 1$. $\| \underline{u} - I^1 \underline{u} \|_{L^2(\Omega)} \le C h_{\mathcal{T}_h} (| \underline{u} |_{H^1(\Omega)} + | \operatorname{curl} \underline{u} |_{H^1(\Omega)})$

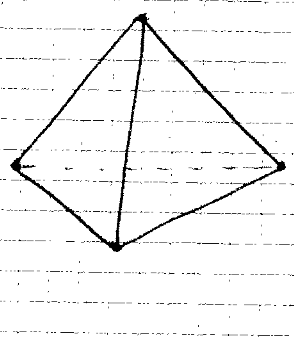
$l = 2$. $\| \underline{u} - I^2 \underline{u} \|_{L^2(\Omega)} \le C h_{\mathcal{T}_h} | \underline{u} |_{H^1(\Omega)}$

$l = 3$. $\| u - I^3 u \|_{L^2(\Omega)} \le C h_{\mathcal{T}_h} | u |_{H^1(\Omega)}$

Remark: bridge the gap $\begin{cases} \text{enriched local spaces (enrichment)} \\ \text{localized global bases} \end{cases}$

2nd family of edge elements!

Attempt: try gradients of globally continuous $\mathcal{T}_h$-piecewise quadratic functions ($\longrightarrow$ 0-forms)



local 0-form basis functions

4 associated vertices. linear (standard local basis function for $W^0(T)$)

6 associated edges. Let $\{b_e\}_{e \in \mathcal{E}(T)}$

For $e = [a_i, a_j]$ $\quad b_e = 4\lambda_i \lambda_j$

$\rightarrow$ Additional global basis function

$$b_e = b_x b_y \quad \text{if } e \in \mathcal{S}^1(\mathcal{T}_h) \text{ with endpoints } x, y.$$

▷ Augment $W^1(\mathcal{T}_h)$ by span $\{\text{grad}\, b_e\}_{e \in \mathcal{F}^1(\mathcal{T})}$ "1-facet set"

$$\widetilde{W}^1(\mathcal{T}_h) = W^1(\mathcal{T}_h) + \text{span}\{\text{grad}\, b_e\}$$

provide locally supported basis function

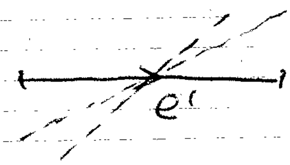
Check: $\widetilde{W}^1(\mathcal{T}_h) \overset{?}{=} (P_1(T))^3$

grad be is invariant for dof.

Proof: Step 1: $+$ is a direct sum.

$$\int_{e'} \text{grad}\, b_e \cdot d\vec{s} = 0. \quad \forall e \in \mathcal{F}^1(T) \Big\} \ \text{grad}\, b_e \notin W^1(T)$$

Step 2. $\int_{e'} \chi_{e'}\, \text{grad}\, b_e \, d\vec{s} = \begin{cases} \neq 0 & \text{if } e = e' \\ = 0 & \text{if } e \neq e'. \end{cases}$

$\chi_{e'} =$  $\int_{e'} \chi_{e'} ds = 0$ $\chi_{e'}$ linear in e'.

grad be $\cdot\, \tau$ (tangent component of grad be)

$\Rightarrow \{\text{grad}\, b_e\}_{\mathcal{F}^1(T)}$ linearly independent

▷ basis $\widetilde{W}^1(T) = \{ b_e^1, \ \text{grad}\, b_e \}_{e \in \mathcal{F}^1(T)}$

basis 1-form

$$\dim \widetilde{W}^1(T) = 12$$
$$= 3 \dim(P_1(T)) = \dim\left((P_1(T))^3\right)$$

▷ $\| \underline{u} - \widetilde{I}^1 \underline{u} \|_{L^2(\Omega)} \leq C\, h_{\mathcal{T}_h}^2 \, |u|_{H^2(\Omega)}$

Interpolation onto $\widetilde{W}^1(\mathcal{T}_h)$

Recall that: magnetostatics : $\underline{a}$ – based

(A drawback) $\rightarrow$ quantity of interest : curl $\underline{a}$

Yet: $\operatorname{curl} \tilde{W}^1(T_k) = \operatorname{curl} W^1(T)$

Thus we won't get better approximation for curl.

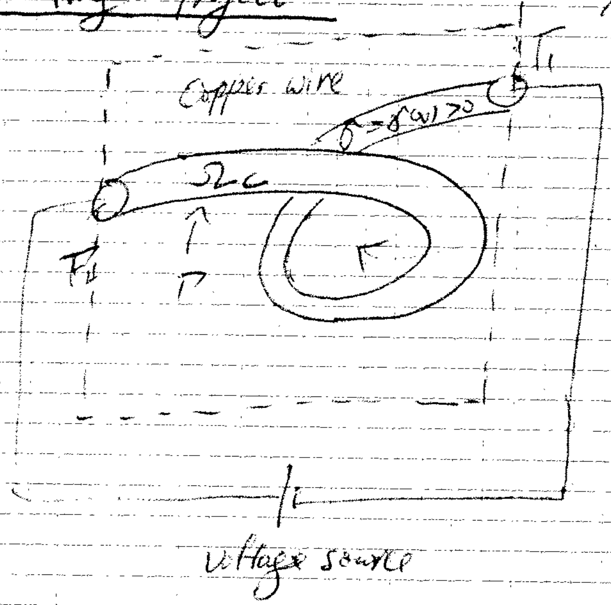
$$\| \operatorname{curl}(\underline{u} - \hat{I}^1 \underline{u}) \|_{L^2(\Omega)} \le C h_{T_k} | \operatorname{curl} \underline{u} |_{H^1(\Omega)}$$

Ex1: define $\hat{I}^1$ in detail.

commute with grad (midpoint. vertex) point-based interpolation.

Ex2: $\tilde{W}^2(T) = W^2(T) + \{ \operatorname{curl} b_{f=[i,j,k]} \} \Rightarrow$

<u>Modelling Project:</u>



Task:

<u>Compute magnetic fields in Ω</u>

compute (Modeling Step)

Step I: compute the stationary current.

$$\operatorname{div}(\sigma \operatorname{grad} v) = 0 \quad \text{in } \Omega_c$$

$$\sigma \operatorname{grad} v \cdot n = 0 \quad \text{on } \Gamma$$

$$v = u \quad \text{on } \Gamma_1$$

$$v = 0 \quad \text{on } \Gamma_2$$

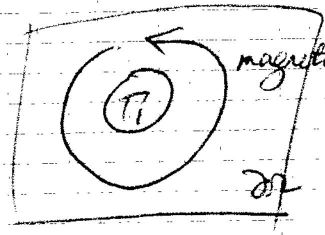
Step II: Compute magnetic field by magnetostatic modeling

$$\operatorname{curl}(\mu^{-1} \operatorname{curl} \underline{a}) = j := \sigma \operatorname{grad} v \quad (\text{on } \mathbb{R}^3 ??)$$

<u>but j has to div-free</u> everywhere

put contact on the surface of an artificial box.

$\Rightarrow$ Considered on 'cut-off domain' Ω s.t. Γ_1, Γ_2 on $\partial\Omega$



magnetic field $\Longrightarrow$ zero magnetic flux condition

$$\underline{a} \times \underline{n} = 0$$

Assmp: wire stretch out in an perpendicular way

Assmp 2: Decay of $\underline{a}$ away from Ω_c $\Longrightarrow$ $\underline{a} \times \underline{n} = 0$ on

Remark: If $W^q(\mathcal{T}_h)$ is used used for stationary current problem.
(V-based formulation)

$$\longrightarrow \underbrace{\underline{\sigma}\ \mathrm{grad}\, v}_{\underline{j}_h} \notin H(\mathrm{div}, \Omega)$$

($\underline{j}_h$) (remedy $\Rightarrow$ mixed discretization)

this $\underline{j}_h$ has no exact magnetic vector potential.

$\rightarrow$ We need formulation for magnetostatics robust w.r.t inconsistent currents $\Longrightarrow$ $\underline{a}$-based formulation.

$$\begin{cases} \int \mu^{-1}\, \mathrm{curl}\, \underline{a}\ \mathrm{curl}\, \underline{a}'\, dx + \int \underline{a}' \cdot z\, dx = \int \underline{j} \cdot \underline{a}'\, dx & \forall \underline{a} \in H_0(\mathrm{curl}, \Omega) \\ \int_\Omega \underline{a} \cdot z'\, dx = 0 & \forall z \in H_0(\mathrm{curl}\, 0, \Omega) \end{cases}$$

☆ : $\left(\begin{array}{l} v_h \longrightarrow \underline{a}_h \\ \underline{j}_h \text{ (mixed)} \longrightarrow \underline{h}_h \end{array}\right)$ primal view / dual view

By mean value theorem:

$$\int_0^1 D^2 E_{el}\left(-\mathrm{grad}\, v_h - \tau(-\mathrm{grad}(u - v_h))\right)\left(-\mathrm{grad}(v - v_h), \mathrm{grad}\, v_h'\right) d\tau = 0$$

$$\gamma \|\mathrm{grad}(v - v_h)\|_{L^2}^2 \le \int_0^1 D^2 E_{el}(\cdots)(\mathrm{grad}(v - v_h), \underset{\underset{w_h \text{ (by G.O.)}}{\uparrow}}{\mathrm{grad}(v - v_h)}) d\tau$$

$$\le C \|\mathrm{grad}(u - u_h)\|_{L^2} \|\mathrm{grad}(u - w_h)\|_{L^2}$$

From continuity $D^2 E_{el}(\underline{e})(\underline{u}, \underline{w}) \le C \|\underline{u}\|_{L^2} \|\underline{w}\|_{L^2}$.

$\Rightarrow$ Quasi-optimality.

· $\underline{d}$-based V.F.

Abstract framework. Space V: $C: V \times V \to \mathbb{R}$ conti. bilinear form.

$V_h \subset V \leftarrow$ F.E. space (no ellipticity of $C(\cdot,\cdot)$)

$\exists C > 0$ $\forall f \in V'$. $\exists u_h \in V_h$: $C(u_h, v_h') = f(v_h')$ and

$$\|u_h\|_V \le C \|f\|_{V_h'}$$

Choose $f(\cdot) = C(u_h, \cdot)$ $\Rightarrow \|u_h\|_V \le C \inf_{v_h \in V_h} \dfrac{\|C(u_h, v_h)\|}{\|v_h\|_V}$ (discrete stability)

if $u \in V$ solves $C(u, v) = f(v)$ $\forall v \in V$

$$\|u - u_h\|_V \le \|u - v_h\|_V + \|v_h - u_h\|_V$$

$$\le \quad '' \quad + \sup_{w \in V_h} \frac{C(u_h - v_h, w_h)}{\|w_h\|_V}$$

$$\le \quad '' \quad + \sup_{w \in V_h} \frac{|C(u - v_h, w_h)|}{\|w_h\|_V}$$

$$\le (1 + C) \|u - v_h\|_V.$$

$\triangleright$ $\|u - u_h\|_V \le (1 + C) \inf_{v_h \in V_h} \|u - v_h\|_V$.

3.7. FE error estimates for electrostatics

• For local, linear material law. v - based, PEC.

$$v \in H_0^1(\Omega): \quad \underbrace{\int_\Omega \underline{\underline{\varepsilon}} \operatorname{grad} v \cdot \operatorname{grad} v' \, dx}_{c(v, v')} = \int_\Omega \rho v' \, dx \quad \forall v' \in H_0^1(\Omega)$$

▷ $c(v, v) \geq \gamma(\Omega, \underline{\underline{\varepsilon}}) \|v\|^2_{H^1(\Omega)} \quad \forall v \in H_0^1(\Omega)$

$$v_h \in W^0(\mathring{\mathcal{T}}_h): \int_\Omega \underline{\underline{\varepsilon}} \operatorname{grad} v_h \cdot \operatorname{grad} v_h' \, dx = \int \rho v_h' \quad \forall v_h' \in W^0(\mathring{\mathcal{T}}_h)$$

vanishing boundary values (trace)

G. O. $c(u - u_h, v_h) = 0 \ \forall v_h$

$$\Rightarrow \gamma \|u - u_h\|^2_{H^1(\Omega)} \leq c(u - u_h, u - u_h) = c(u - u_h, u - w_h)$$

$$\leq C(\varepsilon) \|u - u_h\|_{H^1(\Omega)} \|u - w_h\|_{H^1(\Omega)}$$

▷ Quasi-optimality

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{C(\varepsilon)}{\gamma(\Omega, \varepsilon)} \inf_{w_h \in W^0(\mathring{\mathcal{T}}_h)} \|u - w_h\|_{H^1(\Omega)}$$

$$\leq \frac{C}{\gamma} \|u - I^0 u\|_{H^1(\Omega)}$$

• For nonlocal, nonlinear material law

$$\langle DE_{el}(-\operatorname{grad} v), \operatorname{grad} v' \rangle = \int \rho v' \, dx \quad v' \in H_0^1(\Omega)$$

Assume $\exists \gamma > 0$ s.t. $D^2 E_{el}(\underline{e})(\underline{u}, \underline{u}) \geq \gamma \|\underline{u}\|^2_{L^2(\Omega)} \quad \forall \underline{u} \in \mathcal{F}^1(\Omega)$

at point $\underline{x}$ $\qquad \forall (\underline{e})$

Galerkin $\langle DE_{el}(-\operatorname{grad} v_h), \operatorname{grad} v_h' \rangle = \int \rho v_h'$

G. O. $\langle DE_{el}(-\operatorname{grad} v) - DE_{el}(-\operatorname{grad} v_h), \operatorname{grad} v_h' \rangle = 0$

Seek $\underline{d}_h \in W^2(\mathcal{T}_h)$, $v_h \in C^3(\mathcal{T}_h)$

$$\int_\Omega \varepsilon^{-1} \underline{d}_h \cdot \underline{d}_h' \, dx + \int_\Omega \operatorname{div} \underline{d}_h' \cdot v_h \, dx = 0 \quad \forall \underline{d}_h' \in W^2(\mathcal{T}_h)$$

$$\int_\Omega \operatorname{div} \underline{d}_h \cdot v_h' \, dx = \int_\Omega \rho \, v_h' \, dx \quad \forall v_h' \in W^3(\mathcal{T}_h)$$

$\Rightarrow$ $V = H(\operatorname{div}, \Omega) \times L^2(\Omega)$

$V_h = W^2(\mathcal{T}_h) \times W^3(\mathcal{T}_h)$

$$c\left(\begin{pmatrix} d \\ v \end{pmatrix}, \begin{pmatrix} d' \\ v' \end{pmatrix} \right) = \int_\Omega \varepsilon^{-1} \underline{d} \cdot \underline{d}' + \int_\Omega \operatorname{div} \underline{d}' \, v + \int \operatorname{div} \underline{d} \cdot v'$$

$\Rightarrow$ Continuity $C(\underline{\underline{\varepsilon}})$

Establish (discrete) stability by the same policy as in Sec. 3.2.2

for the continuous problem ?)

Key tool: Lift mapping : $W^3(\mathcal{T}_h) \to W^2(\mathcal{T}_h)$
L_h

$\forall w_h \in W^3(\mathcal{T}_h)$

s.t. ① $\operatorname{div} \circ L_h = Id$ ② $\|L_h w_h\|^2_{L^2(\Omega)} \le C(\Omega, \mathcal{T}_h) \|w_h\|^2_{L^2}$

(Idea: use a lifting $L : L^2(\Omega) \to \underline{H}^1(\Omega)$ continuous $\operatorname{div} \circ L = Id$

Ex: construct such Lift.

$\triangleright$ define $L_h = I^2 \circ L$

① Identity: $\underbrace{\operatorname{div} \circ L_h w_h}_{=w_h} = \operatorname{div} I^2 \circ L w_h = I^3 (\operatorname{div} L) w_h = I^3 w_h$

② Continuity: $\|L_h w_h\| = \|I^2 \circ L w_h\|_2 \le C(\mathcal{T}_h) \|L w_h\|_{H^1(\Omega)} \le C(\Omega, \mathcal{T}_h) \|w_h\|_{L^2(\Omega)}$

3.8. FE error estimates for magnetostatics

- $\underline{a}$-based V.F. (zero flux B.C.)

$\underline{a}_h \in W^1(\mathring{\mathcal{T}}_h)$, $z_h \in W^1(\mathrm{curl}\,0, \mathring{\mathcal{T}}_h)$

$$\begin{cases} \int_\Omega \underline{\underline{\mu}}^{-1} \,\mathrm{curl}\,\underline{a}\; \mathrm{curl}\,\underline{a}'\,dx + \int_\Omega \underline{a}_h' \cdot z_h\, dx = 0 & \forall \underline{a}_h' \in W^1(\mathring{\mathcal{T}}_h) \\ \int_\Omega \underline{a}_h \cdot z_h'\, dx = 0 & \forall z_h' \in W^1(\mathrm{curl}\,0, \mathring{\mathcal{T}}_h) \end{cases}$$

(discrete stability $\Rightarrow$ Quasi-orthogonality)

$\longrightarrow$ same arguments as in Sec. 3.7

$\triangleright$ To show (uniform) discrete stability. $\longrightarrow$ see Sect. 3.4.1.

Key estimates: $\|\underline{u}\|_{L^2(\Omega)} \le C \|\mathrm{curl}\,\underline{u}\|_{L^2(\Omega)} \quad \forall \underline{u} \in \ker(\mathrm{curl})^\perp$

(Poincaré–Friedrichs type inequality) $\|\underline{u}_h\|_{L^2(\Omega)} \le C \|\mathrm{curl}\,\underline{u}_h\|_{L^2(\Omega)} \rightarrow u_h \perp W^1(\mathrm{curl}\,0, \mathcal{T}_h)$ (3.8.6)

$\boxed{\text{Lemma 3.8.A.}}$ (3.8.6) holds true

Proof: Tool: Lifting $L : \mathrm{curl}\, H(\mathrm{curl}, \Omega) \longrightarrow H_0^1(\Omega)$

$$\|u_h\|^2_{L^2(\Omega)} = \Big(u_h, \underbrace{u_h - I^1 L\,\mathrm{curl}\, u_h}_{\text{curl-free}} + I^1 L\,\mathrm{curl}\, u_h\Big)_{L^2}$$

$$\mathrm{curl}(u_h - I^1 L\,\mathrm{curl}\,u_h) = \mathrm{curl}\,u_h - I^2 (\underbrace{\mathrm{curl}\,L\,\mathrm{curl}}_{Id})u = 0.$$

$\mathrm{curl}\, u.$

$$= (u_h, I^1 L\,\mathrm{curl}\,u_h)$$

$$\le \|u_h\|_{L^2(\Omega)} \cdot \|I^1 L\,\mathrm{curl}\,u_h\|_{L^2(\Omega)}$$ (? But I^1 not well defined for $H^1(\Omega)$)

But $\mathrm{curl}\,u_h$ is piecew. const.

$\boxed{\text{Lemma 3.8.B}}$ if $\underline{u} \in \underline{H}^1(T)$ $\mathrm{curl}\,\underline{u} = \text{const.}$ (T: tetrahedron)

$$\Rightarrow \|\underline{u} - I^1 \underline{u}\|_{L^2(T)} \le C(\rho_T)\, h_T\, |\underline{u}|_{H^1(T)}$$

Proof: Tool: Poincaré map for 2-form $O \in T$ (O is one vertex) T origin

$$(k\underline{\omega})(\underline{x}) = \int_0^1 t\, \underline{\omega}(t\underline{x}) \times \underline{x}\, dt$$

$$= \frac{1}{2} \boxed{\underline{\omega} \times \underline{x}} \quad (\text{for const. } \underline{\omega})$$

$\Rightarrow$ $\operatorname{curl} k\underline{\omega} = \underline{\omega}$ ($\operatorname{div} \underline{\omega} = 0$ $\underline{\omega}$ - div-free)

$\Rightarrow$ $\underline{\|k\underline{\omega}\|_{L^2(T)} \le C h_T |\underline{\omega}|_{L^2(T)}}$ (3.8.c)

Note: $\operatorname{curl}(\underline{u} - k\operatorname{curl}\underline{u}) = 0 \Rightarrow \underline{u} - k\operatorname{curl}\underline{u} = \operatorname{grad}\phi$

$$\underline{u} - I^1\underline{u} = \operatorname{grad}\phi + k\operatorname{curl}\underline{u} - I^1(\operatorname{grad}\phi + k\operatorname{curl}\underline{u})$$

Note: ϕ $\operatorname{curl}\underline{u} = \text{const.} \Rightarrow k\operatorname{curl}\underline{u} \in W^1(T)$ $\quad \underline{C \times x}$

$$\Rightarrow I^1 k\operatorname{curl}\underline{u} = k\operatorname{curl}\underline{u}$$

$$= \operatorname{grad}\phi - I^1\operatorname{grad}\phi$$

$$= \operatorname{grad}(\phi - I^0\phi) \quad \underline{(\text{by CD property})}$$

$$|\phi|_{H^2(T)} = |\operatorname{grad}\phi|_{H^1(T)} \le |\underline{u}|_{H^1(T)} + |\underbrace{k\operatorname{curl}\underline{u}}_{\text{linear}}|_{H^1}$$

Inverse estimate

$$\le |u|_{H^1(T)} + C(\phi_T)\, |k\operatorname{curl}\underline{u}|_{L^2} \cdot h^{-1}$$

Inv

$$\le |u|_{H^1(T)} + C(\phi_T)\, h_T^{-1} \cdot \overset{\text{by 3.8.c}}{h_T |\operatorname{curl}\underline{u}|_{L^2(T)}}$$

$$\le C(\phi_T)\, |u|_{H^1(T)}$$

$$\Rightarrow \|u - I^1 u\|_{L^2(T)} \le \|\operatorname{grad}(\phi - I^0\phi)\|_{L^2} \le C(\phi_T) h_T |\phi|_{H^2(T)}$$

$$\le C(\phi_T) \cdot h_T \|u\|_{H^1(T)}$$

Proof 3.8.A (cont'd) $\quad k\operatorname{curl} u_R$ is pw. const. $\in \underline{H}^1(\Omega)$

$$\|u_R\|_{L^2} \le \|I^1 k\operatorname{curl} u_R\|_{L^2} \overset{\text{By 3.8.B}}{\underset{(h_T \le 1)}{\le}} C(\phi_T)\, \|k\operatorname{curl} u_R\|_{H^1} \le C(\phi_T, \Omega) \cdot \|\operatorname{curl} u_R\|_{L^2}$$

- $\underline{h}$-based V.F. is easier and similar.

3.9. Regularized formulation

3.9.1. Discrete regularization

Start from $\underline{a}$-based V.F. for magnetostatics.

$$W^1(\mathrm{curl}\,0, \mathring{\mathcal{T}}_h) = \mathrm{grad}\,\underbrace{\{ v_h \in W^0(\mathcal{T}_h) \quad v_h|_{\Gamma_i} = \mathrm{const} \}}_{W^0_\Gamma(\mathcal{T}_h)}$$

$\Gamma_i \hat{=}$ connected components of $\partial\Omega$

$\underline{a} \in W^1(\mathring{\mathcal{T}}_h)$. $\varphi_h \in W^0_\Gamma(\mathcal{T}_h)$

$$\int_\Omega \underline{\underline{\mu}}^{-1} \mathrm{curl}\,\underline{a}\, \mathrm{curl}\,\underline{a}'\, dx + \int_\Omega \underline{a}_h' \,\mathrm{grad}\, \varphi_h\, dx = \int \underline{j}\, \underline{a}_h' \, dx \quad \forall \underline{a}_h' \in W^1(\mathring{\mathcal{T}}_h)$$

$$\int_\Omega \underline{a}_h \cdot \mathrm{grad}\,\varphi_h'\, dx \overset{\sigma}{=} - \int_\Omega \varphi_h \varphi_h' = 0 \quad \forall \varphi_h' \in W^0_\Gamma(\mathcal{T}_h)$$

If $\underline{j} \in \mathrm{curl}\, H(\mathrm{curl}, \Omega)$ (r.h.s is consistent)

$$\rightarrow \int_\Omega \underline{j}\, \mathrm{grad}\,\varphi_h'\, dx = 0 \quad \forall \varphi_h' \in W^0_\Gamma(\mathcal{T}_h)$$

Take $\underline{a}_h' = \mathrm{grad}\,\varphi_h' \Rightarrow \int_\Omega \mathrm{grad}\,\varphi_h' \cdot \mathrm{grad}\,\varphi_h\, dx = 0 \quad \forall \varphi_h' \in W^0_\Gamma(\mathcal{T}_h)$

$\Rightarrow \varphi_h = 0$ ("dummy unknown")

Matrix formulation:

$$\underline{\underline{C}}^T \underline{\underline{M}}_{\mu^{-1}} \underline{\underline{C}}^T \vec{a}_h + \underline{\underline{M}}_1 \underline{\underline{G}}\, \vec{\varphi}_h = \vec{j}$$

$$\underline{\underline{G}}^T \underline{\underline{M}}_1 \vec{a}_h - \underline{\underline{M}}_0 \vec{\varphi}_h = 0$$

"$\underline{\underline{C}}$: curl-matrix = Incidence matrix "edge-face"

$\underline{\underline{M}}_{\mu^{-1}}$: matrix matrix

$\underline{\underline{M}}_1$: mass matrix for edge element $\underline{\underline{M}}_0$: mass matrix for nodal element

G : "gradient-matrix" = incidence matrix "vertex-face"

Eliminate $\vec{p}_h$.

$$\underbrace{\left(\underline{\underline{C}}^T \underline{\underline{M}}_{\mu^{-1}} \underline{\underline{C}}^T + \underline{\underline{M}}_1 \underline{\underline{G}}\, \underline{\underline{M}}_0^{-1} \underline{\underline{G}}^T \underline{\underline{M}}_1\right)}_{SPD} \vec{a}_h = \vec{j}$$

If $\vec{j}$ is not consistent $\Rightarrow$ "dangerous solution"

Remark Replace $\int_\Omega \ell_h \cdot \ell_h' \, dx = \sum_{p \in \mathcal{T}^0(\mathcal{T}_h)} \ell_h(p) \cdot \ell_h'(p)$

$\Rightarrow \underline{\underline{M}}_0$ will be diagonal " otherwise "$\underline{\underline{M}}_0^{-1}$ will be dense".

Ex: Find the sparsity pattern of "matrix"

which pair of entry (i,j) gives nonzero value

3.9.2 Continuous ~~discretization~~ regularization.
(grad-div) - regularization.

$$\begin{cases} \operatorname{curl}(\underline{\underline{\mu}}^{-1} \operatorname{curl} \underline{a}) = j & \text{in } \Omega \\ a \times n = 0 & \text{on } \partial\Omega \end{cases}$$

$$\operatorname{div} \underline{a} = 0 \quad \text{in } \Omega$$

(Coulomb gauge, Ω : trivial topology)

$$\operatorname{curl}(\underline{\underline{\mu}}^{-1} \operatorname{curl} \underline{a}) - \underbrace{\operatorname{grad}(\operatorname{div} \underline{a})}_{\triangle} = j \quad \text{in } \Omega$$

add grad(div) term. drop "gauge" term

Seek $\underline{a} \in \underline{H_0(\mathrm{curl},\Omega) \cap H(\mathrm{div},\Omega)} = X_T$

$$\int_\Omega \underline{\underline{\mu}}^{-1} \operatorname{curl}\underline{a} \cdot \operatorname{curl}\underline{a}'\,dx + \int_\Omega \operatorname{div}\underline{a}\,\operatorname{div}\underline{a}'\,dx \qquad (3.9.f)$$

$$+ \cancel{\int_\Omega \underline{a}' \underbrace{\underline{z}}_{0}\,dx} = \int_\Omega j \cdot \underline{a}'\,dx \qquad \forall \underline{a}' \in H_0(\mathrm{curl},\Omega) \cap H(\mathrm{div},\Omega)$$

$$\cancel{\int_\Omega \underline{a}\,\underline{z}' = 0}$$

- on trivial Ω. (3.9.f) has a solution.
- $\underline{a}$ solves $\underline{a}$-base V.P. $\Rightarrow$ $\underline{a}$ solves (3.9.f)
- a solves (3.9.f) $\Longrightarrow$ $\int \operatorname{div}\underline{a} \cdot \Delta\varphi\,dx = 0$

  Let $a' = \operatorname{grad}\varphi$

  $\Delta\varphi \in L^2(\Omega)$ $\Rightarrow$ $\operatorname{div}\underline{a} = 0$

  $\varphi \in H_0^1(\Omega)$

  $\Rightarrow a$ solves $\underline{a}$-base. V.P.

FE - discretization. $V_h \subset X_T \iff$ require ~~tangential~~ normal continuity.

$\Uparrow$

p.w. polynomials.

Simplest choice:

$$N_h = \left(W^0(\mathcal{T}_h)\right)^3 \cap H_0(\mathrm{curl},\Omega)$$ of component wise continuity.

Num. Test. (2D magnetostatics)

$$\begin{cases} \underset{\sim}{\operatorname{curl}}_{2d} \operatorname{curl}_{2d} \underset{\sim}{a} = j & \text{in } \Omega \\ a_t = 0 & \text{on } \partial\Omega \end{cases}$$

$$\operatorname{div}\underline{a} = 0 \qquad \text{gauge}$$

$$\underset{\sim}{\operatorname{curl}}_{2d}\varphi = \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}\varphi \qquad \operatorname{curl}_{2d}\underline{a} = \left(\frac{\partial a_1}{\partial x_2} - \frac{\partial a_2}{\partial x_1} \right)$$

Discrete reg. 2D edge element.

Conti. reg. Nh

$\Omega = \{$