Review (up to now)

Static EM., V.F., analysis, Function spaces, well-posedness (existence, uniqueness)

Focus (of next step): Stability of variational problems

seek $u \in V$: $c(u, v) = f(v) \quad \forall v \in V$

Lipschitz continuity dependence of solution on the given data.

- Linear case: $\exists C > 0$ s.t. $\|u\|_V \leq C \|f\|_{V'}$

Supplement to 3.3 on Lifting

Lifting: $L.\ \mathcal{F}^l(\Omega) \to \mathcal{F}^{l-1}(\Omega)$ is right inverse of d on $d\mathcal{F}^{l-1}(\Omega)$

$\hookrightarrow \quad d \circ L = Id \quad$ on $d\mathcal{F}^{l-1}(\Omega)$

Concern continuity w.r.t. "~~[illegible]~~ natural norms".

- Poincaré mapping (on star-shaped domain Ω)

$k_a: \quad D\mathcal{F}^l(\Omega) \to D\mathcal{F}^{l-1}(\Omega) \qquad$ (cf. 1.5.9)

See [6] Continuity $\|k_0 \omega\|_{L^2(\Omega)} \leq C(\Omega) \|\omega\|_{L^2(\Omega)} \quad \forall \omega \in L^2(\Lambda^l(\Omega))$

if ω is closed

But! $\|k_0 \omega\|_{H^1(\Omega)} \leq C\|\omega\|_{L^2(\Omega)}$ (Wrong!)

Component-wise H^1-norm for vector proxy

- Fourier Lifting: $(d=3 \quad l=2)$

i) $L : \mathrm{curl}(H(\mathrm{curl},\Omega)) \subset L^2(\Omega) \longrightarrow H(\mathrm{curl},\Omega)$

① Extension: $\quad \underline{\omega} = \mathrm{curl}\,\underline{u} \in H(\mathrm{div}\,0,\Omega)$

$$\downarrow \text{Neumann } \Delta\text{-extension}$$

② $\quad \underline{\hat{\omega}} \in H(\mathrm{div}\,0, \mathbb{R}^3)$

$$\downarrow \text{Fourier transformation}$$

$$\hat{\underline{\Psi}}(\underline{\xi}) = \frac{1}{|\underline{\xi}|^2}\,\hat{\underline{\omega}}(\underline{\xi}) \times \underline{\xi}$$

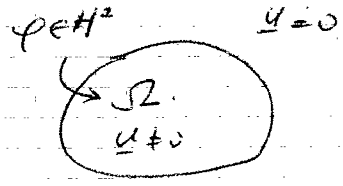
$\mathrm{curl}\,\underline{\Psi} = \hat{\omega}$

$\|\underline{\Psi}\|_{H^1(\mathbb{R}^3)} \leq \|\hat{\omega}\|_{L^2(\mathbb{R}^3)}$

ii) $L : \mathrm{curl}(H_0(\mathrm{curl},\Omega)) \longrightarrow \underset{\sim}{H}_0^1(\Omega)$

① zero-extension $\quad \underline{\omega} = \mathrm{curl}\,\underline{u} \in H_0(\mathrm{div}\,0,\Omega)$

$$\downarrow \text{zero-extension}$$



$$\hat{\underline{\omega}} \in H(\mathrm{div},\Omega)$$

$$\downarrow \text{Fourier}$$

$\underline{\Psi} \in \underset{\sim}{H}^1(\mathbb{R}^3) \qquad \mathrm{curl}\,\underline{\Psi} = \hat{\omega}$

$\mathrm{curl}(\underline{\Psi} - \underset{\uparrow}{\hat{\underline{u}}}) = 0$ on $\mathbb{R}^3 \Rightarrow \underline{\Psi} - \hat{\underline{u}} = \mathrm{grad}\,\varphi \quad \varphi \in H^1(\mathbb{R}^3)$

zero-extension of u.

$\Rightarrow \underline{\Psi} = \mathrm{grad}\,\varphi$ on $\mathbb{R}^3 | \Omega$.

Notation: $H^2(\Omega) = \{ v \in L^2(\Omega) : \mathrm{grad}\,v \in \underline{H}^1(\Omega) \}$

$\Rightarrow \varphi|_{\mathbb{R}^3 \setminus \Omega} \in H^2(\mathbb{R}^3 | \Omega)$

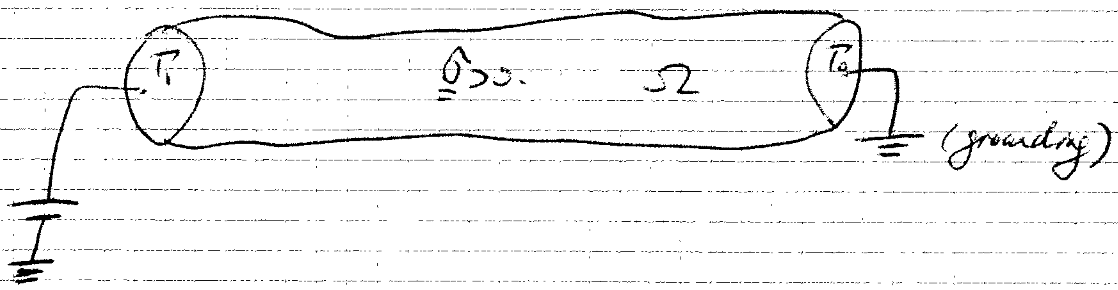
$\ast$ H^2-extension into Ω $\quad$ (? cf.

$\tilde{\varphi} \in H^2(\mathbb{R}^3)$

$L\,\underset{\sim}{w} = \underline{\Psi} - \mathrm{grad}\,\tilde{\varphi}$

3.5 Stationary currents

$$\left.\begin{cases} \underline{e} = -\operatorname{grad} v & \to \text{F.L.} \\ \underline{j} = \underline{\underline{\sigma}}\, \underline{e} & \to \text{Ohm's Law} \\ \text{by } \operatorname{div} \underline{j} = 0 & \to \text{A.L} \end{cases}\right\} \quad -\operatorname{div}(\underline{\underline{\sigma}} \operatorname{grad} v) = 0$$



Voltage excitation through contacts

$$\begin{cases} \underline{j} \cdot \underline{n} = 0 & \text{on } \partial\Omega \setminus (\Gamma_0 \cup \Gamma_1) \\ v = 0 & \text{on } \Gamma_0 \\ v = V & \text{on } \Gamma_1 \end{cases}$$

(Assume contact surfaces are equipotential

Ex: V-based V.F.

Seek $v \in H^1_{\Gamma_0}(\Omega) = \{ v \in H^1(\Omega) \quad v|_{\Gamma_0} = 0 \} \ \& \ v|_{\Gamma_1} = V \}$

$$\int_\Omega \underline{\underline{\sigma}} \operatorname{grad} v \cdot \operatorname{grad} v' \, dx = 0 \qquad \forall v' \in H^1_{\Gamma_0}(\Omega) \cap H^1_{\Gamma_1}(\Omega)$$

Current excitation

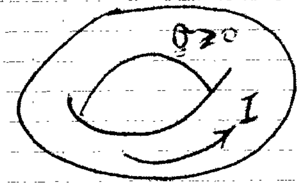
$$\begin{cases} \underline{j} \cdot \underline{n} = 0 & \text{on } \partial\Omega \setminus (\Gamma_0 \cup \Gamma_1) \\ v = 0 & \text{on } \Gamma_0 \\ v = \text{const} & \text{on } \Gamma_1 \quad (\text{equi-potential}) \end{cases}$$

$$\int_{\Gamma_1} \underline{j} \cdot \underline{n} \, dS = I \int_{\Gamma_1} dS \text{ on } \Gamma_1 \qquad (\text{current}$$

Seek $v \in \{ H^1_{\Gamma_0}(\Omega) \ \& \ v|_{\Gamma_1} = \text{const} \} = V$

$$\int_\Omega \underline{\underline{\sigma}} \operatorname{grad} v \cdot \operatorname{grad} v' \, dx = I \cdot v'|_{\Gamma_1} \qquad \forall v' \in V.$$

Ex:



Find V_h? (by push a total current into the current,)
(to find the distribution of current)

3.6 Interpolation estimates for Whitney forms

Answer the question "How well can Whitney forms ~~can~~ approximate a field?"

Ref. Interpolation operator (Ch. 2)

$$I^l := W^l \circ S_l : \mathcal{F}^l \to W^l(\mathcal{T}_h)$$

Whitney map   Sampling map $\longrightarrow$ (For $d=3$ Not continuous for $l \le 2$)

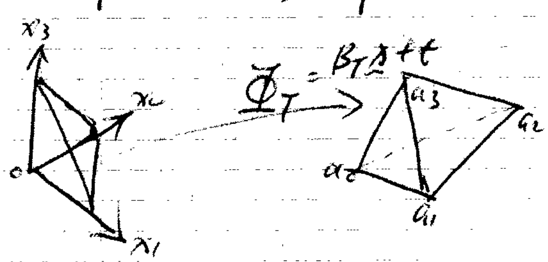
$\mathcal{T}_h$: tetrahedral mesh

Intuitive example: Interpolation error estimate for I^3. $d=3$
(only for $l=3$)

$$I_T^3 u = I^3 u\Big|_T = \frac{1}{|T|}\int_T u(x)\,dx \longrightarrow \text{Local operator.}$$

vector proxy for 3-form   basis 3-form.

Transformation technique: $\hat{T} \xrightarrow{\Phi_T} T$



$B = (\underline{a}_1 - \underline{a}_0,\ \underline{a}_2 - \underline{a}_0,\ \underline{a}_3 - \underline{a}_0)$
$t = \underline{a}_0$

Note $\Phi_T^* \circ I_T^3 = I_{\hat{T}}^3 \circ \Phi_T^*$

$$\| u - I_T^3 u\|_{L^2(T)}^2 = \cancel{\int_{\hat{T}} \hat{u} - \cdots} \ \frac{1}{|\det B_T|}\| \Phi_T^*(u - I_T^3 u)\|_{L^2(T)}^2 \quad (*)$$

$$\Phi_T^* u = |\det D\Phi_T| \cdot u(\Phi(\hat{x})) = 6|T| \cdot \hat{u}(\hat{x})$$

$=$

$$\|\Phi_T^* v\|^2_{L^2(\hat{T})} = \int_{\hat{T}} |\det B_T|\ v(\Phi(\hat{x}))^2\, d\hat{x}$$

$$= \int_T |\det B_T|^2\ v(x)^2\, (\det B_T)^{-1}\, dx$$

$$= |\det(B_T)| \cdot \|v\|_{L^2(\hat{T})}$$

~~$(*) = |\det B_T|^{-1} \|$~~

$$(*) = \|\underbrace{\Phi_T^* u}_{\hat{u}} - I_T^3 \Phi_T^* u\|^2_{L^2(T)} = \|\underbrace{(Id - T_{\hat{T}}^3)(\hat{u} - c)}_{\text{vanishing}}\|_{L^2(\hat{T})} \quad \forall c \in \mathbb{R}$$

$$\le C \|\hat{u} - c\|_{L^2(\hat{T})}$$

continuity of $T_{\hat{T}}^3$ $\quad c = \int_{\hat{T}} \hat{u}\, d\hat{x} / |\hat{T}|$ (vanishing average)

$$\|T_{\hat{T}}^3 \hat{u}\|_{L^2(\hat{T})} \le \underbrace{C(\hat{T})} \|\hat{u}\|_{L^2(\hat{T})}$$

Compute best (minimal) const)

Poincaré (3.2.4)

$$\le C(\hat{T})\, |\hat{u}|_{H^1(\hat{T})} \qquad (\text{Assume } \hat{u} \in H^1(\hat{T}))$$

$$|\hat{u}|_{H^1(\hat{T})} = \cancel{|\det B_T|} \cdot \int_T (B_T^T \cdot \operatorname{grad} u)^2 \cdot |\det B_T|^{-1}\, dx$$

$$\le |B_T|^2 \cdot |u|_{H^1(T)}\, |\det B_T|^{-1}$$

$$\|u - I_T u\|^2_{L^2(T)} \le C(\hat{T})\, |B_T| \cdot |u|_{H^1(T)}$$

Note: For other interpolation estimates, we need higher order Sobolev norms

$$\|v\|^2_{H^m(\Omega)} = \|v\|_{L^2(\Omega)} + \|\operatorname{grad} v\|_{H^{m-1}(\Omega)}$$

$$|v|_{H^m(\Omega)} = |\operatorname{grad} v|_{H^{m-1}(\Omega)}$$

Purpose: measure the smoothness of functions

Tool: transformation of norms under affine ~~mapping~~ pullback

$$\longrightarrow \Phi_T(\hat{x}) = B_T \hat{x} + t$$

- pull-back of 0-form $\quad \hat{u} = (\Phi_T^* u)(\hat{x}) = u(\Phi_T(\hat{x}))$

 — — 1- — $\quad \hat{\underset{\sim}{u}} = (\Phi^* \underset{\sim}{u})(\hat{x})$

Ex:
$$|\hat{u}|^2_{H^m(\hat{T})} \le C(m) \underbrace{|\det B_T|^{-1} |B_T|^{2m}}_{h_T^{2m-3}} |u|^2_{H^m(T)} \qquad m \ge 0$$

prove by induction

$$|u|^2_{H^m(T)} \le C(m) \underset{\underset{|\det(B_T)|}{\parallel}}{|\det B_T^{-1}|^{-1}} |B_T^{-1}|^{2m} \cdot |u|^2_{H^m(\hat{T})} \qquad m \ge 0$$

- 1-form

$$\hat{\underset{\sim}{u}} = \Phi_T^* \underset{\sim}{u}(\hat{x}) = D\Phi_T^T \underset{\sim}{u}(\Phi(\hat{x})) = B_T^T \underset{\sim}{u}(\Phi(\hat{x}))$$

$$|\hat{u}|^2_{H^m(\hat{T})} \le C(m) \underbrace{|\det B_T|^{-1} |B_T|^{2m+2}}_{h_T^{2m-1}} |u|_{H^m(T)}$$

- 2-form

$$\hat{\underset{\sim}{u}} = \Phi_T^* \underset{\sim}{u}(\hat{x}) = |\det B_T| \cdot B_T^{-1} \cdot \underset{\sim}{u}(\Phi(\hat{x}))$$

$$|\hat{u}|^2_{H^m(\hat{T})} \le C(m) \underbrace{|\det B_T| \, |B_T|^{2m} |B_T^{-1}|^2}_{h_T^{2m+1}} |u|^2_{H^m(T)}$$

- 3-form

$$\hat{\underset{\sim}{u}} = \Phi_T^* u(\hat{x}) = |\det B_T|^{-1} \cdot u(\Phi(\hat{x}))$$

$$|\hat{u}|^2_{H^m(\hat{T})} \le C(m) \underbrace{|\det B_T| \cdot |B_T|^{2m}}_{h_T^{2m+3}} |u|^2_{H^m(T)}$$

Def. 3.6.A. T: tetrahedron $h_T = \cancel{\text{diam}}(T)$ ~~$r_T = \max r(B_r \subset T)$~~

$r_T = \max\{ r > 0,\ \exists x \in T \quad B_r(x) \subset T \}$.

$\rho_T = h_T / r_T$: shape-regularity measure.

Lemma 3.6.B $\Phi_T(\hat{x}) = B\hat{x} + t$. affine mapping maps reference $\hat{T}$ to T

Then $|\det B_T| \le C h_T^3$

$|\det B_T^{-1}| \le C \rho_T^3 \cdot h_T^{-3}$

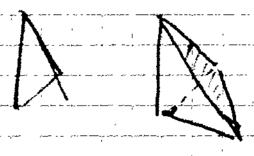
$|B_T| \le C h_T \qquad |B_T^{-1}| \le C \rho_T h_T^{-1}$

Interpolation estimate for 2-forms

$$\| \underline{u} - I_T^2 \underline{u} \|^2_{L^2(T)} \le C\, h_T^{(\ldots)} \rho_T^2 \, \| \underset{\uparrow}{\hat{\underline{u}}} - I_{\hat{T}}^2 \hat{\underline{u}} \|^2_{L^2(\hat{T})}$$

(transform to ref. element) $\quad \hat{\underline{u}} = \Phi^* \underline{u}$

Continuity:



$$\int_{\hat{F}} v \, \cancel{\ldots}\, dS \le C \int_{\hat{F}} v \, \underline{x} \cdot n \, dS = C \int_{\partial \hat{T}} v \, \underline{x} \cdot \underline{n} \, dS$$

$$\underline{\underline{x} \cdot \underline{n} \ge \tfrac{1}{C}}$$

$$= C \int_{\hat{T}} \operatorname{div}(v \underline{x}) \, d\underline{x}$$

$$\le \int_{\hat{T}} (\underline{x} \cdot \operatorname{grad} v + 3v) \, d\underline{x}$$

$$\le C \left(\| \operatorname{grad} \underset{\tilde{}}{v} \|_{L^2(\hat{T})} + \| \underset{\tilde{}}{v} \|_{L^2(\hat{T})} \right)$$

$$\Rightarrow \int_{\hat{T}} \hat{\underline{u}} \cdot \underline{n} \, dS \le \sum_{i=1}^{3} \int_{\hat{F}} \hat{\underline{u}}_i \, dS \le C(\hat{T}) \, \| \underline{u} \|_{H^1(\hat{T})}$$

$$\| I^2_{\hat T} \hat{\underline u} \|_{L^2(T)} = \sum_{F:\text{ face of }\hat T} \left| \int_F \hat{\underline u}\cdot n \, dS \right| \cdot \underbrace{\|\hat b_F\|_{L^2(\hat T)}}_{\text{const.}}$$

$$\le C(\hat T) \| \hat{\underline u} \|_{H^1(\hat T)}$$

▷ $\| \hat{\underline u} - I^2_{\hat T} \hat{\underline u} \|_{L^2(\hat T)} = \| (Id - I^2_{\hat T})(\hat{\underline u} - \underline c) \|_{L^2(\hat T)}$

with $\underline c = \frac{1}{|\hat T|} \int_{\hat T} \hat{\underline u} \, d\hat{\underline x}$

$$\overset{\text{contin.}}{\le} (1 + C(\hat T)) \| \hat{\underline u} - \underline c \|_{L^2(\hat T)}$$

$$\overset{\text{Poincaré}}{\le} \hat C(\hat T) \, | \hat{\underline u} |_{H^1(\hat T)}$$

Assume $\hat u \in H^1(\hat T)$

$$\overset{\text{transform back}}{\le} \hat C(\hat T) \ell_T^{u} h_T^{3/2} \, | \underline u |_{H^1(T)}$$

▷ $\| \underline u - I^2_T \underline u \|_{L^2(T)} \le C \tilde\ell_T h_T \, | u |_{H^1(T)}$

Interpolation estimates for 0-forms

$$\| u - I^0_T u \|_{L^2(T)} \le C \ell_T^{k} h_T^{+3/2} \| \hat u - I^0_{\hat T} \hat u \|_{L^2(\hat T)}$$

$$= C \ell_T^{k} h_T^{+3/2} \| (\hat u - I^0_{\hat T})(\hat u - \hat q) \|_{L^2(\hat T)} \quad \forall \hat q \in P_1(\hat T)$$

choose $\hat q$ s.t. $\hat u - \hat q$ has vanishing average

$\operatorname{grad}(\hat u - \hat q)$ has componentwise vanishing average

Use result that point evaluation is continuous on $H^2(\hat T)$.

(Find elementary proof to show $I^0_{\hat T}$-continuity)

$$\le C(\hat T) \, \ell_T \, h_T^{+\frac{3}{2}} \| \hat u - \hat q \|_{L^2(\hat T)}$$

$$\overset{\text{Poincaré twice}}{\le} C(\hat T) \cdot \ell_T \cdot h_T^{+\frac{3}{2}} \, | \hat u |_{H^2(\hat T)}.$$

$$\le C(\hat T) \, \ell_T h_T^{\frac{3}{2}} h_T^{\frac{1}{2}} \cdot | u |_{H^2(T)} \le C(\hat T) \cdot \ell_T \, h_T^2 \, | u |_{H^2(T)}$$

Interpolation error estimates for 1-forms:

Key: continuity of edge path integrals:

Tool: Smooth Poincaré mapping

$$k_a(\underline{u})(\underline{x}) = \int_0^1 t\, \underline{u}(a+t(x-a)) \times (x-a)\, dt.$$

for domain star-shaped w.r.t. $\underline{a}$

$$\operatorname{div} \underline{u} = 0 \Rightarrow \boxed{\operatorname{curl} k_a(\underline{u}) = \underline{u}}$$

This lifting operator is not so good for lack of continuity.

For Ω convex. $\chi \in C_0^\infty(\Omega)$. $\int_\Omega \chi(x)\,dx = 1$.

$$\boxed{k := \int_\Omega k_{\underline{a}}\, \chi(a)\, d\underline{a}}$$ (average over all possible $\underline{a}$)

$$(k\,\underline{u})(\underline{x}) = \underbrace{\int_{R^3}}_{\chi \in C_0^\infty(\Omega)} \int_0^1 t\, \underline{u}(\underline{a}+t(\underline{x}-\underline{a})) \times (x-a)\, dt\, \chi(\underline{a})\, d\underline{a}$$

$$\overset{\substack{y = a+t(x-a) \\ \tau = \frac{1}{1-t}}}{=} \int_{R^3} \int_1^\infty \tau(1-\tau)\, \underline{u}(y) \times (x-y)\, \chi(y+\tau(y-x))\, d\tau\, dy$$

$$= \int_{R^3} \underbrace{K(x, y-x)}_{\text{kernel}} \times \underline{u}(y)\, dy.$$

convolution-type integral operator.

$$K(x, z) = \int_1^\infty \tau(1+\tau)\, \chi(x+\tau z)\, d\tau$$

$$= \frac{z}{|z|^2} \int_1^\infty \xi\, \chi\!\left(x+\xi \frac{z}{|z|}\right) d\xi + \frac{z}{|z|^3} \int_1^\infty \xi^2\, \chi\!\left(x+\xi\frac{z}{|z|}\right) d\xi.$$

pseudo diff operator

Thus $K(x,\xi) \leq \underset{\text{smooth}}{k(x)} \frac{1}{|\xi|}$ ← Continuity depends on singularity of kern

By theory of pseudo differential operators.

$$k : \underset{\sim}{H}^{m}(\mathbb{R}^3) \longrightarrow \underset{\sim}{H}^{m+1}(\mathbb{R}^3) \quad \text{continuous.}$$

$$\hat{\underline{u}} = \underbrace{\hat{\underline{u}} - k \operatorname{curl} \hat{\underline{u}}}_{\substack{\text{curl-free} \\ = \operatorname{grad} \varphi}} + k \operatorname{curl} \hat{\underline{u}} \quad \text{on } \hat{T}$$

$$I^1_{\hat{T}} \hat{\underline{u}} = \operatorname{grad} I^0_{\hat{T}} \varphi + I^1_{\hat{T}} k \operatorname{curl} \hat{\underline{u}}$$

$I^1_{\hat{T}} \circ \operatorname{grad} = \operatorname{grad} \circ I^0_{\hat{T}}$

$$\| I^1_{\hat{T}} \hat{\underline{u}} \|_{L^2(\hat{T})} \leq | I^0_{\hat{T}} \varphi |_{H^1(\hat{T})} + \| I^1_{\hat{T}} k \operatorname{curl} \hat{\underline{u}} \|_{L^2(\hat{T})}$$

? Minimal smoothness of $\hat{\underline{u}}$

Require $\varphi \in H^2(\Omega) \Rightarrow \hat{\underline{u}} \in H^1(\hat{T})$

Require $k \operatorname{curl} \hat{u} \in \underline{H}^2(\Omega) \Rightarrow \operatorname{curl} \hat{\underline{u}} \in H^1(\hat{T})$

$\} \; \hat{\underline{u}} \in H^1(\operatorname{curl}, \hat{T})$

$$\leq C\left(\|\varphi\|_{H^2(\hat{T})} + \|k \operatorname{curl} \hat{\underline{u}}\|_{H^2(\hat{T})} \right)$$

$\hat{u} = \operatorname{grad} \varphi$ conti. of k

$$\leq C(\hat{T}) \left(\|\hat{\underline{u}}\|_{H^1(\hat{T})} + \|\operatorname{curl} \hat{\underline{u}}\|_{H^1(\hat{T})} \right)$$

$$\leq C(\hat{T}) \left(\|\hat{\underline{u}}\|_{H^1(\operatorname{curl}, \hat{T})} \right)$$

$$\triangleright \; \|\underline{u} - I^1_T \underline{u}\|_{L^2(T)} \leq C(f_T)\, h_T^{1/2} \|\hat{\underline{u}} - I^1_{\hat{T}} \hat{\underline{u}}\|_{L^2(\hat{T})}$$

$$\leq C(f_T) \cdot h_T^{1/2} \| (Id - T^1_{\hat{T}})(\hat{\underline{u}} - \underline{c}) \|_{L^2(\hat{T})}$$

conti.

$$\leq C(f_T)\, h_T^{1/2} \| \hat{\underline{u}} - \underline{c} \|_{H^1(\operatorname{curl}, \Omega)}$$

P.F.: $\leq C(f_T)\, h_T^{1/2} \left(|\hat{\underline{u}}|_{H^1(\hat{T})} + \|\mathrm{curl}\,\hat{\underline{u}} - \underline{c}\|_{H^1(\hat{T})} \right)$

$$= \|\underbrace{\Phi_T^*}_{\uparrow}\,\mathrm{curl}\,\underline{u}\|_{H^1(\hat{T})}$$

pull-back of 2-forms

$\mathrm{curl}\,\Phi_T^* u = \Phi^*\,\mathrm{curl}\,u \Downarrow \hat{\underline{u}}$

: full-$H^1(T)$-norm

$$\leq C(f_T)\, h_T^{1/2} \left(C(f_T)\, h_T^{1/2}\, |u|_{H^1(T)} + h_T^{1/2}\, \|\mathrm{curl}\,\underline{u}\|_{H^1(T)} \right)$$

$$\leq C(f_T) \left(h_T\, |\underline{u}|_{H^1(T)} + h_T\, \|\mathrm{curl}\,\underline{u}\|_{H^1(T)} \right)$$

- **Summary**: $\quad f_{\mathcal{T}_h} = \max_{T \in \mathcal{T}_h} f_T \quad h_{\mathcal{T}_h} = \max_{T \in \mathcal{T}_h} h_T$

- $\|u - I^0 u\|_{L^2(\Omega)} \leq C(f_{\mathcal{T}_h})\, h_{\mathcal{T}_h}^2\, |u|_{H^2(\Omega)}$

$$\|\underline{u} - I^1 \underline{u}\|_{L^2(\Omega)} \leq C(f_{\mathcal{T}_h})\, h_{\mathcal{T}_h} \left(|\underline{u}|_{H^1(\Omega)} + h_{\mathcal{T}_h}\, \|\mathrm{curl}\,\underline{u}\|_{H^1} \right)$$

$$\|\underline{u} - I^2 \underline{u}\|_{L^2(\Omega)} \leq C(f_{\mathcal{T}_h})\, h_{\mathcal{T}_h}\, |\underline{u}|_{H^1(\Omega)}$$

$$\|u - I^3 u\|_{L^2(\Omega)} \leq C\, h_{\mathcal{T}_h}\, |u|_{H^1(\Omega)}$$

By commuting diagram.

$$\operatorname{grad} I^0 = I^1 \operatorname{grad}$$

$$|u - I^0 u|_{H^1(\Omega)} = |\operatorname{grad}(u - I^0 u)|_{L^2(\Omega)} = \|(Id - I^1)\operatorname{grad} u\|_{L^2(T)}$$

$$\leq C(f)\, h_{\mathcal{T}_h}^1\, |\underline{u}|_{H^2(\Omega)}$$

$$|\mathrm{curl}(u - I^1 u)|_{L^2(\Omega)} = \|(Id - I^2)\,\mathrm{curl}\,\underline{u}\|_{L^2(T)} \leq C(f)\, h_{\mathcal{T}_h}\, |\mathrm{curl}\,\underline{u}|_{H^1(\Omega)}$$

! No loss of order when considering the interpolation error in $\|\cdot\|_{H(d_\ell,\Omega)}$, $\ell = 1, 2$.