Lecture VII   31-10-2007

3.2.2. (Cont'd) (Review)

$\underline{d}$-based V.F.: Seek $\underline{d} \in H(\mathrm{div}, \Omega)$, $v \in L^2(\Omega)$

$$\int_\Omega \underline{\underline{\varepsilon}}^{-1} \underline{d} \cdot \underline{d}' \, dx + \int_\Omega \mathrm{div}\, \underline{d}' \, v \, dx = 0 \qquad \forall \underline{d}' \in H(\mathrm{div}, \Omega)$$

(PEC $\Rightarrow$ natural B.C in Mixed V.F)

$$\int_\Omega \mathrm{div}\, \underline{d} \cdot v' dx = \int \rho \cdot v' dx \quad \forall v' \in L^2(\Omega)$$

3-step procedure:

i) Find $\underline{d}_\perp$: $\mathrm{div}\, \underline{d}_\perp = \rho$ $\quad \| \underline{d}_\perp \|_{H(\mathrm{div}, \Omega)} \le C \| \rho \|_{L^2(\Omega)}$

ii) Find $\underline{d}_0$ $\quad \mathrm{div}\, \underline{d}_0 = 0$ $\quad \int_\Omega \underline{\underline{\varepsilon}}^{-1} (\underline{d}_0 + \underline{d}_\perp) \cdot \underline{d}_0' = 0$

$$\| \underline{d}_0 \| \le C \| \underline{d}_\perp \|_{L^2(\Omega)}$$

iii) Find v.

$$\int_\Omega \mathrm{div}\, \underline{d}' \cdot v = - \int_\Omega \underline{\underline{\varepsilon}}^{-1} (\underline{d}_0 + \underline{d}_\perp) \, \underline{d}'$$

$$\| v \|_{L^2(\Omega)} \le C \| \underline{d}_0 + \underline{d}_\perp \|_{L^2(\Omega)}$$

$$\Rightarrow \quad \| v \|_{L^2(\Omega)} + \| \underline{d} \|_{H(\mathrm{div}, \Omega)} \le C \| \rho \|_{L^2(\Omega)}$$

Note: $v \in \mathcal{F}^0(\Omega)$ should belong $H^1(\Omega)$ in the table of next page

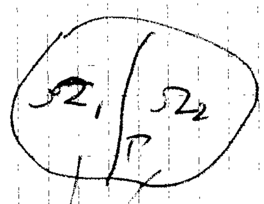
But $\int_\Omega \mathrm{div}\, \underline{d}' \, v = \int_\Omega d\, d' \wedge \star v$

thus v is still a 0-form.

Function spaces of V.F. may not be the ones suggested by the forms.

3.3. More about Function spaces

D.F.	Function spaces	Norm	Transmission conditions *
$\mathcal{F}^0(\Omega)$	$H^1(\Omega)$	$\lVert u\rVert^2_{H^1(\Omega)} = \lVert u\rVert^2_{L^2} + \lVert \operatorname{grad} u\rVert^2_{L^2}$	$[u]_\Gamma = 0$
$\mathcal{F}^1(\Omega)$	$H(\operatorname{curl},\Omega)$	$\lVert \underline{u}\rVert^2_{H(\operatorname{curl},\Omega)} = \lVert \underline{u}\rVert^2_{\underline{L}^2} + \lVert \operatorname{curl} \underline{u}\rVert^2_{\underline{L}^2}$	$[n\times \underline{u}]_\Gamma = 0$
$\mathcal{F}^2(\Omega)$	$H(\operatorname{div},\Omega)$	$\lVert \underline{u}\rVert^2_{H(\operatorname{div},\Omega)} = \lVert \underline{u}\rVert^2_{\underline{L}^2(\Omega)} + \lVert \operatorname{div} \underline{u}\rVert_{L^2}$	$[n\cdot \underline{u}]_\Gamma = 0$
$\mathcal{F}^3(\Omega)$	$L^2(\Omega)$	$\lVert u\rVert^2_{L^2} = \lVert u\rVert^2_{L^2}$	
$\mathcal{F}^\ell(\Omega)$	$H(d_\ell,\Omega)$	$\lVert \omega\rVert^2_{H(d,\Omega)} = \lVert \omega\rVert^2_{L^2} + \lVert d\omega\rVert^2_{L^2}$	$[\gamma\omega]=0$ "continuity of trace"



u smooth up to boundary in Ω_1 and Ω_2

RK ① Sampling operator S_ℓ is not bounded in $H(d\ell, \Omega)$

(Example) $x \to \log|\log|x||$ $x < \frac{1}{2}$ in
$\ell = 0$. $\in H^1(B(0, \frac{1}{2}))$

Ex.1 ~~Formal~~ example $\ell = 1$ $\ell = 2$

but it holds true for $\ell = 3$

② Smooth functions $C^\infty(\bar{\Omega})$ / $C^\infty(\bar{\Omega})^3$ are dense in $H(d\ell, \Omega)$

3.3.1 potentials in function spaces (Poincare)

Thm 3.3.A If the topology of Ω is trivial, then the sequence

$$\{\text{const}\} \xrightarrow{Id} H^1(\Omega) \xrightarrow{grad} H(curl, \Omega) \xrightarrow{curl} H(div, \Omega) \xrightarrow{div} L^2(\Omega) \xrightarrow{Id} \{0\}$$

is exact.

proof: (for star-shaped Ω) w.r.t the origin) Poincaré map:

$$k_\ell(\omega)(v_1, \cdots v_{\ell-1}) = \int_0^1 t^{\ell-1} \omega(tx)(x, v_1, \cdots v_{\ell-1})\, dt$$

$$\boxed{d \circ k + k \circ d = Id}$$

For $\ell = 1$ $k(\underline{u})(x) = \int_0^1 \underline{u}(tx) \cdot x\, dt$

$\underline{u}$ v.p. of 1-form

if $\underline{u} \in (C^\infty(\bar{\Omega}))^3 \Rightarrow \text{grad}\, k(\underline{u}) = \underline{u} \iff \text{curl}\, \underline{u} = 0$

To show $\|k_0(\underline{u})\|_{H^1(\Omega)} \leq C(\Omega) \|\underline{u}\|_{H(curl, \Omega)}$ $\forall \underline{u} \in (C^\infty(\bar{\Omega}))^3$ & $\text{curl}\, \underline{u} = 0$.

obvious $\|\text{grad}\, k(\underline{u})\|_{L^2} \leq \|u\|_{L^2}$

$$\|k(\underline{u})\|_{L^2}^2 \leq \|(\int_0^1 \underline{u}(tx) \cdot x\, dt)^2\|_{L^1(\Omega)} \leq \int_0^1 \int_\Omega |u(tx)|^2 |x|^2\, dt\, dx$$

Ex.2 $\ell = 2$.

Thm 3.3.B. same assumption as in Thm 3.3.A.

$$\{0\} \xrightarrow{Id} H_0^1(\Omega) \xrightarrow{grad} H_0(curl, \Omega) \xrightarrow{curl} H_0(div, \Omega) \xrightarrow{div} L^2_*(\Omega) \xrightarrow{Id} \{0\},$$

const. funct. $\qquad\qquad$ with vanishing mean

Sketch of Proof: need tools: extensions for Lipschitz domains.

Step 1. extension to the whole R^3 $\quad H(dc, \Omega) \quad$ $U_\Omega \Rightarrow W|_{R^n}$ $\; div = 0$

2. Fourier transformation $\quad \hat{v} \quad$ $w \to \hat{w}$

3. Solve Fourier eqn $\Rightarrow f \quad$ $\hat{w}$?

4. Inverse Fourier transformation $\quad v \quad$ $\hat{w} \to w$

5. Show $w|_\Omega \in H(dc, \Omega)$ $\quad$ $w|_\Omega \in H_0(dc, \Omega)$

1. define a map from outside Ω to inside (& leave $\partial\Omega$ invariant)

2. use smooth partition of unity to glue together reflection mappings

$\Rightarrow$ global reflection mapping R:

R: {exterior neighborhood of $\partial\Omega$} $\to$ {interior neighborhood of $\partial\Omega$}

$|DR|$ $|DR^{-1}|$ bounded matrix norm uniformly.

$R|_{\partial\Omega} = Id$

3. Define $\hat{\omega} = R^* \omega$.

$R|_{\partial\Omega} = Id \Rightarrow \hat{\omega}|_{\partial\Omega} = \omega|_{\partial\Omega}$ (equality of traces)

$\Rightarrow$ local extension

$$\omega_{ext} = \begin{cases} \hat{\omega} & \text{in exterior} - - - \\ \omega & \text{in interior neighbor of } \partial\Omega \end{cases}$$

[larger extension by cut-off ? $\to$ partition of unity $\Rightarrow$

curl-free property may lost here

$curl(\varphi\omega) = \nabla\varphi \times \omega + \varphi\, curl\, \omega$ $\quad \|\hat{\omega}\|_{L^2(ext)} \le C(\partial\Omega) \|\omega\|_{L^2(int)}$

Recall $d \circ R^* = R \circ d$

$$\Rightarrow \quad \| d\tilde{\omega} \|_{L^2(ext)} = \| d \circ R^* \omega \|_{L^2(ext)}$$

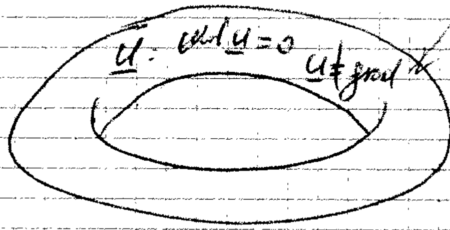
$$\lesssim \| R \circ d\omega \|_{L^2(int)} \lesssim \| d\omega \|_{L^2(int)}$$

This $\Rightarrow$ preserves the kernel of exterior derivatives

Ex: ? Is it possible to construct kernel-preserving extension operator ?

No. for general topology

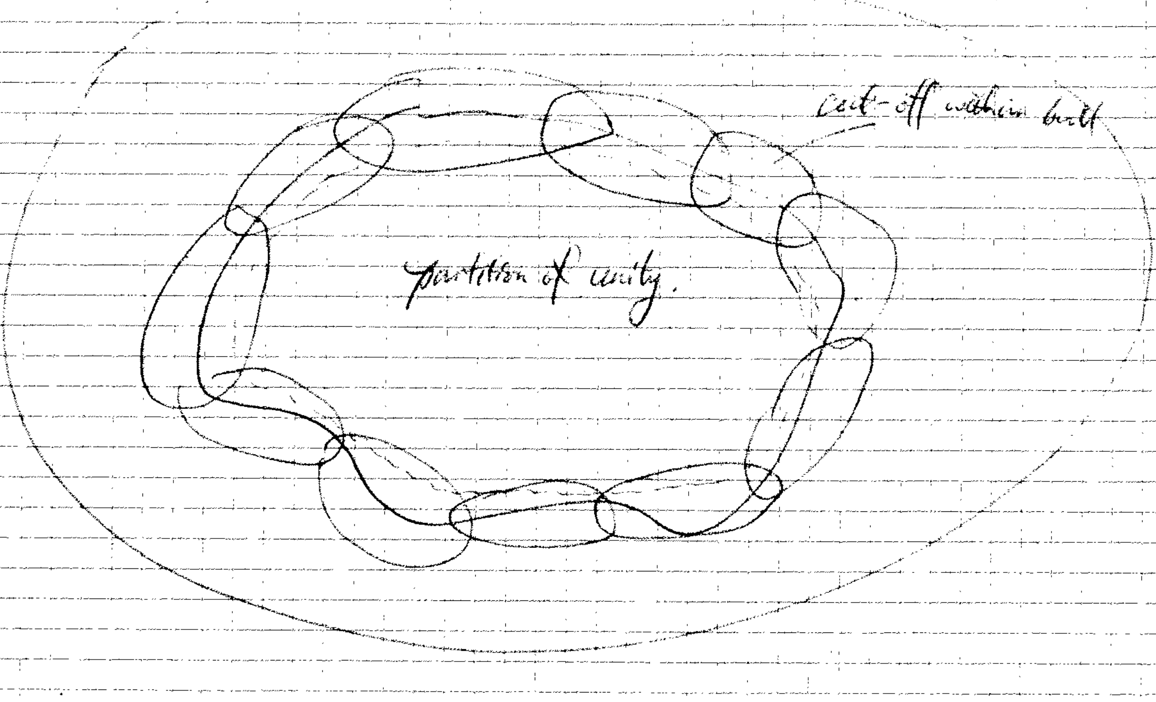
topological invariant should be kept to keep curl-free property



if it is possible to extend curl-free extension $\underline{u}$ to a large ball

Then $\quad E\underline{u} = \text{grad}\,\phi \quad$ in B

$$\Rightarrow \quad E\underline{u}|_{\Omega} = \text{grad}\,\phi|_{\Omega}$$



$\underline{u} \in H_0(\mathrm{curl}, \Omega) \xrightarrow[\text{by zero}]{\text{extend}} \hat{\underline{u}} \in H(\mathrm{curl}, R^3)$, $\mathrm{curl}\,\hat{\underline{u}} = 0$

$\mathrm{curl}\,\underline{u} = 0$

$$\exists\, \varphi \in H^1(R^3) \Rightarrow \hat{\underline{u}} = \mathrm{grad}\,\varphi$$

thus $\mathrm{grad}\,\varphi\big|_{R^3 \setminus \Omega} = \hat{\underline{u}}\big|_{R^3\setminus\Omega} = 0 \Rightarrow \varphi = \text{const on } R^3\setminus\Omega$

W.L.O.G. set $\varphi \equiv 0$ on $R^3\setminus\Omega \Rightarrow \varphi|_\Omega \in H_0^1(\Omega)$

Check $\beta_2 = 0$ for torus $\Rightarrow$ with vanishing boundary condition)

curl, $l = 1$ $\qquad \beta^{n-l} = \beta^{3-1} = \beta_2 = 0$

torus works but ball with hole does not work !

3.4 Magnetostatics

$$\begin{cases} \mathrm{div}\,\underline{b} = 0 \\ \mathrm{curl}\,\underline{h} = \underline{j} \end{cases} \quad \text{in } \mathbb{R}_3$$

$\left[\begin{array}{l} \mathrm{curl}\,\underline{e} = 0 \\ \mathrm{div}\,\underline{d} = \rho \quad \underline{\mathrm{div}: H_0(\mathrm{div},\Omega) \xrightarrow{\text{onto}} L^2(\Omega)} \\ \underline{e} = -\mathrm{grad}\,v \quad \underline{v \text{ is unique up to a const}} \end{array}\right.$

$\underline{b} = \mathrm{curl}\,\underline{a}$ ($\underline{a}$: non-unique) we need gauging condition.

Two issues: i) $\underline{j}$ cannot be arbitrary $\underline{\mathrm{div}\,\underline{j} = 0}$ plus $\underline{\text{more}}$ (4)

ii) $\underline{a}$ is unique up to $\mathrm{grad}\,\phi$

On boundary Ω D) $\underline{\text{PMC}}$: $\underline{h}\times\underline{n} = 0$ on $\partial\Omega \longrightarrow \underline{j}\cdot\underline{n} = 0$ and $\int_{\Sigma_1} \underline{j}\cdot\underline{n}\,dS = 0$

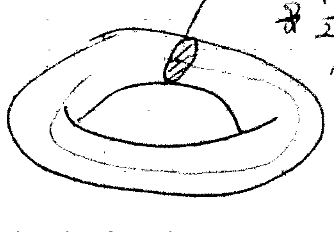
(No constraint on $\underline{a}$)

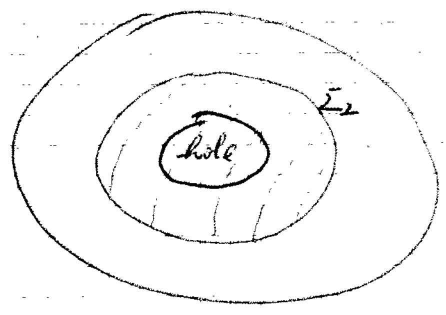
$\underline{\text{zero flux B.C.}}$ $\underline{b}\cdot\underline{n} = 0$ on $\partial\Omega \longrightarrow \int_{\Sigma_2} \underline{j}\cdot\underline{n}\,dS = 0$

$(\underline{a}\times\underline{n} = 0)$

For torus

cut surface Σ_1 $\partial\Sigma \subset \partial\Omega$
& Σ does not split Ω into two parts.





$\int_{\Sigma_2} j \cdot n \, dS = 0$ for all connected components of $\partial\Omega$.

4 $\operatorname{div} j = 0$ in Ω $\quad j \cdot n = 0$ on $\partial\Omega$ $\quad \int_{\Sigma_1} j \cdot n \, dS = 0$ on cut $\quad \int_{\Sigma_2} j \cdot n \, dS = 0$ on surface separat P_i

3.4.1 $\underline{a}$ - based V.F.

$\underline{b} = \operatorname{curl} \underline{a}$ strongly & $\operatorname{curl} \underline{h} = \underline{j}$ weakly & Material Law

$$\langle DE_{mag}(\underline{b}), \underline{b}' \rangle = \int_\Omega \underline{h} \cdot \underline{b}' \, dx \quad \forall \underline{b}' \in \mathcal{F}(\Omega)$$

$$\triangleright \langle DE_{mag}(\operatorname{curl} \underline{a}), \operatorname{curl} \underline{a}' \rangle + \int_{\partial\Omega} (\underline{h} \times \underline{n}) \, \underline{a}' \, dS$$

$$= -\int_\Omega \underline{j} \cdot a' \, dx \quad \forall a' \in \mathcal{F}'(\Omega)$$

For linear local material laws: Seek $\underline{a} \in H(\operatorname{curl}, \Omega)$

$$\int_\Omega \underline{\underline{\mu}}^{-1} \operatorname{curl} \underline{a} \cdot \operatorname{curl} \underline{a}' + \int_{\partial\Omega} (\underline{h} \times \underline{n}) \underline{a}' \, dS \qquad \boxed{3.4.a}$$

$$= -\int_\Omega \underline{j} \cdot \underline{a}' \, dx \qquad \forall \underline{a}' \in H(\operatorname{curl}, \Omega)$$

For zero flux B.C. $\Rightarrow \underline{a} \times n = 0$ ~~the~~ $H(\operatorname{curl}, \Omega) \Rightarrow H_0(\operatorname{curl}, \Omega)$.

Add constraints:

$$\int_\Omega \underline{a} \cdot \underline{z} \, dx = 0 \quad \forall \underline{z} \in H_0(\operatorname{curl} 0, \Omega) = \{ \underline{v} \in H_0(\operatorname{curl}, \Omega) \; \operatorname{curl} \underline{v} = 0 \}$$

$\boxed{3.4.a}$ $\quad \underline{a} \in \ker(\operatorname{curl})^\perp$

a closed subspace of $H(\operatorname{curl}, \Omega)$

<u>Thm 3.4.A</u> If $\underline{j} \in \mathrm{curl}\, H_0(\mathrm{curl}, \Omega)$ then (3.4.a) ~~(3.4.b)~~ has a unique solution.

Proof. (PMC) restrict first equation to $(H(\mathrm{curl}\,0, \Omega))^\perp$

orth. complement ~~of~~ in $H(\mathrm{curl}, \Omega)$.

$$\underbrace{\int_\Omega \underline{\underline{\mu}}^{-1} \mathrm{curl}\,\underline{a} \cdot \mathrm{curl}\,\underline{a}'}_{\Downarrow} = \int \underline{j} \cdot \underline{a}'\, dx \quad \forall a' \in H(\mathrm{curl}\,0, \Omega)^\perp$$

To show: $c(\underline{a}, \underline{a}')$ is an inner product in $H(\mathrm{curl}\,0, \Omega)$

$$\Downarrow$$

$$\exists \gamma > 0, \text{ s.t. } \underline{\|\underline{u}\|_{L^2} \le \gamma \|\mathrm{curl}\,\underline{u}\|_{L^2}} \quad \forall \underline{u} \in H(\mathrm{curl}\,0, \Omega)^\perp$$

Let $\underline{w} = \mathrm{curl}\,\underline{u} \in H(\mathrm{div}\,0, \Omega)$.

together with $\underline{\int_{\Gamma_i} \underline{\omega}\, \underline{n}\, dS = 0}$ $(*)$ Γ_i's connected components of $\partial\Omega$

For all connected components D of $\mathbb{R}^3 \setminus \Omega$ solve

$$\begin{cases} -\Delta \psi = 0 & \text{on } D \\ \mathrm{grad}\,\psi \cdot n = \omega \cdot n & \text{on } \partial D \end{cases} \longmapsto \text{existence of } \psi \text{ follows from } (*)$$

$$\widetilde{\omega} = \begin{cases} \omega & \text{on } \Omega \\ \mathrm{grad}\,\psi & \text{in } \mathbb{R}^3 \setminus \Omega \end{cases} \Rightarrow \underline{\widetilde{\omega} \in H(\mathrm{div}\,0, \mathbb{R}^3)}$$

Now by Fourier transformation

$$\mathrm{div}\,\widetilde{\omega} = 0 \Rightarrow \xi \cdot \hat{\omega} = 0$$

Solve $\hat{\Phi}(\xi) \times \xi = \hat{w}(\xi) \Rightarrow \hat{\Phi}(\xi)$

$$\left(\begin{array}{c} \text{Inverse} \Downarrow \text{Fourier Transform} \\ \mathrm{curl}\,\Phi(x) = \hat{\omega}(x) \end{array} \right) \quad \int_\Omega (1 + |\xi|^2) |\hat{\Phi}(\xi)|\, d\xi < \infty$$

$$\Rightarrow \Phi \in H^1(\mathbb{R}^3)$$

$\triangleright\ \phi \in H^1(\mathbb{R}^3) \quad \operatorname{curl}\phi = \widetilde{\omega}$

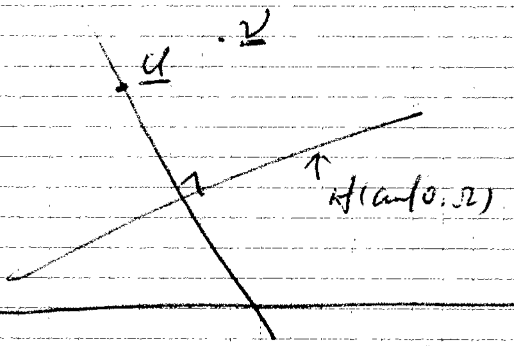
$$\|\Phi\|_{L^2(\mathbb{R}^3)} \le C\|\widetilde{\omega}\|_{L^2(\mathbb{R}^3)} \le C\|\omega\|_{L^2(\Omega)}$$

So we have found $\Phi \in H(\operatorname{curl},\Omega) \quad \operatorname{curl}\Phi = \operatorname{curl}\underline{u}$

$$\|\phi\|_{L^2(\Omega)} \le C\|\operatorname{curl}\underline{u}\|_{L^2(\Omega)}$$

If $\underline{u} \in H(\operatorname{curl}0,\Omega)^\perp$ then $\|u\|_{L^2} = \inf\limits_{\operatorname{curl}\underline{u}=\operatorname{curl}v} \|\underline{v}\|_{L^2(\Omega)} \le \|\Phi\|_{L^2(\Omega)}$

$$\le C\|\operatorname{curl}\underline{u}\|_{L^2(\Omega)}$$



Does this $\underline{a}$ satisfy the first variational equation for general a'?

also cover case $a' \in H(\operatorname{curl}0,\Omega)$. (LHS of (3.4a) $= 0$)

$$\Downarrow$$

$$\int_\Omega j \cdot a'\,dx = 0$$

<u>Note</u>: $j \in \operatorname{curl} H_0(\operatorname{curl},\Omega)$ indeed

$$j = \operatorname{curl}\underline{h}_0 \Rightarrow \int_\Omega \operatorname{curl}\underline{h}_0\, a'\,dx = \int_\Omega \underline{h}_0 \overset{0}{\cancel{\operatorname{curl}\underline{a}'}}\,dx = 0.$$

<u>Note</u>: We have shown

$$\|a\|_{H(\operatorname{curl},\Omega)} \le C(\Omega,\mu)\|j\|_{L^2(\Omega)}$$

In case j is not compatible (may from intermediate result of computing)

<u>Extended V.F</u>: (for any $j \in L^2(\Omega)$) Seek $a \in H(\operatorname{curl},\Omega)$ $z \in H(\operatorname{curl}0,\Omega)$

$$\int_\Omega \underline{\mu}^{-1}\operatorname{curl}a\cdot\operatorname{curl}a'\,dx + \int_\Omega \underline{z}\, a'\,dx = -\int_\Omega j\cdot a'\,dx \quad \forall a' \in H(\operatorname{curl},\Omega) \tag{3.4b}$$

with constraint $\int_\Omega \underline{a} \cdot \underline{z}' = 0 \quad \forall \underline{z}' \in H(\mathrm{curl}\, 0, \Omega)$

$\underline{a} \in (\ker(\mathrm{curl}))^\perp$

Rk: $\mathrm{grad}\, H^1(\Omega) \subset H(\mathrm{curl}\, 0, \Omega)$

$\Downarrow$

For PMC $\int_\Omega \underline{a} \cdot \mathrm{grad}\,\phi = 0 \quad \forall \phi \in H^1(\Omega)$

implicit Coulomb gauging

Let $\phi \in C_0^\infty(\Omega) \Rightarrow \int_\Omega \mathrm{div}\, \underline{a}\, \phi\, dx = 0 \Rightarrow \boxed{\mathrm{div}\, \underline{a} = 0}$ in Ω

Let $\phi \in C^\infty(\Omega) \Rightarrow \int_{\partial\Omega} \underline{a} \cdot n\, \phi\, dS = 0 \Rightarrow \underline{a} \cdot \underline{n} = 0$ on $\partial\Omega$.

Ex: Show well-posedness of (3.4.6)

Consider zero-flux boundary condition.

Find implicit constraints of $\underline{a}$.

3.4.2 $\underline{h}$-based variational formulation

$\underline{b} = \mathrm{curl}\, \underline{a}$ weakly & $\mathrm{curl}\, \underline{h} = j$ strongly & M.L.

$\downarrow$

$\langle DE_{mag}(\underline{h}), \underline{h}' \rangle = \int_\Omega \underline{b} \cdot \underline{h}'\, dx \quad \forall \underline{h}'$

$$\int_\Omega \underline{b} \cdot \underline{h}' = \int_\Omega \underline{a}\, \mathrm{curl}\, \underline{h}' - \int_{\partial\Omega} \underline{a} \cdot (\underline{h}' \times \underline{n})\, dS$$

Seek $\underline{h} \in H(\mathrm{curl}, \Omega)$, $\underline{a} \in \mathrm{curl}(H(\mathrm{curl}, \Omega)) \subset L^2(\Omega)$

PMC

$$\begin{cases} \langle DE_{mag}(\underline{h}), \underline{h}' \rangle - \int_\Omega \underline{a} \cdot \mathrm{curl}\, \underline{h}'\, dx + \int_{\partial\Omega} (\underline{a} \times \underline{n})\, \underline{h}'\, dS = 0 \quad \forall \underline{h}' \in H_0(\mathrm{curl}, \Omega) \\ \int_\Omega \mathrm{curl}\, \underline{h} \cdot \underline{a}' = \int_\Omega \underline{j} \cdot \underline{a}'\, dx \quad \underline{a}' \in \mathrm{curl}(H(\mathrm{curl}, \Omega)) \end{cases}$$

Ex: For linear local M.L.

"not a popular formulation"

Collapsing the variational problem:

Step 1: Find $\underline{h}_\perp$ s.t. $\operatorname{curl} \underline{h}_\perp = \underline{j}$

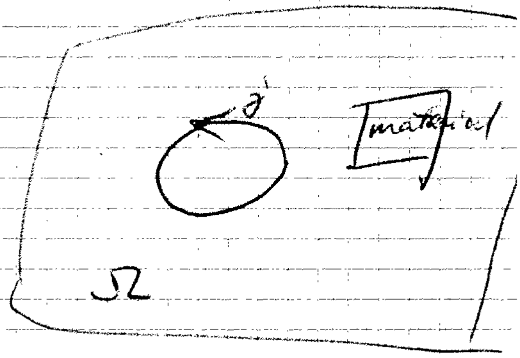
with $\|\underline{h}_\perp\|_{L^2(\Omega)} \le C \|\underline{j}\|_{L^2(\Omega)}$

Option: Biot-Savart formula

$$\underline{h}_\perp(x) = \operatorname{curl} \int_{R^3} \underline{j}(y) \frac{1}{4\pi|x-y|}\, dy$$

$\uparrow$ div free extend $\underline{j}$ (possible for PMC case and if $\operatorname{supp}(\underline{j}) \subset \Omega$

In reality $\underline{j}$ is some current along some coil inside Ω



$$\operatorname{curl} \underline{h}_\perp(x) = \operatorname{curl}\operatorname{curl} \int_{R^3} \sim\, dy$$

$$= (-\Delta - \operatorname{grad}\operatorname{div}) \int_{R^3} \frac{\underline{j}(x)}{4\pi|x-y|}\, dy$$

$\Uparrow$ this improper integral exists for bounded $\underline{j}$

<u>RK Δ</u>: $\quad -\Delta \int_{R^3} \frac{\underline{j}(y)}{4\pi|x-y|}\, dy = \underline{j}(x)$

$$\operatorname{div}_x \int_{R^3} \frac{\underline{j}(y)}{4\pi|x-y|}\, dy = \int_{R^3} \underline{j}(y) \cdot \cancel{\operatorname{grad}}\, \frac{1}{4\pi|x-y|}\, dy \quad \wedge \text{ by antisymmetry}$$

$$= -\int_{R^3} \cancel{\operatorname{div}}\, \underline{j}(y) \cdot \operatorname{grad}_y \frac{1}{4\pi|x-y|}\, dy$$

$$= \int_{R^3} \underbrace{\operatorname{div}_y \underline{j}(y)}_{=0} \frac{1}{4\pi|x-y|}\, dy = 0$$

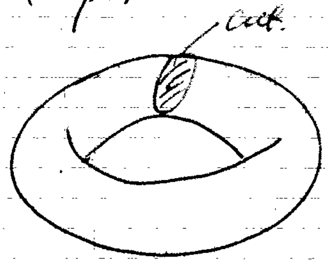
thus $\operatorname{curl} \underline{h}_\perp = \underline{j}$

<u>Step 2:</u> $\langle DE_{mag}(\underline{h}_1 + \underline{h}_0), \underline{h}_0' \rangle = 0 \quad \forall \underline{h}_0' \in H_0(\mathrm{curl}\,0, \Omega)$

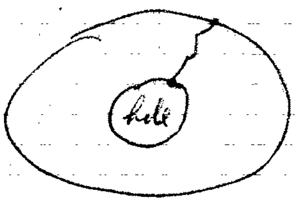
Seek $\underline{h}_0 \in H_0(\mathrm{curl}, \Omega)$ [PMC]-case

- use representation of $\left\{ \begin{array}{l} H(\mathrm{curl}\,0, \Omega) \\ H_0(\mathrm{curl}\,0, \Omega) \end{array} \right. \xrightarrow[\text{topo}]{\text{for trivial}} \left\{ \begin{array}{l} \mathrm{grad}\, H^1(\Omega) \\ \mathrm{grad}\, H_0^1(\Omega) \end{array} \right.$

- general topo.



$$H(\mathrm{curl}\,0, \Omega) = \{ \mathrm{grad}\,\phi \quad \phi \in H^1(\Omega \mid \Sigma), \quad [\phi]_\Sigma = \text{const. on } \Sigma \}$$



$$H_0(\mathrm{curl}\,0, \Omega) = \{ \mathrm{grad}\,\phi, \quad \phi \in H^1(\Omega), \quad \phi = \text{const on connected component of } \partial\Omega \}$$