Lecture VI 26-10-2008

- Review

FE space for

$\underline{0\text{-forms}} \hat{=} W^0(M) = \operatorname{span}_{x \in \mathcal{F}_0(M)} \{b_x\}$

$b_x|_T = \lambda_i$ if $a_i = x$ (for tetrahedral mesh)   dof. $u \to u(x)$

$\underline{1\text{-forms}} \hat{=} W^1(M) = \operatorname{span}_{e \in \mathcal{F}_1(M)} \{b_e\}$

$b_e|_T = \lambda_i \nabla\lambda_j - \lambda_j \nabla\lambda_i$ if $e = [a_i, a_j]$ dof $u \to \int_e u \cdot ds$

$\underline{2\text{-forms}} \hat{=} W^2(M) = \operatorname{span}_{f \in \mathcal{F}_2(M)} \{b_f\}$

$b_f|_T = 2(\lambda_i \nabla\lambda_j \times \nabla\lambda_k + \lambda_j \nabla\lambda_k \times \nabla\lambda_i + \lambda_k \nabla\lambda_i \times \nabla\lambda_j)$

if $f = [a_i\ a_j\ a_k]$ dof. $u \to \int_f u \cdot n\, ds$

$\underline{3\text{-forms}} \hat{=} W^3(M) = \operatorname{span}_{T \in \mathcal{F}_3(M)} \{b_T\}$

$b_T|_T = \frac{1}{|T|}$ $u \to \int_T u\, dx$

Chap. 3. Stationary electromagnetics

3.1 Stationary field eqns

$$\left\{\begin{array}{l} \operatorname{curl} \underline{e} = -\partial_t \underline{b} \\ \operatorname{curl} \underline{h} = \partial_t \underline{d} + \underline{j} \end{array}\right. \xRightarrow[\partial_t = 0]{\text{Stationary case}} \left\{\begin{array}{l} \boxed{\operatorname{curl} \underline{e} = 0} \\ \underline{\operatorname{curl} \underline{h} = \underline{j}} \end{array}\right. \quad (\operatorname{div} \underline{j} = 0)$$

compatibility cond.

$$\left\{\begin{array}{l} \operatorname{div} \underline{b} = 0 \\ \operatorname{div} \underline{d} = \rho \Rightarrow -\partial_t \rho = \operatorname{div} \underline{j} \end{array}\right. \Rightarrow \left\{\begin{array}{l} \boxed{\operatorname{div} \underline{b} = 0} \\ \boxed{\operatorname{div} \underline{d} = \rho} \end{array}\right\} \text{new independent eqns.}$$

new source

Observation: decoupling of electric & magnetic fields.

——— $\hat{=}$ electric eqns

——— $\hat{=}$ magnetic eqns

3.2 Electrostatics

3.2.1. v-based variational formulation

- $\mathrm{curl}\,\underline{e} = 0$ on A_3 $\Rightarrow$ $\underline{e} = -\mathrm{grad}\, v$
- material law (energy oriented form)

$$\langle DE_{el}(\underline{e}), \underline{e}' \rangle = \int_\Omega \underline{d} \wedge e' \, dx \qquad \forall e' \in \mathcal{F}(\Omega)$$

not A_3

- $\mathrm{div}\,\underline{d} = \rho$ weakly

$$\int_\Omega \mathrm{div}\,\underline{d}\, v = -\int_\Omega \underline{d} \wedge \mathrm{grad}\, v' + \int_{\partial\Omega} \underline{d} \cdot \underline{n}\, v' dS$$

$$= -\langle DE_{el}(e), \mathrm{grad}\, v' \rangle + \int_{\partial\Omega} \underline{d} \cdot \underline{n} \cdot v' dS$$

$$= \int_\Omega \rho v' dx \qquad \forall v' \in \mathcal{F}^0(\Omega)$$

$$\langle DE_{el}(-\mathrm{grad}\, v), -\mathrm{grad}\, v' \rangle + \int_{\partial\Omega} \underline{d} \cdot n\, v' dS = \int_\Omega \rho v' dx \qquad (3.2.a)$$

For ~~local~~ linear material law:

$$\boxed{\int_\Omega \underline{\underline{\varepsilon}}\, \mathrm{grad}\, v\, \mathrm{grad}\, v' + \int_{\partial\Omega} \underline{d} \cdot \underline{n}\, v' dS = \int_\Omega \rho v' dx} \qquad (3.2.b)$$

B.C. for electrostatics:

PEC $\quad \underline{e} \times \underline{n} = 0$ on $\partial\Omega$ $\Rightarrow$? $v = 0$ on $\partial\Omega$

Strong B.C. imposed on trial and test functions

$$v|_{\partial\Omega} = 0 \qquad v'|_{\partial\Omega} = 0$$

surface charge: $\quad \underline{d} \cdot n = \rho_{\partial\Omega}$ weakly enforced.

(3.2.b) $\Rightarrow$ linear variational problem

$$u \in V \qquad a(u, v) = f(v) \qquad \forall v \in V \qquad (3.2.c)$$

$a(\cdot,\cdot)$ symmetric coercive continuous bilinear form

$f(\cdot)$ continuous linear form

$$a(u, v) = \int_\Omega \underline{\underline{\varepsilon}}\, \mathrm{grad}\, u \cdot \mathrm{grad}\, v$$

Try $V = C^\infty(\bar\Omega)$ or ($C_0^\infty(\bar\Omega)$ for PEC case) ⑤

Lemma 3.2.A. (Poincaré – Friedrichs Inequality)

$$\|u\|_{L^2(\Omega)} \leq C(\Omega) [\|\operatorname{grad} u\|_{L^2(\Omega)} \qquad u \in C_0^\infty(\bar\Omega)$$

$$\|u\|_{L^2(\Omega)} \leq C(\Omega) \|\operatorname{grad} u\|_{L^2(\Omega)} \qquad u \in C^\infty(\bar\Omega) \quad \int_\Omega u\, dx = 0$$

proof (if Ω is convex)

$\|v\|_A = \sqrt{a(v,v)}$ is a norm of

• Construction of solutions to (3.2.b) $\quad C_*^\infty(\bar\Omega) = \{ v \in C^\infty(\bar\Omega) \ \int_\Omega v = 0 \}$

• $f \in L^2(\Omega) \Rightarrow v \to \int_\Omega f v\, dx$ is linear conti. on V

• $\ker(f) = \{ v \in V \ f(v) = 0 \} \neq V$ for $f \neq 0$ ▷ $\ker(f)^\perp \neq \{0\}$

does not exist for inner product space

$\Rightarrow \exists\, w \neq 0. \ w \in \ker(f)^\perp$

$$f(\underbrace{f(v) w - f(w) v}_{\in \ker(f)}) = 0 \qquad \forall v \in W$$

▷ $a(w, f(v) w - f(w) v) = 0$

Ex.

▷ By linearity $\quad a\left(\frac{w f(w)}{\|w\|_A^2}, v\right) = f(v) \quad \forall v \in V$

← Riesz representation thm

"we find the unique solution"

Example: $u(x,y) = \sin(x) \sin(y)$ Solves

$$-\Delta u = u \quad \text{on } \Omega = [0, \pi]^2, \quad u = 0 \ \partial\Omega$$

$$\Rightarrow \int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v\, dx = \int u v\, dx \quad \forall v \in C_0^\infty(\Omega)$$

But $u(x,y) \notin C_0^\infty(\Omega)$

We need $(C_*^\infty(\Omega), \|\cdot\|_A)$ to be complete by completion.

$$C_*^\infty(\bar\Omega) \longrightarrow H_*^1(\Omega) \qquad C_0^\infty(\Omega) \longrightarrow H_0^1(\Omega)$$

put the proof in the setting of Sobolev Spaces

Thm 3.2.c (3.2.b) has a unique solution $u \in \begin{cases} H^1_*(\Omega) & \text{for zero charge B.C.} \\ H^1_0(\Omega) & \text{for PEC} \end{cases}$

RK: (1) $C^\infty(\bar{\Omega})$ is dense in $H^1(\Omega)$

Assertion of Sobolev function $\Rightarrow$ reduces to relations of norms

Procedure: i) prove first for continuous case

ii) Take limit by density

(2) $H^1(\Omega) = \{ u \in L^2(\Omega) \mid \text{grad}\, u \in \underset{\sim}{L}^2(\Omega) \}$

- For nonlocal nonlinear material Law

(3.2.a) reduces to: Seek $u \in H^1_0(\Omega)$

$$\langle DE_{el}(-\text{grad}\, v), -\text{grad}\, v' \rangle = \int_\Omega \rho\, v'\, dx \qquad \forall v' \in H^1_0(\Omega) \qquad \underline{(3.2.f)}$$

L Recall: $E_{el}: \underset{\sim}{L}^2(\Omega) \to R$

$DE_{el}: \underset{\sim}{L}^2(\Omega) \longrightarrow \underbrace{(\underset{\sim}{L}^2(\Omega))'}_{}$ space of $\underbrace{\text{linear}}_{\text{continuous}}$ functionals on $(\underset{\sim}{L}^2(\Omega))$

Define $\hat{E}_{el}(v) = E_{el}(-\text{grad}\, v)$

$$D\hat{E}_{el}(v) = D\tilde{E}_{el}(-\text{grad}\, v)$$

$$\langle D\tilde{E}_{el}(v), v' \rangle = \langle DE_{el}(-\text{grad}\, v), -\text{grad}\, v' \rangle$$

$\in (H^1_0(\Omega))' = H^{-1}(\Omega)$

Astract chain rule:

$f: X \to Y \qquad g: Z \to X$

$h = f \circ g$

$$\langle Dh(z), y \rangle = \langle D(f(g(z))), Dg(z) y \rangle$$

Assume
- E_{el} smooth finite

- bounded $\quad D^2 E_{el}(\underline{e})(\underline{e}', \underline{e}'') \le T \, \|\underline{e}'\| \cdot \|\underline{e}''\|$ (3.2.d)

$\forall \, \underline{e}', \underline{e}'', \underline{e}$ with const $T > 0$

$= a(\cdot, \cdot)$ for linear material law.

- coercive $\quad D^2 E_{el}(\underline{e})(\underline{e}', \underline{e}') \ge \gamma \, \|\underline{e}'\|^2$ (3.2.e)

$\forall \, \underline{e}, \underline{e}'$ with $\gamma > 0$.

▷ well-posed of nonlinear problem:

1. Let $T : H^{-1}(\Omega) \to H_0^1(\Omega)$ be defined by

$$a(Tf, v) = f(v) \quad \forall v \in H_0^1(\Omega)$$

for $a(u, v) = \int_\Omega \gamma \operatorname{grad} u \cdot \operatorname{grad} v \, dx. \quad \| \cdot \|_A = \sqrt{a(\cdot, \cdot)}$

T: is a solution operator (bijective)

$$\|Tf\|_A^2 = f(Tf) \le \|f\|_{H^{-1}(\Omega)} \cdot \|Tf\|_A \Rightarrow \|T\| = 1$$

A'

$\sup_{w \in H_0^1(\Omega)} \frac{|f(w)|}{\|w\|_A}$

Scheme Iteration:
~~Newton Method~~
Fixed Point Iteration
Newton

$$v^{(k+1)} = v^{(k)} - \alpha T(g - \underbrace{D\hat{E}_{el}(v^{(k)})}_{H^{-1}}) \quad (3.2.g) \quad \alpha > 0$$

$g(v) = \int_\Omega g v \, dx$

- Suppose $v^{(k)} \to v^*$ then $v^* = v^* - \alpha T(g - D\hat{E}_{el}(v^*))$

$\iff \quad D\hat{E}(v^*) = g \quad$ in $H^{-1}(\Omega)$

$\iff \quad \langle D\hat{E}(v^*), v' \rangle = \langle g, v' \rangle$ for $v' \in H_0^1(\Omega)$

To show Existence & uniqueness of limits of (3.2.g)

$v^{(k+1)} = \phi(v^{(k)}) \quad \phi(v) = v - \alpha T(g -$

$$\|\phi(u)-\phi(v)\|_A^2 = \|u-v+\alpha T(D\hat{E}_{el}(u) - D\hat{E}_{el}(v))\|_A^2$$ (55)

$$= \|u-v\|_A^2 + 2\alpha \underbrace{\langle D\hat{E}_{el}(u) - D\hat{E}_{el}(v), u-v\rangle}_{\mathrm{II}} +$$

$$\alpha^2 \underbrace{\|D\hat{E}_{el}(u) - D\hat{E}_{el}(v)\|_A^2}_{\mathrm{III}}$$

mean value theorem

$$\langle D\hat{E}_{el}(u) - D\hat{E}_{el}(v), w\rangle = \int_0^1 D^2\hat{E}_{el}(v+t(u-v))(u-v, w)\,dt$$

$$\mathrm{II} \geq \gamma \|\mathrm{grad}(u-v)\|_{L^2}^2 = \|u-v\|_A^2 \quad \text{by (3.2.e)}$$

$$\mathrm{III} \leq \frac{\Gamma}{\gamma}\|u-v\|_A^2 \quad \text{by (3.2.d)}$$

$$\|\phi(u)-\phi(v)\|_A^2 \leq \|u-v\|_A^2 + 2\alpha\|u-v\|_A^2 + \alpha^2\frac{\Gamma}{\gamma}\|u-v\|_A^2 \quad (\alpha<0)$$

$$= \underbrace{\left(1+2\alpha+\alpha^2\frac{\Gamma}{\gamma}\right)}_{\rho}\|u-v\|_A^2$$

choose $\alpha \to 0$ s.t. $\rho < 1$

By Contraction mapping principle $\Rightarrow \exists!\, u^*$
"Banach fixed point theorem"

3.2.2 $\underline{d}$ - based variational formulation

- $\operatorname{div} \underline{d} = f$ strongly
- $\underline{e} = -\operatorname{grad} v$ weakly

$$\int_\Omega v \operatorname{div} \underline{d}' - \int_{\partial\Omega} v \, \underline{d}' \underline{n} \, dS = \int_\Omega \underline{e} \cdot \underline{d}' \, dx$$

- material law $\langle DE_{el}(\underline{d}), \underline{d}' \rangle = \int_\Omega \underline{e} \cdot \underline{d}' \, dx$

$$\triangleright \begin{cases} \langle DE_{el}(\underline{d}), \underline{d}' \rangle - \int_\Omega v \operatorname{div} \underline{d}' \, dx = -\int_{\partial\Omega} v \, \underline{d}' \underline{n} \, dS & (3.2.h) \\ -\int_\Omega \operatorname{div} \underline{d} \, v' \, dx = -\int_\Omega f v' \, dx & \forall v' \in \mathcal{F}^0(\Omega) \end{cases}$$

- for local linear

$$\begin{cases} \underbrace{\int_\Omega \underline{\underline{\varepsilon}}^{-1} \underline{d} \cdot \underline{d}'}_{a} - \underbrace{\int \operatorname{div} \underline{d}' \, v \, dx}_{b} = \underbrace{-\int_{\partial\Omega} v \, \underline{d}' \underline{n} \, dS}_{g} & \forall \underline{d}' \in \mathcal{F}^2(\Omega) \quad (3.2.i) \\ \underbrace{-\int_\Omega \operatorname{div} \underline{d} \, v' \, dx}_{b} = \underbrace{-\int_\Omega f v' \, dx}_{f} & \forall v' \in \mathcal{F}^0(\Omega) \end{cases}$$

$\hat{=}$ linear variational problem in saddle point form.

Abstract setting

$$\begin{cases} \underline{a(d, d')} + \underline{b(d', v)} = g(d') & \forall d' \in V \\ \underline{b(d, v')} = f(v') & \forall v' \in Q \end{cases} \quad (3.2.j)$$

- Hilbert space framework for (3.2.i)
  - find norms s.t. $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are continuous
  - find weak norms to guarantee ellipticity.

$$V = H(\operatorname{div}, \Omega) \qquad Q = L^2(\Omega)$$

By PEC B.C. require $g \to \int_{\partial\Omega} v\, \underline{d}' n\, dS = 0$

zero surface charge condition $\underline{d} \cdot \underline{n} = 0$. (strongly enforced)

$$V = H_0(\mathrm{div}, \Omega) \text{ to treat B.C.}$$

- Well-posedness of abstract saddle point problem.

Heuristic (3.2.j) ~ block matrix $\begin{pmatrix} A & B^T \\ B & \end{pmatrix}\begin{pmatrix} \vec{d} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \vec{g} \\ \vec{f} \end{pmatrix}$

$\mathrm{kern}(B) = \{ \underline{d} \in V \;\; b(\underline{d}, v') = 0 \;\; \forall v' \in Q \} \subset$ closed subspace of Q.

Orthogonal decomposition: $V = \mathrm{kern}(B) \oplus_V (\mathrm{kern}(B))^\perp$.

Then (3.2.j) $\Rightarrow$: Seek $\underline{d}_0 \in \mathrm{Ker}(B)$. $\underline{d}_{0,1} \in \mathrm{ker}(B)^\perp$. $v \in Q$

Test $d_0' \in \mathrm{Ker}(B)$

$$\begin{cases} a(\underline{d_0}, \underline{d_0}') + a(\underline{d_1}, \underline{d_0}') = g(\underline{d_0}') & \underline{d_0} \in \mathrm{Ker}(B) \\ a(\underline{d_0}, \underline{d_1}') + a(\underline{d_1}, \underline{d_1}') + b(\underline{d_1}', v) = g(\underline{d_1}') & \underline{d_1} \in \mathrm{Ker}(B)^\perp \\ b(\underline{d}_1, v') = f(v') & \forall v' \in Q \end{cases}$$

Solve $\underline{d}_1' \longrightarrow \underline{d}_0' \longrightarrow v$

For (3.2.i) & PEC

① find $\underline{d}_1 \in H(\mathrm{div}, \Omega)$ $\mathrm{div}\, \underline{d}_1 = \rho$ (this $\underline{d}$ has d_1 and a part from $\mathrm{ker}(B)$ but it does not blow up.)

Let $\underline{d}_1 = \mathrm{grad}\, \phi$. $\Rightarrow \begin{cases} \Delta\phi = \rho & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$

$\Rightarrow \|\underline{d}_1\|_{L^2} \lesssim \|\rho\|_{L^2(\Omega)}$ $\hookrightarrow$ convex Ω

② Seek $\underline{d_0} \in \mathrm{Ker}(B) = H(\mathrm{div}\, 0, \Omega) = \{ v \in H(\mathrm{div}\, \Omega) \;\; \mathrm{div}\, v = 0 \}$

$$\int_\Omega \underline{\underline{\varepsilon}}^{-1} \underline{d}_0 \cdot \underline{d}_0' \, dx = -\int \underline{\underline{\varepsilon}}^{-1} \underline{d}_1' \, \underline{d}_0' \, dx \qquad \forall \underline{d}_0' \in H(\mathrm{div}\, 0, \Omega)$$

④ $\triangleright$ $\| \underline{d}_0 \|_{L^2} \lesssim C(\underline{\underline{\varepsilon}}) \| \underline{d}_1 \|_{L^2(\Omega)}$

⑤ Find $v \in L^2(\Omega)$ s.t.

$$\int_\Omega \underline{\underline{\varepsilon}}^{-1} (\underline{d}_0 + \underline{d}_1) \, \underline{d}_1' \, dx + \int_\Omega \mathrm{div}\, d_1' \, v \, dx = 0 \qquad \forall d_1' \in \ker(B)^\perp$$

$\ker(\mathrm{div})^\perp$ [illegible]

from ③ we get a map $\mathbb{L} : L^2(\Omega) \to H(\mathrm{div}, \Omega)$ and $\mathrm{div} \circ L = Id$

Set $\underline{d}_1' = Lv'$

$$\triangleright \int_\Omega \underline{\underline{\varepsilon}}^{-1} (\underline{d}_0 + \underline{d}_1) \cdot Lv' \, dx + \int_\Omega v' \, v \, dx = 0$$

$$\Longrightarrow \quad \underline{d} = \underline{d}_0 + \underline{d}_1 \text{ and } v \text{ solve } (3.2.i)$$

$$H(\mathrm{div}, \Omega) = H(\mathrm{div}\, 0, \Omega) + \mathbb{L}(L^2(\Omega))$$

Properties: $\|\underline{d}_1\|_{L^2(\Omega)} \leq C \|\varphi\|_{L^2}$

$\|d_0\|_{L^2(\Omega)} \leq \|\underline{d}_1\|_{L^2}$

$$\|v\|^2_{L^2(\Omega)} \leq \|\underline{d}_0 + \underline{d}_1\|_{L^2} \, \underbrace{\|\mathbb{L} v\|_{L^2}}_{\leq C \|v\|_{L^2}}$$

$$\triangleright \; \|d\|_{L^2} + \|v\|_{L^2} \lesssim \|\varphi\|_{L^2}$$

$\|\mathrm{div}\, d\|_{L^2}$

$\leq \|\phi\|_{H^1} \leq \|\varphi\|_{L^2}$ (in step ①)