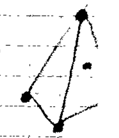
<u>2.5 Whitney forms and Galerkin approach</u> Lecture V 17-10-20

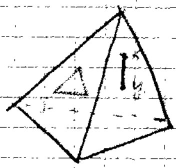
- Review of Whitney maps.

2.5.1 Simplicial Whitney forms (cont'd)

$l=0.$ $\quad x = \sum_i \lambda_i(x) a_i$ $\qquad \underline{W^0\omega_h(x) = \sum_i \lambda_i(x)\, \omega_h(a_i)}$



$l=1.$ $\quad [x,y] = \sum_{i<j} \left(\lambda_i(x)\lambda_j(y) - \lambda_j(x)\lambda_i(y)\right)[a_i, a_j] = \sum_i \sum_j \lambda_i(x)\lambda_j(y)[a_i, a_j]$



$$\underline{\int_{[x,y]} W^1\omega_h = \sum_i \sum_j \lambda_i(x)\lambda_j(y)\, \omega_h([a_i, a_j])}$$

$l=2.$ $$\int_{[x,y,z]} W^2\omega_h = \sum_i \sum_j \sum_k \lambda_i(x)\lambda_j(y)\lambda_k(z)\, \omega_h([a_i, a_j, a_k])$$

With $W^l\omega_h$ well defined for all l-facet inside T. we can define

<u>Local Reconstruction of differential forms</u>

Recall by <u>recovery formula</u> $(*)$:

$$\omega \in DF^l. \qquad \omega(x)(v_1, \dots v_l) = \lim_{t\to 0} \frac{l!}{t^l} \int_{\Sigma_t} \omega$$

$$\Sigma_t = \text{convex hull}\,(x,\ x+tv_1, \ \dots\ x+tv_l)$$

$l=1.$ $$(W^1\omega_h)(x)(v) \overset{(*)}{=} \lim_{t\to 0} \frac{1}{t} \int_{[x,x+tv]} W^1\omega_h \qquad \int_{\partial[x,x+tv]} \lambda_j = \int_{[x,x+tv]} d\lambda_j$$

$$\overset{\text{def. of } W^1}{=} \lim_{t\to 0} \sum_{i<j} \Big[\lambda_i(x)\frac{\lambda_j(x+tv) - \lambda_j(x)}{t} - \lambda_j(x)\frac{\lambda_i(x+tv) - \lambda_i(x)}{t}$$

(trick: always add/subtract $\sum_i\sum_j\sum_k \lambda_i(x)\lambda_j(x)\cdot\lambda_k(x)$)

$$\overset{(*)}{=} \sum_{i<j} \left(\lambda_i(x)\, d\lambda_j(x)(v) - \lambda_j(x)\, d\lambda_i(x)(v)\right) \omega_h[a_i, a_j]$$

$\triangleright$ $$(W^1\omega_h)(x) = \sum_{i<j} \underbrace{\left(\lambda_i(x)\, d\lambda_j(x) - \lambda_j(x)\, d\lambda_i(x)\right)}_{\text{basis 1-form}} \omega_h([a_i, a_j])$$

$$(W^1\omega_h)(x) = \sum_i \underbrace{\lambda_i(x)}_{\text{basis 0-form}}\ \omega_h(a_i)$$

$$(W^2\omega_h)(x) = \sum_{i_0<i_1<i_2} \sum_{j=0}^{2} (-1)^j \lambda_{i_j}(x) \prod \Big(\bigwedge_{\substack{k=0\\k\neq j}}^{2} d\lambda_{i_k}(x)\Big)\, \omega_h([a_i, a_j, a_k])$$

$$(W^2\omega_R)(x) = 2\sum_{i<j<k} \underbrace{(\lambda_i\, d\lambda_j\wedge d\lambda_k + \lambda_j\, d\lambda_k\wedge d\lambda_i + \lambda_k\, d\lambda_i\wedge d\lambda_j)}_{\text{basis 2-form}}\,\omega_R([a_i,a_j,a_k])$$

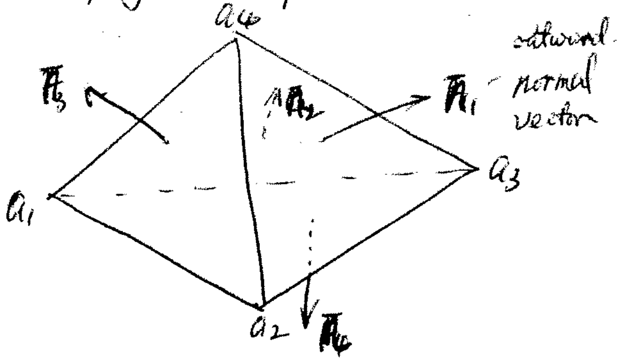
$$(W^\ell\omega_R)(x) = \underbrace{\ell!}_{\substack{\text{normalization factor}\\ \text{for } \ell\text{-facet}}} \sum_{i_0<i_1<\dots<i_\ell} \underbrace{\left(\sum_{j=0}^{\ell}(-1)^j\,\lambda_{i_j}\, d\lambda_{i_0}\wedge\cdots\wedge \widehat{d\lambda_{i_j}}\wedge\cdots\wedge d\lambda_{i_\ell}\right)}_{\text{basis } \ell\text{-form}}\,\omega_R([a_{i_0}\dots a_{i_\ell}])$$

Note: W^ℓ commutes with trace γ onto facets of a complex.

2.5.2 Simplicial Whitney finite elements.

Perspective: vector proxies

- Local polynomial spaces:



Recall: $\lambda_i(x) = \dfrac{|F_i|}{3|T|}(x-a_j)\cdot n_i \quad \forall (j\neq i)$ $\left\{\begin{array}{l}\text{linear}\\ \lambda_i(a_j)=\delta_{ij}\end{array}\right.$

$$\operatorname{grad}\lambda_i(x) = \frac{|F_i|}{3|T|}\, n_i$$

$\ell=0$. VP $W^0\mathcal{C}^0(T) = \{x\mapsto \alpha + b\cdot x,\ \alpha\in\mathbb{R},\ b\in\mathbb{R}^3\}$ dim $=4$ $\quad a\times(b\times c) = b(a\cdot c) - c(a\cdot b)$

$\ell=1$. basis 1-form associated with $[a_i, a_j]$. $b_{ij} = \lambda_i \operatorname{grad}\lambda_j - \lambda_j\operatorname{grad}\lambda_i$.

$$b_{ij} = \frac{|F_i|\cdot|F_j|}{9|T|^2}\Big[\underbrace{((x-a_k)\cdot n_i)n_j - ((x-a_k)\cdot n_j)n_i}_{(n_i\times n_j)\times(x-a_k)}\Big]$$

$$\hat{=}\ \underline{a} + \underline{b}\times\underline{x}$$

VP $W^1\mathcal{C}^1(T) = \{x\mapsto \underline{a}+\underline{b}\times\underline{x},\ \underline{a},\underline{b}\in\mathbb{R}^3\}$ dim $=6$

$(\nabla\times(b\times x) =$

$\ell=2$.

basis 2-form associated with $[a_i, a_j, a_k]$

$$b_{ijk} = \lambda_i\nabla\lambda_j\times\nabla\lambda_k + \lambda_j\nabla\lambda_k\times\nabla\lambda_i + \lambda_k\nabla\lambda_i\times\nabla\lambda_j$$
$$= \frac{|F_i|\cdot|F_j|\cdot|F_k|}{27|T|}\big[((x-a_l)\cdot n_i)\, n_j\times n_k + ((x-a_l)\cdot n_j)\, n_k\times n_i + ((x-a_l)\cdot n_k)\, n_i\times n_j\big]$$

$$b_{ijk} = \frac{|F_i||F_j||F_k|}{27|T|^3}(x-a_l)\det(n_i, n_j, n_k) \quad \underline{Ex.}$$

(45)

$$= \frac{1}{6}\frac{|F_l|}{|T|}(x-a_l)$$

$V.P.\ W^2\mathcal{C}^2(T) = \{ x \to \underline{a} + \beta x,\ \underline{a} \in R^3,\ \beta \in R \}$. $\dim = 4$.

$l=3$. $V.P.\ W^3\mathcal{C}^3(T) = \{ x \to c,\ c \in \mathbb{R} \}$

- <u>degrees of freedom</u> For local polynomial space $W^l = V.P.\ W^l\mathcal{C}^l(T)$

$l=0$. point evaluation $u \to u(a_i)$

$l=1$ edge path integral $\underline{u} \to \int_e \underline{u}\cdot d\vec{s}$ e: edge of T.

$l=2$ face flux integral $\underline{u} \to \int_f \underline{u}\cdot\underline{n}\, dS$ f face of T

$l=3$ cell mass integral $u \to \int_T u\, d\cancel{x}$

Check validity of dofs ?

(i) unisolvent : $dof = 0 \Rightarrow$

$S_l w = 0$ for $w \in W^l\mathcal{C}^l(T) \Rightarrow \underbrace{S_l W^l}_{I_d} w_h = 0$.

$\Rightarrow w_h = 0 \Rightarrow w = 0$

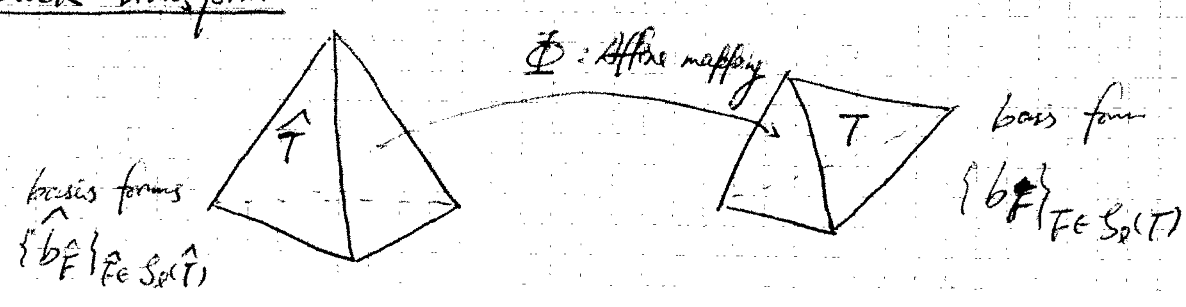
(ii) trace fixing to show if all dof associated with $f \in S_l(T)$

vanishes $\Rightarrow w = 0$ for $w \in W^l\mathcal{C}^l(T)$

(1-dim lower unisolvent condition)

$\triangleright$: unique trace of whitney forms on shared faces of two adjacent tetrahedron with the same d.o.f.

- <u>pullback transform</u>



2.6. Whitney form Galerkin Discretization

Recall: variational formulation of Sect. 1.7.

Galerkin discretization: replace $\mathcal{F}^l(\Omega)$ with $W^l(\mathcal{T}_h)$

mesh over Ω.

R.K. ① For strong B.C.s $\longleftrightarrow$ trial space $\{\omega \in \mathcal{F}^l(\Omega),\ \omega|_{\partial\Omega} = 0\ \gamma_{\partial\Omega}\omega = 0\}$

$\Rightarrow$ replace $\leftarrow$ with $\{\omega_h \in W^l(\mathcal{T}_h)$, basis functions ~~[illegible]~~ associated with ~~boundary~~ facets are dropped$\}$

② $$W^l(\mathcal{T}_h) \overset{\substack{\text{Whitney map} \\ W^l\text{map}}}{\cong} \mathcal{C}^l(\mathcal{T}_h) \cong \mathbb{R}^{N_l}$$

isomorphism

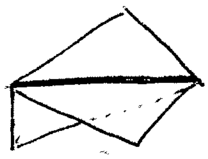
$$\begin{cases} \mathbb{R}^{N_l} \longrightarrow W^l(\mathcal{T}_h) \\ (\alpha_i)_{i=1}^{N_l} \longrightarrow \omega = \sum_i \alpha_i b_i^l \end{cases}$$

basis form associated with i-th l-facet

$\longrightarrow$ Matrix representation of $d \overset{\vee}{=}$ matrix representation of $d_l = \underline{\underline{D}}^l$

$$\left(d b_i^l = \sum_{j=1}^{N_{l+1}} (\underline{\underline{D}}^l)_{ji}\, b_j^{l+1} \right)$$ Ex: verify in terms of V.F.s

$l = 1$.



Example: ($\underline{e}$ - based V.F., PEC, local linear material Law)

$\underline{e}_h \in W^1(\mathring{\mathcal{T}}_h)$ $\longleftarrow$ use only basis ~~funct~~ 1-forms of interior edges

$$\int_\Omega \underline{\underline{\mu}}^{-1} \operatorname{curl} \underline{e}\ \operatorname{curl} \underline{e}'\, dx + \partial_t^2 \int \underline{\underline{\varepsilon}}\, \underline{e}_h \cdot \underline{e}_h'\, dx = -\partial_t \int_\Omega \underline{j} \cdot \underline{e}'\, dx$$

$\Downarrow$ Assume ordering of basis 1-forms / basis functions

$$\underline{e}_h = \sum_{\substack{e \in \mathcal{S}_1(\mathcal{T}_h) \\ \text{edge}}} \xi_e \underline{b}_e$$

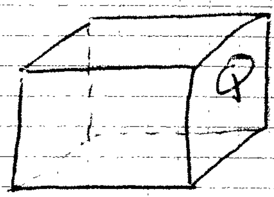
$\triangleright \quad \hat{b}_{\hat{F}} = \Phi^* b_F$: Whitney Finite elements are affine equivalent

proof: i) $\hat{\lambda}_k = \Phi^* \lambda_k$

ii) $\Phi^* \circ d = d \circ \Phi^*$

2.5.3 Non-simplicial discrete differential forms

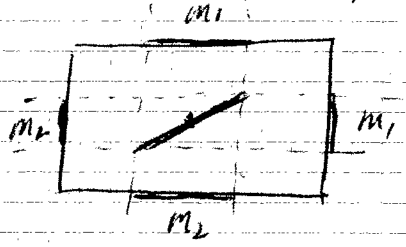
- Hexahedral Whitney forms $\quad W^0(Q) = (P_1 \otimes P_1 \otimes P_1)$



$l = 0$. $W^0(Q) = \{ x \mapsto \prod_{i=1}^{3} (\alpha_i + \beta_i x), \ \alpha_i, \beta_i \in \mathbb{R} \}$

(trilinear polynomial) dim = 8 (# vertices)

$l = 1$. (follow simplicial case.



midpoint (m_1, m_2)

in 3D. / project onto face

/ $\longrightarrow$ onto boundary edges with midpoint (m_1, m_2, m_3)

$\longrightarrow$ for basis function with edge $[0,1] \times \{0\} \times \{0\}$.

$$\underline{b}(x) = \begin{pmatrix} (1-x_2)(1-x_3) \\ 0 \\ 0 \end{pmatrix}$$

$$W^1(Q) = \begin{pmatrix} P_0 \otimes P_1 \otimes P_1 \\ P_1 \otimes P_0 \otimes P_1 \\ P_1 \otimes P_1 \otimes P_0 \end{pmatrix}^{1 \times 3 \times 2} \quad \text{dim} = 12.$$

$l = 2$
$$W^2(Q) = \begin{pmatrix} P_1 \otimes P_0 \otimes P_0 \\ P_0 \otimes P_1 \otimes P_0 \\ P_0 \otimes P_0 \otimes P_1 \end{pmatrix}^{3 \times 1 \times 1} \quad \text{dim} = 6$$

$\int_\Omega$ $\boxed{\underline{\underline{D}}'^T \underline{\underline{M}}_{\mu^{-1}} \underline{\underline{D}}' \vec{\xi} + \partial_t^2 \underline{\underline{M}}_\varepsilon \vec{\xi} = -\partial_t \vec{\varphi}}$

$\in \mathbb{R}^{\mathring{N}_2 \times \mathring{N}_2}$ ($\mathring{N}_2$ = # of interior faces) $\qquad \underline{\underline{M}}_{f f'} = \int \underline{\underline{\mu}}^{-1} b_f^2 \cdot b_{f'}^2$

material matrix

$\underline{\underline{M}}_{e e'} = \int_\Omega \underline{\underline{\varepsilon}}\, b_e^1 \cdot b_{e'}^1 \qquad (\vec{\varphi})_e = \int_\Omega \vec{j} \cdot b_e^1$

$\vec{\xi}, \vec{\varphi} \in \mathbb{R}^{\mathring{N}_1}$: $\mathring{N}_1$: # interior edges

• Computation of $\underline{\underline{M}}_\varepsilon$. say $\underline{\underline{\varepsilon}} = \varepsilon \underline{\underline{I}}$:

$(\underline{\underline{M}}_\varepsilon)_{ee'} = \sum_{T \in \mathcal{T}_h} \int_\Omega \varepsilon\, b_e^1 \cdot b_{e'}^1$ $\qquad \underline{\underline{M}}_\varepsilon^T \leftarrow$ local cell

① Compute element material matrix $\left(\int_T \varepsilon\, b_e^1 \cdot b_{e'}^1 dx \right)_{e, e' \in \mathcal{E}_1(T)} \in \mathbb{R}^{6 \times 6}$

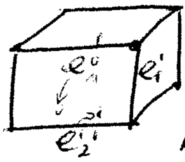
② Assemble $\underline{\underline{M}}_\varepsilon$ following local-to-global edge index mapping with relative orientation.

$$\int_T \varepsilon_0 \underbrace{(\lambda_i \nabla\lambda_j - \lambda_j \nabla\lambda_i) \cdot (\lambda_k \nabla\lambda_l - \lambda_l \nabla\lambda_k)}_{(\text{quadratic})} dx =$$

$$\int_T \lambda_i \lambda_j = |T| \cdot \overline{(3+2)!} \begin{cases} 1 \cdot 1 & i \neq j & \frac{1}{120} \\ 2 & i = j & \frac{1}{60} \end{cases}$$

RK: $\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}(x)$ uses numerical quadrature to evaluate $\underline{\underline{M}}_\varepsilon^T$ (at least exact for linear functions)

• Mass lumping:



$b_e \perp b_{e'}$

$b_e \cdot b_{e_2'} = 0$ on (a_i)

• Consider $Q \triangleq$ cube $\qquad \int_Q f(x) dx \approx \frac{|Q|}{8} \sum_{i=1}^{8} f(a_i)$ $\quad a_i$: eight vertices.

$$(\underline{\underline{M}}_\varepsilon)_{ee'} = \frac{|Q|}{8} \sum_{i=1}^{8} b_e(a_i)\, b_{e'}(a_i) = \begin{cases} 0 & \text{for all } e \neq e' \\ ? & (\text{Ex}) \end{cases}$$

▷ $\underline{\underline{M}}_\varepsilon$ is diagonal, when computed with vertices-based quadrature.

▷ FVM ⟹ Whitney form Galerkin discretization + special numerical quadrature

* consider $T \hat{=}$ tetrahedral mass lumping (open elusive ?)

$$\fint_T b_f \cdot b_{f'} \, dx = \begin{cases} 0 & \text{if } f \neq f' \\ ? \end{cases}$$

integration by means of numerical quadrature.

Constraint: ↳ this Mass lumping quadrature has to be exact for linear p. constant

$$\fint_T \underline{c}_1 \cdot \underline{c}_2 = \int_T \underline{c}_1 \cdot \underline{c}_2 \, dx \quad \forall \underline{c}_1, \underline{c}_2 \in \mathbb{R}^3$$

$$\Rightarrow \fint_T \underbrace{\operatorname{curl} \underline{b}_e}_{\text{const}} \cdot \underbrace{\operatorname{curl} b_{e'}}_{\text{const}} \, dx = \underbrace{\int_T \operatorname{curl} b_e \cdot \operatorname{curl} b_{e'}}_{(*)} \quad \forall e, e' \in \mathcal{S}_1(T)$$

(Ex: find geo cond. that (*) is zero for e. e' opposite to it.

? mass-lumping for 2-forms is impos.

Ex: "mass lumping holds for 1-forms" at least exact for constants. (open problem)

Rk: Diagonal material matrices are important for explicit timestepping