Lecture IV 10-10-2008

II. Co-chains and Whitney Forms

2.1. Meshes / Triangulations

(Review)

2.2. Co-chains $\mathcal{C}^l(\mathcal{T}_h)$

$$
\begin{array}{ccccccccc}
\mathcal{F}^0(\Omega) & \xrightarrow{d} & \mathcal{F}^1(\Omega) & \xrightarrow{d} & \mathcal{F}^2(\Omega) & \xrightarrow{d} & \mathcal{F}^3(\Omega) & \longrightarrow & \{0\} \\
\downarrow S_0 & & \downarrow S_1 & & \downarrow & & \downarrow & & \\
\mathcal{C}^0(\mathcal{T}_h) & \xrightarrow{d_h} & \mathcal{C}^1(\mathcal{T}_h) & \xrightarrow{d_h} & \mathcal{C}^2(\mathcal{T}_h) & \xrightarrow{d_h} & \mathcal{C}^3(\mathcal{T}_h) & \longrightarrow & \{0\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{R}^{N_v} & \xrightarrow{\underline{\underline{D}}_0} & \mathbb{R}^{N_e} & \xrightarrow{\underline{\underline{D}}_1} & \mathbb{R}^{N_f} & \xrightarrow{\underline{\underline{D}}_2} & \mathbb{R}^{N_c} & \longrightarrow & \{0\}
\end{array}
$$

2.3 Discrete cohomology

2.3.1 Discrete potentials

If in Ω all ~~edges~~ cycles are boundaries, the sequence (*) are exact.

2.3.2 Cohomology

$$
\boxed{\begin{array}{l} B_l(\mathcal{T}_h) \subsetneq Z_l(\mathcal{T}_h) \\ B_l(\Omega) \subsetneq Z_l(\Omega) \end{array}}
$$

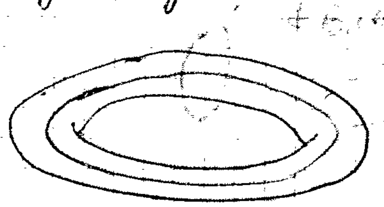
Ω: bdd. domain

$\mathcal{T}_h$: mesh of Ω

B_l: set of boundaries of all oriented $l+1$-surface

Z_l: set of l-dim oriented surface with empty boundary $:= \{\Sigma \in \mathcal{S}_l(\Omega),\ \partial\Sigma = \emptyset\}$

Example: torus



$\to\ \in Z_l(\Omega)$

$\to\ \notin B_l(\Omega)$

Define Equivalence relation of $Z_\ell(\Omega)$

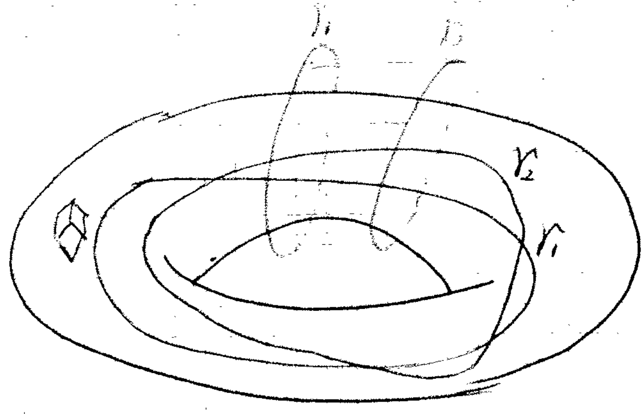
$\gamma_1, \gamma_2 \in Z_\ell(\Omega)$: $\gamma_1 \sim \gamma_2 \iff \gamma_1 \pm \gamma_2 \in B_\ell(\Omega)$

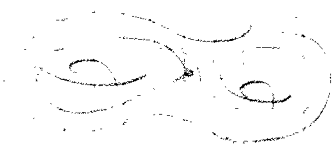
more precisely

$z_1 \gamma_1 + z_2 \gamma_2 \in B_\ell(\Omega)$

$z_1, z_2 \in \mathbb{Z}$ $\quad z_1 \neq 0, z_2 \neq 0$

(chain equivalence)





$\Rightarrow$ Equivalence Class $[Z_\ell(\Omega)]_N$

$$\dim [Z_\ell(\Omega)]_N = \underline{\ell\text{-th Betti number of } \Omega.}$$

$\underline{\beta_\ell(\Omega)}$: finite & topologically invariant.

torus:

$\beta_0(\Omega) = 1$ $\quad \beta_0(A_3|\Omega) = 1$ $\quad$ # of connected component.

$\beta_1(\Omega) = 1.$ $\quad \beta_1(A_3|\Omega) = 1.$ $\quad$ # of

$\beta_2(\Omega) = 0$ $\quad \beta_2(A_3|\Omega) = 0$ $\quad$ #

Theorem : $\beta_\ell(\mathcal{T}_h) = \beta_\ell(\Omega)$ if $\mathcal{T}_h$ triangulates Ω

Def: Cohomology - space : $\mathcal{H}_\ell(\Omega) = \text{kernel}(d|_{\mathcal{F}^\ell(\Omega)}) / \text{Image}(d|_{\mathcal{F}^{\ell-1}(\Omega)})$

ℓ-cohomology space of Ω $\quad$ quotient space

Theorem : $\dim \mathcal{H}_\ell(\Omega) = \beta_\ell(\Omega)$

$\ell = 1.$ if $\beta_1(\Omega) > 0 \iff \exists$ closed curves with no boundary

Find a representatives, $\gamma_1, \cdots \gamma_M$ $\quad M = \beta_1(\Omega)$ of equivalence class.

in $[Z_1(\Omega)]_N$.

Find $\xi_i \in \mathcal{F}^1(\Omega)$ s.t. $d\xi_i = 0$ & $\int_{\gamma_j} \xi_i = \delta_{ij}$

Pick $\omega \in \mathcal{F}^1(\Omega)$. $d\omega = 0$

$$\hat{\omega} = \omega - \sum_{j=1}^{N} \left(\int_{\gamma_j} \omega\right) \xi_j \quad \Rightarrow \quad \int_{\gamma_j} \hat{\omega} = 0 \quad \forall j$$

Consider a general cycle $\gamma \in \mathcal{Z}_1(\Omega)$

$$\exists\, z_j \in \mathbb{Z} : \quad \gamma + \sum_{j=1}^{M} z_j \gamma_j \in \mathcal{B}_1(\Omega)$$

$$d\hat{\omega} = 0 = 0 \;\Rightarrow\; 0 = \int_{\gamma + \sum_{j=1}^{M} z_j \gamma_j} \hat{\omega} = \int_\gamma \hat{\omega}$$

(lack of rigor)

Then $\exists\, \eta \in \mathcal{F}^0(\Omega)$ $d\eta = \hat{\omega}$

$$d\eta - \omega \in \operatorname{span}\{\xi_1 \cdots \xi_M\} \hat{=} \mathcal{H}_1 \qquad \#$$

Note: $\dim \mathcal{H}_\ell(\mathcal{T}_h) = \dim \mathcal{H}_\ell(\Omega)$

$$\Rightarrow \quad \underline{\dim \mathcal{H}_\ell(\mathcal{T}_h) = \dim \mathcal{B}_1(\mathcal{T}_h)} = \dim \left({\ker(\mathbb{D}_\ell)}/{\operatorname{Range}(\mathbb{D}_{\ell-1})} \right)$$

Def. Cohomology space of ℓ-cochains

$$\mathcal{H}_\ell(\mathcal{T}_h) = \operatorname{kernel}\left(d_h\big|_{\mathcal{C}^\ell(\mathcal{T}_h)}\right) / \operatorname{Image}\left(d_h\big|_{\mathcal{C}^{\ell-1}(\mathcal{T}_h)}\right)$$

2.3.3 Relative Cohomology (taking into account boundaries)

Skip

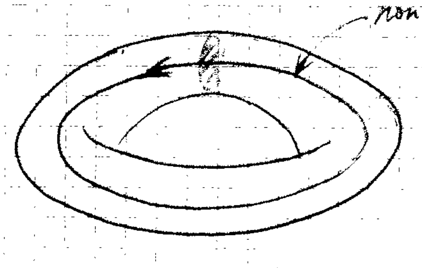
2.3.3 Representative of co-homology spaces

$\Rightarrow$ We seek closed ℓ-cochains / ℓ-forms that do not have a potential

$\exists$ non vanishing integral for some cycle!

Consider 3D: Ω with triangulation $\mathcal{T}_h$

Example: torus

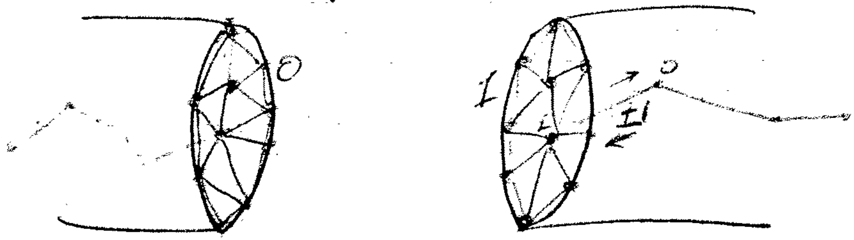


- Find* (dual) "cut" Σ

① surface consisting of 2-facets. ② $\partial\Sigma \subset \partial\Omega$ ③ Σ does not split Ω into two

* Poincaré duality theorem ; (difficult proof)

Consider $\hat{\mathcal{T}}_h$ as a triangulation of $\Omega \setminus \Sigma$



Define $\eta \in \mathcal{C}^0(\hat{\mathcal{T}}_h)$ $\quad \hat{\mathcal{T}}_h$: divide faces on Σ into "two"

— $\eta(x) = 0$ $\quad \forall$ vertices of $\hat{\mathcal{T}}_h$ ~~and~~ not on cut.

— $\eta(x) = 0$ $\quad$ on left side of $\hat{\mathcal{T}}_h$ of the cut Σ

— $\eta(x) = 1$ $\quad$ on right side of $\hat{\mathcal{T}}_h$ of the cut Σ

Consider $d\eta$:

— $d_h\eta(e) = 0$ $\quad \forall e$ on either sides on Σ
(constant on left (0) or on right (1))

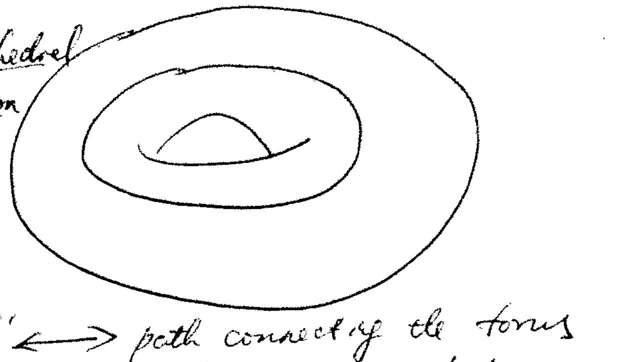
$\Rightarrow$ $d_h\eta \in \mathcal{C}^1(\mathcal{T}_h)$ ~~& $d_h\eta(e)$~~
* is a well-defined 1-cochain before $\mathcal{T}_h$ is cut into $\hat{\mathcal{T}}_h$.

$\Rightarrow$ $\int_\gamma d_h\eta = \pm 1$

$\Rightarrow$ $d_h(d_h\eta) = 0$ first considered as $\mathcal{C}^1(\hat{\mathcal{T}}_h)$. also as $\mathcal{C}^1(\mathcal{T}_h)$.

Ex.3 $\Omega \triangleq$ complement of a torus; $\mathcal{T}_h \triangleq$ tetrahedral triangulation
inside a big ball.

Find closed 2-cochain that is not the derivative of 1-cochain



Hint: Poincaré's Duality. "cut" $\longleftrightarrow$ path connecting the torus and the surface of the big ball

2.4. Finite Volume Discretization

Recall: discrete topological electromagnetic Laws (Sect 2.2)
also called "Network Equation"

$$(2.4.a) \quad d_h \underline{e}_h = -\partial_t \underline{b}_h$$
$$(2.4.b) \quad d_h \underset{\uparrow \text{1-cochain}}{\underline{h}_h} = \partial_t \underset{\uparrow \text{2-cochain}}{\underline{d}_h} + \underline{j}_h$$
] "disconnected"

Material Laws: connect $\underline{e}_h \Longleftrightarrow \underline{d}_h \qquad \underline{h}_h \Longleftrightarrow \underline{b}_h$ (bijective)

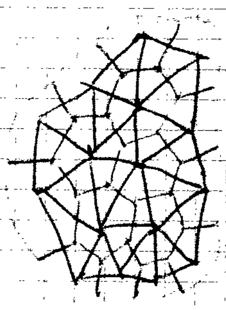
But: $\dim(\mathcal{C}^1(\mathcal{T}_h)) \neq \dim(\mathcal{C}^2(\mathcal{T}_h))$

Idea: Consider (2.4.a) (2.4.b) on different triangulations $\mathcal{T}_h$, $\hat{\mathcal{T}}_h$
for which: $N_\ell = \hat{N}_{3-\ell}$. $\quad N_\ell = \# \mathcal{S}_\ell(\mathcal{T}_h)$
$\hat{N}_\ell = \# \mathcal{S}_\ell(\hat{\mathcal{T}}_h)$

Special tool: (dual mesh technique)

Choose $\mathcal{T}_h$, $\hat{\mathcal{T}}_h$ as dual meshes.

2D example:



cell point: $\hat{N}_0 = N_2$ (# of cells)
cell edges: $\hat{N}_1 = N_1$ (# of edges)

$\mathcal{T}_h$ $\qquad \mathring{\hat{\mathcal{T}}}_h$ - interior of dual mesh (excluding all green ones)

$$\dim(\mathcal{C}^\ell(\mathcal{T}_h)) = \dim(\mathcal{C}^{2-\ell}(\mathring{\hat{\mathcal{T}}}_h))$$

2-ℓ-cochains with vanishing traces on the boundary of $\hat{\mathcal{T}}_h$.

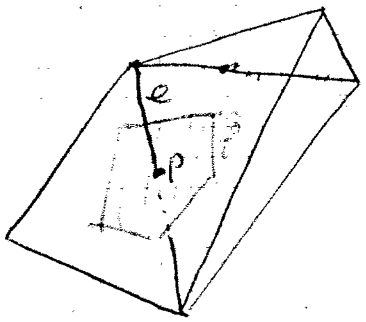
Ex: $\mathcal{T}_h \to \hat{\mathcal{T}}_h$.
Compute incident matrix
for $\mathcal{T}_h$, $\hat{\mathcal{T}}_h$. Check $\boxed{\underline{D}_\ell = \pm \underline{\hat{D}}^T_{d-\ell}}$

Discrete Material Laws: on dual meshes?

Assumption: Consider linear isotropic material Laws. $\underline{d} = \underline{\varepsilon}\, \underline{e}$

d_h defined on interior facets of dual mesh $\tilde{T}_h$

e_h defined on edges of primal mesh T_h.



$\underline{e}_h(e) = \int_e \underline{e} \cdot d\vec{s}$ — $\underline{e}$ - tangential component of $\underline{e}$

$\underline{d}_h(\hat{f}) = \int_f \underline{d} \cdot \underline{n}\, dS$ — $\hat{f}$ - normal component of $\underline{d}$

☺ : $\underline{e}_t$ match $\underline{d}_n$ if $\boxed{e \perp \hat{f}}$

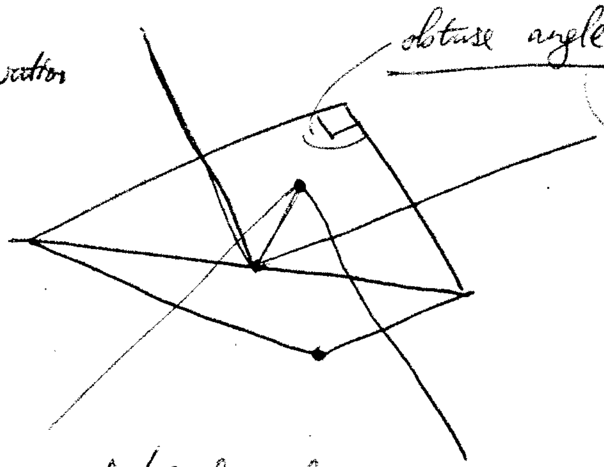
(2.4.c) $$\boxed{\underline{d}_h(\hat{f}) = \frac{|\hat{f}|}{|e|}\, \varepsilon(p) \cdot \underline{e}_h(e)}$$

Conclusion: orthogonal dual mesh is required.

For tetrahedral T_h: vertices of $\tilde{T}_h$ = centers of circumscribed sphere of cells of T_h

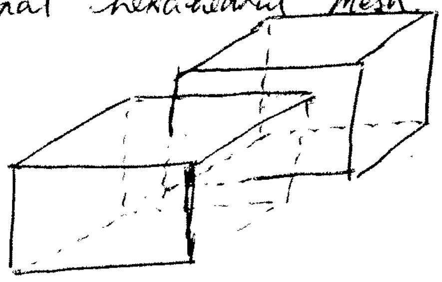
! Geometric Observation

obtuse angle → Delaunay triangulation



Extreme simple mesh/dual mesh

Orthogonal hexahedral mesh ⇒ Yee's Scheme (FIT) (finite integration technique)



▷ Discrete Maxwell equation: (FVM approach on dual meshes)

$$\begin{cases} \underline{D}_1 \vec{\underline{e}} = -\partial_t \vec{\underline{b}} \\ \widehat{\underline{D}}_1 \vec{\underline{h}} = \partial_t \vec{\underline{d}} + \vec{\underline{j}} \end{cases}$$

$$\vec{\underline{b}} = \underline{\underline{M}}_\mu \vec{\underline{h}} \qquad \underline{\underline{M}}_\mu \in \mathbb{R}^{N_{\ell+1} \times N_\ell} \text{ diagonal}$$

$$\vec{\underline{d}} = \underline{\underline{M}}_\varepsilon \vec{\underline{e}} \qquad \underline{\underline{M}}_\varepsilon \in \mathbb{R}^{N_{\ell+1} \times N_\ell}$$

$$\Rightarrow \quad \widehat{\underline{D}}_1 (\underline{\underline{M}}_\mu)^{-1} \underline{D}_1 \vec{\underline{e}} = -\partial_t^2 \underline{\underline{M}}_\varepsilon \vec{\underline{e}} - \partial_t \vec{\underline{j}}$$

For dual meshes $\quad \underline{D}_1^T (\underline{\underline{M}}_\mu) \underline{D}_1 \vec{\underline{e}} = -\partial_t^2 \underline{\underline{M}}_\varepsilon \vec{\underline{e}} - \partial_t \vec{\underline{j}}$

$$\underline{D}_\ell = \widehat{\underline{D}}^T_{n-(\ell+1)} \qquad n: \text{dim of space}$$

$$\mathbb{R}^{N_{\ell+1} \times N_\ell} \qquad \in \mathbb{R}^{\tilde{N}_{n-\ell} \times \tilde{N}_{n-\ell-1}}$$

$$\left(\because \widehat{\underline{D}}_{n-(\ell+1)} \in \mathbb{R}^{\tilde{N}_{n-(\ell+1)+1} \times \tilde{N}_{n-(\ell+1)}} \right)$$

2.5. Whitney Forms

(Geometry Integration Theory) 57 → 73 → 98 < Bossavit, Nedelec

Goal: from cochains ⟶ forms

⟶ seek linear Whitney maps $\mathcal{W}^\ell : \mathcal{C}^\ell(\mathcal{T}_h) \to \mathcal{F}^\ell(\Omega)$

($\mathcal{T}_h \hat{=}$ mesh of Ω)

Requirements of such maps :

(2.5.a) $\quad S_\ell \circ \mathcal{W}^\ell = Id \iff \int_\Sigma \mathcal{W}^\ell \omega_h = \omega_h(\Sigma) \quad \forall \Sigma \in \mathcal{S}_\ell(\mathcal{T}_h), \ \forall \omega_h \in \mathcal{C}^\ell(\mathcal{T}_h)$

$\mathcal{W}^\ell$ is a right inverse of S_ℓ

(2.5.b) $\quad \boxed{d \circ \mathcal{W}^{\ell+1} = \mathcal{W}^{\ell+1} \circ d_h}$

(2.5.c) $\quad$ Strict locality

$$\omega_h(f) = 0 \quad \forall f \in \mathcal{S}_\ell(\mathcal{T}_h),\ f \subset \underset{\text{tetrahedral}}{T}$$

$$\Rightarrow \mathcal{W}^\ell \omega_h \big|_T = 0 \qquad T: \text{cell of } \mathcal{T}_h \text{ in } \mathcal{S}_n(\mathcal{T}_h)$$

(2.5.d) $\quad VP(W^l \omega_h)$ is $\mathcal{T}_h$-piecewise polynomial

▷ Def 2.5.A. Space of Whitney l-forms

$$\mathcal{W}_0^l(\mathcal{T}_h) = W^l(\mathcal{C}^l(\mathcal{T}_h)) \qquad 0: \text{ lowest order Whitney form}$$

Def. 2.5.B. Nodal interpolation $I^l : \mathcal{F}^l(\Omega) \longrightarrow \mathcal{W}_0^l(\mathcal{T}_h)$

$$\boxed{I^l := W^l \circ S_l}$$

Lemma 2.5.C (Commuting Diagram Property)

$$d \circ I^l = I^{l+1} \circ d$$

proof: $d \circ I^l = d \circ W^l \circ S_l = W^{l+1} \circ d_h \circ S_l = W^{l+1} \circ S_{l+1} \circ d = I^{l+1} \circ d$

Note $\quad - \ (I^l)^2 = I^l \quad$ by (2.5.a)

$\qquad - \ (2.5.c) \Rightarrow \omega|_T = 0 \Rightarrow I^l \omega|_T = 0.$

2.5.1. Simplicial Whitney Forms

tetrahedron $T = [a_1 \ a_2 \ a_3 \ a_4]$ $\quad a_i \in A_3$

$l = 0$: $\quad W^0 \rightarrow$ linear interpolation (local on T)

$$(W^0 \omega_h)(x) = \underbrace{\sum_{i=1}^{4} \lambda_i(x)}_{\text{"barycentric coordinate functions"}} \omega_h(a_i) \qquad \omega_h \in \mathcal{C}^0(\mathcal{T}_h)$$

~~l=1~~: (change our perspective)

Note: $\quad \boxed{p = \sum_{i=1}^{4} \lambda_i(p)\, a_i}$ $\quad$ p is a weighted sum of vertices a_i

p: 0-facet in $\mathcal{T}_h$; weighted sum: same weights; a_i: 0-facets in $\mathcal{T}_h$

$l = 1$.

$l=1$: $[x,y] \subset T$ line segment

$$[x,y] = \{ tx + (1-t)y,\ 0 \le t \le 1 \}$$
$$= \left\{ t \sum_{i=1}^{4} \lambda_i(x) a_i + (1-t) \sum_{j=1}^{4} \lambda_j(y) a_j,\ 0 \le t \le 1 \right\}$$
$$= \left\{ t \sum_{i=1}^{4} \sum_{j=1}^{4} \lambda_i(x) \lambda_j(y) a_i + (1-t) \sum_{j=1}^{4} \sum_{i=1}^{4} \lambda_j(y) \lambda_i(x) a_j,\ 0 \le t \le 1 \right\}$$
$$= \left\{ \sum_{i=1}^{4} \sum_{j=1}^{4} \lambda_i(x) \lambda_j(y) [\, t a_i + (1-t) a_j \,],\ 0 \le t \le 1 \right\}$$

▷ Suggests

$$\int_{[x,y]} W^1 \omega_k = \sum_{i=1}^{4} \sum_{j=1}^{4} \omega_k([a_i, a_j])\, \lambda_i(x) \lambda_j(y)$$
$$= \sum_{1 \le i < j \le 4} (\lambda_i(x) \lambda_j(y) - \lambda_j(x) \lambda_i(y))\, \omega_k([a_i, a_j])$$

<u>Definite of W^1</u>

For $f \in T$ face $f = [a_1, a_2, a_3]$ $\quad \lambda_4(x) = 0$ on f.

▷ $\omega_k([a_k, a_4]) = 0 \quad \forall k = 1, 2, 3$.

⇛ $W^1 \omega_k|_f$ determined by $\omega_k([a_i, a_j])$ $1 \le i < j \le 3$

⇛ $W^1 \omega_k|_f$ is independent to tetrahedra adjacent to face f. (continuity across faces)

▷ $W^1 \omega_k$ with edges patch together indeed defines global 1-forms.

(2.5.a) (2.5.c) ✓

(2.5.b) Ex $\boxed{l=0}$ To show: $\displaystyle\int_{\partial [x,y]} W^0 \omega_k = \int_{[x,y]} W^1 (d_k \omega_k)$

$\Rightarrow$ LHS $= W^0 \omega_k(y) - W^0 \omega_k(x) = \displaystyle\sum_{i=1}^{4} (\lambda_i(y) - \lambda_i(x)) a_i$ $\quad \sum_{j=1}^{4} \lambda_j(x) \quad \sum_{j=1}^{4} \lambda_j(y)$

$\Rightarrow$ RHS $= \displaystyle\sum_{1 \le i < j \le 4} \sum_{j=1}^{4} \lambda_i(x) \lambda_j(y) (d_k \omega([a_i, a_j])) = \sum_i \sum_j \lambda_i(x) \lambda_j(y) (\omega(a_j) - \omega(a_i$

Ex: $\boxed{l=1}$ $\displaystyle\int_{\partial [x,y]} W^1 \omega_k = \int_{[x,y]} W^2 (d_k \omega_k)$

☆1 $$\int_{\Delta[x,y,z]} W^2 \omega_h \triangleq \sum_i \sum_j \sum_k \lambda_i(x)\lambda_j(y)\lambda_k(z)\, \omega_h([a_i, a_j, a_k])$$

$$\omega_h \in \mathcal{C}^2(\mathcal{T}_h)$$

☆3 $$\int_{\triangle[w,x,y,z]} W^3 \omega_h \triangleq \lambda_i(w)\lambda_j(x)\lambda_k(y)\lambda_l(z)\, \omega_h([a_1, a_2, a_3, a_4])$$

$$\omega_h \in \mathcal{C}^3(\mathcal{T}_h)$$

$$= \mathrm{sgn}(i,j,k,l)\,(\lambda_i(w)\lambda_j(x)\lambda_k(y)\lambda_l(z))\, \omega_h([a_1, a_2 \dots a_4])$$

$$= \det \underbrace{\begin{pmatrix} \lambda_1(w) & \lambda_1(x) & \lambda_1(y) & \lambda_1(z) \\ \lambda_2(w) & \lambda_2(x) & \lambda_2(y) & \lambda_2(z) \\ \lambda_3(w) & \lambda_3(x) & \lambda_3(y) & \lambda_3(z) \\ \lambda_4(w) & \lambda_4(x) & \lambda_4(y) & \lambda_4(z) \end{pmatrix}}_{B} \omega_h([a_1 \cdot a_2 \cdot a_4])$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}$$

$$|T| = \det(A) \qquad = \frac{\pm \triangle[w,x,y,z]}{|T|}$$

(± determined by relative orientation of $\triangle[w,x,y,z]$ and T.

$$A^T \cdot B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_w & x_x & x_y & x_z \\ y_w & y_x & y_y & y_z \\ z_w & z_x & z_y & z_z \end{pmatrix} \Rightarrow \det(A^T B) = |\triangle_{[w,x,y,z]}|$$

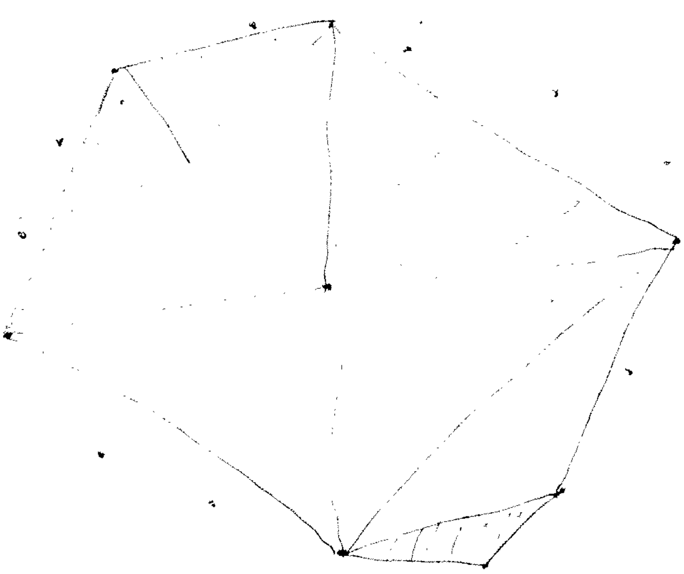
☆2 Answer II: find a surface with a boundary given by a closed loop in a 3d mesh $\mathcal{T}_h$ of Ω (simply-connected)

< Divide + Conquer >:

nested mesh $\mathcal{T}_{h_i}$

$\mathcal{T}_{h_0}$: (coarse) C

very deep knowledge of Graph theory.



1. Closed loops can be represented at different level of mesh
2> Multilevel idea!!
3> Minimum spanning tree Algo. (Prim's Algo)