3-10-2008 Lecture III

Variational forms: (Review)

$\underline{e}$-based VF: seek $\underline{e} \in \mathcal{F}^1(\Omega)$ s.t. for $\forall e' \in \mathcal{F}(\Omega)$

$$< D E_{mag}(d\underline{e}), d\underline{e}'> + \partial_t^2 < D E_{el}(\underline{e}), \underline{e}'> + \int_{\partial\Omega} \partial_t \underline{h} \wedge e' = -\partial_t \int_\Omega \underline{j} \wedge e'$$

$\underline{h}$-based VF: Seek $\underline{h} \in \mathcal{F}^1(\Omega)$ s.t. for $\forall \underline{h}' \in \mathcal{F}(\Omega)$

$$< D E_{el}(d\underline{h} - \underline{j}), d\underline{k}'> + \partial_t^2 < D E_{mag}(\underline{h}), \underline{h}'> + \int_{\partial\Omega} \partial_t \underline{e} \wedge \underline{h}' = 0.$$

Boundary Conditions

	$\underline{e}$-based	$\underline{h}$-based
PEC ($\gamma_{\partial\Omega} \underline{e} = 0$)	strongly	weak
PMC ($\gamma_{\partial\Omega} \underline{h} = 0$)	weakly	strongly

Ex: solution of $\underline{h}$-based formulation without $\int_{\partial\Omega}$ satisfies PEC b.c. $\triangle$
(for linear isotropic materials)

1.9. Potential based Variational forms (Supplement to 1.7)

F.L. $\Rightarrow$ $\boxed{\underline{e} = -\partial_t \underline{a} - dv}$ (1.9.1) $\Rightarrow d\underline{e} = \partial_t(-d\underline{a}) - 0$

plug into $\underline{e}$-based V.F.

Seek $\underline{a} \in \mathcal{F}^1(\Omega)$, $v \in \mathcal{F}^0(\Omega)$ s.t. (*: take ∂_t out)

$$< D E_{mag}(-d\underline{a}), d\underline{a}'> + \partial_t < D E_{el}(-\partial_t \underline{a} - dv), \underline{a}'> \quad (1.9.a)$$
$$- \int_{\partial\Omega} \underline{h} \wedge \underline{a}' = - \int_\Omega \underline{j} \wedge a' \quad \forall a' \in \mathcal{F}(\Omega)$$

In vector proxies & linear material

$$-\int_\Omega \underline{\mu}^{-1} \operatorname{curl} \underline{a} \operatorname{curl} \underline{a}' - \partial_t \int_\Omega \varepsilon(\partial_t \underline{a} + \operatorname{grad} v) \cdot a' \, dx$$
$$- \int_{\partial\Omega} (\underline{h} \times \underline{a}') \cdot n \, dS = - \int_\Omega \underline{j} \cdot a' \, dx \quad \forall a' \quad (1.9.b)$$

Remarks on B.C.

PEC: impose strongly $\gamma_{\partial\Omega} \underline{a} = 0$ $\gamma_{\partial\Omega} v = 0$

PMC: weakly $\int_{\partial\Omega}$ cancels

∇: Need Gauge Condition $\longrightarrow$ nonuniqueness of $\underline{a}$. v

Given $\underline{e}$ determine $\underline{a}$. v in a unique way

- Lorenz gauge:

$$\boxed{\underbrace{d *_{\kappa} \underline{a}}_{\substack{3\text{-form} \\ d(*\,1\text{-form})}} = -\underbrace{*_{\beta} \partial_t v}_{\substack{3\text{-form} \\ *\,0\text{-form}}}} \quad (1.9.2) \quad \Longleftrightarrow \quad \operatorname{div}(\underline{\underline{\kappa}}\, \underline{a}) = \beta\, \partial_t v$$

different Hodge operator (material laws)

material tensor / uniformly positive function

Check uniqueness:

i) In forms: take $d *_{\kappa}$ on both sides of (1.9.1)

$$d *_{\kappa} \underline{e} = -\partial_t\, d *_{\kappa} \underline{a} - d *_{\kappa} dv$$

plug (1.9.2)

$$= -\partial_t^2 *_{\beta} v - d *_{\kappa} dv$$

ii) in V.P. $\operatorname{div}(\underline{\underline{\kappa}}\, \underline{e}) = -\partial_t^2 \beta v - \operatorname{div}(\underline{\underline{\kappa}} \operatorname{grad} v)$

So given $\underline{e}$ with suitable B.C.s $\Rightarrow$ v can be determined uniquely

Strong Lorenz gauge in (1.9.b) with $\underline{\underline{\kappa}} = \underline{\underline{\varepsilon}}$, $\beta = 1$.

(pointwise gauge)

Be cautious!

$$-\partial_t \int_{\Omega} \underline{\underline{\varepsilon}} \operatorname{grad} v \cdot \underline{a}'\, dx \underset{I.B.P.}{=\!=} \partial_t \int_{\Omega} v \operatorname{div}(\underline{\underline{\varepsilon}}\, \underline{a}')\, dx - \int_{\partial\Omega} \underline{\underline{\varepsilon}}\, \underline{a}' \underline{n}\, \partial_t v\, dS$$

$$\underline{\underline{-\partial_t v = \operatorname{div}(\underline{\underline{\varepsilon}}\, \underline{a})}} \quad -\int_{\Omega} \operatorname{div}(\underline{\underline{\varepsilon}}\, \underline{a}) \cdot \operatorname{div}(\underline{\underline{\varepsilon}}\, \underline{a}')\, dx - \int_{\partial\Omega} \operatorname{div}(\underline{\underline{\varepsilon}}\, \underline{a}) \cdot (\underline{\underline{\varepsilon}}\, \underline{a}' \cdot \underline{n})\, dS$$

In (1.9.b)

$$\int_{\Omega} \underline{\underline{\mu}}^{-1} \operatorname{curl} \underline{a} \cdot \operatorname{curl} \underline{a}'\, dx + \int_{\Omega} \operatorname{div}(\underline{\underline{\varepsilon}}\, \underline{a}) \cdot \operatorname{div}(\underline{\underline{\varepsilon}}\, \underline{a}')\, dx - \partial_t^2 \int \underline{\underline{\varepsilon}}\, \underline{a} \cdot \underline{a}'\, dx$$

$$-\int_{\partial\Omega} (\underline{h} \times \underline{a}') \underline{n}\, dS + \int_{\partial\Omega} \operatorname{div}(\underline{\underline{\varepsilon}}\, a) (\underline{\underline{\varepsilon}}\, \underline{a}' \cdot n)\, dS = \int_{\Omega} \underline{j} \cdot \underline{a}'\, dx$$

$\forall\, \underline{a}' \in \mathcal{F}(\Omega)$

Ex 2. fix B.Cs for PEC. PMC b.c.

• Weak Lorenz Gauge

$$\int_\Omega \underline{\underline{\kappa}}\,\underline{a}\cdot \mathrm{grad}\, v'\, dx = \int_\Omega \beta \overset{? \partial_t}{v}\, v'\, dx \quad \forall v' \in \mathcal{F}^0(\Omega) \quad (1.9.d)$$

test (1.9.b) with $\underline{a}' = \mathrm{grad}\, v'$ and use (1.9.d) with $\underline{\underline{\kappa}} = \underline{\underline{\varepsilon}}$, $\beta = 1$.
(extract one ∂_t) $\Rightarrow$ or integrate in time

$\Rightarrow$

$$-\partial_t^2 \int_\Omega \beta v\, v'\, dx - \int_\Omega \underline{\underline{\varepsilon}}\, \mathrm{grad}\, v \cdot \mathrm{grad}\, v'\, dx$$

$$- \underbrace{\int_{\partial\Omega}}_{?\int_0^t} (\underline{h} \times \mathrm{grad}\, v')\cdot n\, dS = - \underbrace{\int_\Omega}_{?\int_0^t} \underline{j}\cdot \mathrm{grad}\, v'\, dx \quad \forall v' \in \mathcal{F}^0(\Omega) \quad (1.9.d.0)$$

(RK: we have to solve v from (1.9.d.0)

but we still have to solve $\underline{a}$ from (1.9.b)

to reconstruct $\underline{e}$, no obvious gain.

Alternatively: Augment (1.9.b) with (1.9.d) ↑ "weak gauge constraint"

(also applies to (1.9.a))

$\Rightarrow$ $\underset{\text{Mixed}}{\text{variational}}$ problem on $\mathcal{F}^1(\Omega) \times \mathcal{F}^0(\Omega)$

• Coulomb Gauge: $\boxed{d *_\kappa \underline{a} = 0}$ (strong form) (1.9.3)

Special case for Lorenz gauge by setting $\beta = 0$

weak: $\int_\Omega \underline{\underline{\kappa}}\, \underline{a}\cdot \mathrm{grad}\, v'\, dx = 0 \quad \forall v' \in \mathcal{F}^0(\Omega)$

• Temporal gauge $\boxed{v = 0}$ (1.9.4)

view $\underline{e}$ of $\underline{a}$

*: further reading materials
rp on comp. EM

(Add voltage current condition for stable low freq. Maxwell eqns)

Chap I #

Chap. II. Co-chains and Whitney Forms

§ 2.1 Volume meshes

Goal: discrete-forms $\Longleftrightarrow$ finite amount of information.

Idea: $\hookrightarrow$ mapping: {finite set of oriented surfaces} $\longrightarrow \mathbb{R}$

$\hookrightarrow$ closed under "boundary operations".

Concepts to be elucidated: (Closedness constraint)

Given: $\Omega \subset A_3$ bounded "computational domain".

Def 2.1.A Mesh / triangulation of $\mathcal{T}_h$ of Ω.

$\hat{=}$ finite collection of (open) oriented cells (– set $\mathcal{S}_3(\mathcal{T}_h)$)

— — — faces $\mathcal{S}_2(\mathcal{T}_h)$

— — — edges $\mathcal{S}_1(\mathcal{T}_h)$

— — — vertices $\mathcal{S}_0(\mathcal{T}_h)$

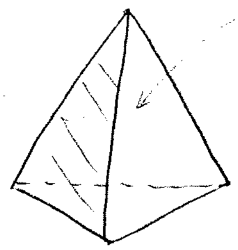
satisfies (constraints)

- Every $T \in \mathcal{S}_3(\mathcal{T}_h)$ is the diffeomorphism image of some polyhedron.
- $\bigcup_{l=0}^{3} \bigcup_{S \in \mathcal{S}_l(\mathcal{T}_h)} S$ is a partition of Ω.
- $\boxed{\forall S \in \mathcal{S}_l(\mathcal{T}_h).\quad \partial S = \bigcup_{i=1}^{P_S} f_i \quad \text{for some } P_S \in \mathbb{N}.\ f_i \in \mathcal{S}_{l-1}(\mathcal{T}_h)}$ closedness constraint
- $0 \le l < 3$. $\forall S \in \mathcal{S}_l(\mathcal{T}_h)$. $\exists T \in \mathcal{S}_{l+1}(\mathcal{T}_h)$ s.t. $S \in \partial T$.

  (No low-dim boundary is isolated)

Example: tetrahedral mesh

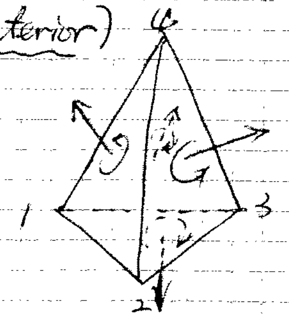
$f \in \mathcal{S}_2(\mathcal{T}_h)$ $\quad$ $S \in \mathcal{S}_3(\mathcal{T}_h)$



(hanging nodes is allowed by Def 2.1.A.)

Orientation:

(Interior)



tetrahedron $[a_1, a_2, a_3, a_4]$

→ orientation fixed by ordering vertices, up to permutation of sign +

e.g. Cyclic permutation ⇒ keep orientation unchanged.

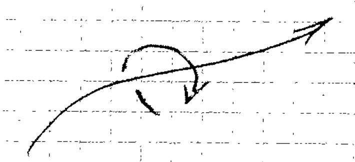
Induced orientation: of face $[a_i, a_j, a_k]$ $i<j<k$

$= (-1)^{l+1}$ where $\{i, j, k, l\} \in \{1, 2, 3, 4\}$

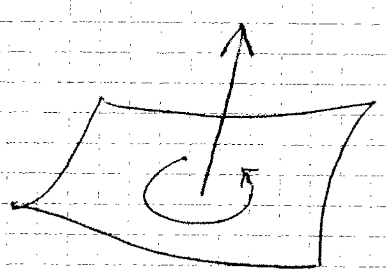
Exterior Orientation:

by (r. h. rule)

$l = 1$

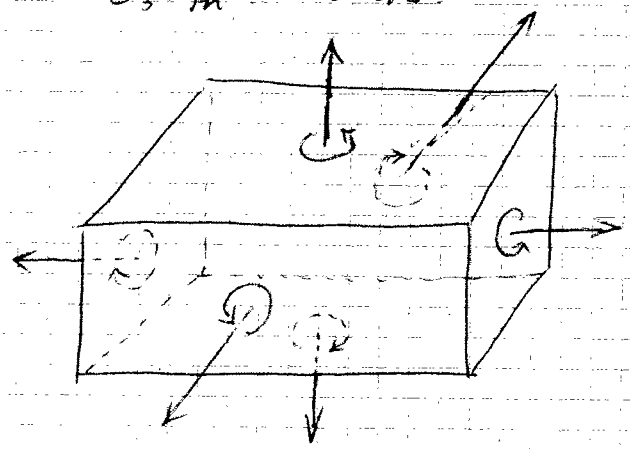


$l = 2$



Example: hexahedral mesh.

$S_3(T_h) \hat{=}$ bricks



Distinguish:

inner orientation ⟷ exterior orientation

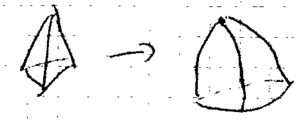
induced orientation (of a boundary)

Data Structure: Class FEM_Topology

{ vector <nodes>

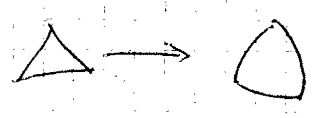
Vector <TET parabolic> → define curved edges

└ get_node(i)



Vector <TRIA parabolic> →

└ only subface triangles.



Vector<edges>

edges: low indexed node ⇒ high-indexed nodes

triangles.

- Geometric infor: coordinates of nodes
- Connection info: network matrix.

TET

{ get - main-node-index (i) $1 \le i \le 4$

FEM_Topol -> get_node(m) -> get_r( )

(node-m) (coordinates)

§ 2.2 Co-chains

Given $\mathcal{T}_h \hat{=}$ mesh of Ω.

Def 2.2.A : l-cochain $\vec{\omega}$, $0 \le l \le 3$. on $\mathcal{T}_h$ is a mapping

mapping : $\mathcal{S}_l(\mathcal{T}_h) \longrightarrow \mathbb{R}$

Vector space $\mathcal{C}^l(\mathcal{T}_h) = \{ l\text{-cochain } \vec{\omega} \}$

$\dim \mathcal{C}^l(\mathcal{T}_h) = \# \mathcal{S}_l(\mathcal{T}_h)$: # of l-facets.

Let $\mathcal{T}_h'$ be a submesh of $\mathcal{T}_h$.
(may be the boundary mesh (submesh tiling the boundary))

$\boxed{\bullet \text{ trace}}$ $(\gamma_{\mathcal{T}_h'} \vec{\omega})(S) := \vec{\omega}(S) \quad \forall S \in \mathcal{S}_l(\mathcal{T}_h') \subset \mathcal{S}_l(\mathcal{T}_h)$

$\longrightarrow \gamma_{\partial\Omega}(\vec{\omega}) \quad (\mathcal{T}_h' \text{ tiling } \partial\Omega)$

- discrete exterior derivative

relative orientation
$$O_r(F, f) = \begin{cases} +1 & \text{if or. of } f = \text{induced or. of } f \text{ w.r.t. } F, \\ -1 & \text{if } \cdots \neq \cdots \\ 0 & \text{if } f \not\subset \partial F. \end{cases} \qquad \begin{array}{l} F \in \mathcal{S}_l(\mathcal{T}_h) \\ f \in \mathcal{S}_{l-1}(\mathcal{T}_h) \end{array}$$

Ex3. implement this function $\sigma_r(F, f)$ in 3D (refer. 2D example first)

$F \triangleq$ tetrahedron described by four integers $(7.\ 30.\ 20.\ 15)$

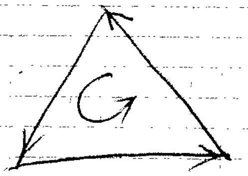
$f \triangleq$ triangle — — — three — — $(1.\ 3.\ 2)$ $(7.\ 15.\ 30)$

Def. 2.2.B. Discrete exterior derivative is a mapping

$$d_h : \mathcal{C}^l(\mathcal{T}_h) \to \mathcal{C}^{l+1}(\mathcal{T}_h)$$

$$d_h \vec{\omega}(F) = \sum_{\substack{f \subset \partial F \\ f \in \mathcal{S}_l(\mathcal{T}_h)}} \sigma_r(F, f)\, \vec{\omega}(f) \qquad \forall F \in \mathcal{S}_{l+1}(\mathcal{T}_h)$$

e.g.



Recall

$$\int_F d\omega = \int_{\partial F} \omega = \sum_{f \subset \partial F} \overbrace{\int_f \omega}^{\sigma_r(F,f)} = \sum_{f \subset \partial F} \sigma(F,f) \overbrace{\int_f \omega}^{\omega(f)}$$

"rewrite standard exterior derivatives for specific test manifold"

Properties: $d_h \triangleq$ linear operator.

Since $\mathcal{C}^l(\mathcal{T}_h) \cong \mathbb{R}^{N_l}$: $N_l \triangleq \# \mathcal{S}_l(\mathcal{T}_h)$ $\to d_h \longleftrightarrow$ matrix

$\to \underline{\underline{D}}^l \in \mathbb{R}^{N_{l+1} \times N_l}$

[$e_f = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ no. of f, $f \in \mathcal{S}^l(\mathcal{T}_h)$ "global"

$$\left(\underline{\underline{D}}^l \cdot e_f\right)_{F \in \mathcal{S}^{l+1}(\mathcal{T}_h)} = (d_h \vec{\omega}_f)(F) = \sigma_r(F, f)$$

$\uparrow$ l-cochains assigning 1 to f and 0 to all others.

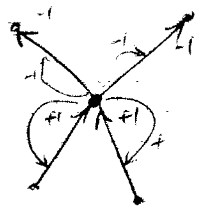
$\underline{\underline{D}}^l \in \{-1, 0, 1\}^{N_{l+1} \times N_l}$

$\hookrightarrow$ "incidence matrix" of oriented l-facets and $l+1$-facets of $\mathcal{T}_h$

$$\underline{\underline{D}}^l_{(F, f)} = \sigma_r(F, f)$$

invariant under deformation of the mesh $\mathcal{T}_h$

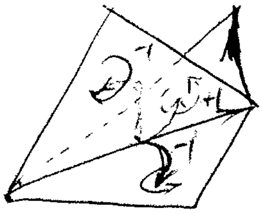
Examples: $\ell = 0$



$$\mathrm{Gr}(\text{node}, \text{edge}) = \begin{cases} -1 & \text{node} = \text{starting node} \\ +1 & \text{node} = \text{end node} \end{cases}$$

$\underline{\underline{D}}^0$: discrete gradient.

$\ell = 1$



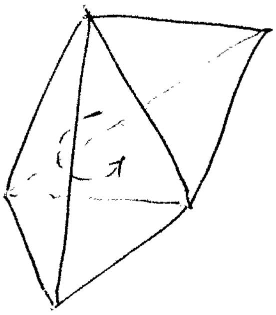
$$\mathrm{Gr}(\text{edge}, \text{face}) = \begin{cases} -1 & \text{intrinsic or } F \neq \text{induced or of } F \text{ w.r.t } f \\ +1 & = \end{cases}$$

blue: intrinsic orientation

red: induced orientation

$\underline{\underline{D}}^1$: discrete curl

$\ell = 2$



$$\mathrm{Gr}(\text{face}, \text{cell}) = \Big\{$$

$\underline{\underline{D}}^2$: discrete divergence.

Lemma 2.2.C. $\quad d_h \circ d_h = 0 \iff \underline{\underline{D}}^{\ell+1} \cdot \underline{\underline{D}}^{\ell} = 0$ Ex. tetra mesh $\ell = 0$

Advantage: discrete topological Laws for Maxwell Eqns.

continuous		discrete	
electric field	$\underline{e}$	1-co-chain	$\vec{e} \in \mathbb{R}^{N_1}$
magnetic induction	$\underline{b}$	2-cochain	$\vec{b} \in \mathbb{R}^{N_2}$
magnetic field	$\underline{h}$	1-co-chain	$\vec{h} \in \mathbb{R}^{N_1}$
displacement current	$\underline{d}$	2-co-chain	$\vec{d} \in \mathbb{R}^{N_2}$.

P30

Continuous Laws	discrete Laws	
$d\underline{e} = -\partial_t \underline{b}$	$\underline{D}^1 \vec{\underline{e}} = -\partial_t \vec{\underline{b}}$	(F.L.)
$d\underline{h} = \partial_t \underline{d} + \underline{j}$	$\underline{D}^1 \vec{\underline{h}} = \partial_t \vec{\underline{d}} + \vec{\underline{j}}$	(A.L.)

RK
$$\begin{cases} \underline{D}^2 \vec{\underline{b}} = 0 & (\text{by take } \underline{D}^2 \text{ on discrete F.L.}) \\ \partial_t \underbrace{\underline{D}^2 \vec{\underline{d}}}_{\vec{\rho}\ \text{discrete charge}} = -\partial_t \underline{D}^2 \vec{\underline{j}} & - \,-\, - \,(A.L.) \end{cases}$$

Def: De Rham Map $\mathcal{S}_l : \mathcal{F}^l(\Omega) \to \mathcal{C}^l(\mathcal{T}_h)$

defined by $(\mathcal{S}_l(\omega))(F) = \int_F \omega$. $F \in \mathcal{S}_l(\mathcal{T}_h)$

De Rham map $:\hat{=}$ Sampling operator

▷ If $\underline{e} \in \mathcal{F}^1(\Omega)$, $\underline{b} \in \mathcal{F}^2(\Omega)$ satisfy F.L. Then
$\mathcal{S}_1 \underline{e}$, $\mathcal{S}_2 \underline{b}$ satisfies the discrete F.L. } perfect Consistency !

Which automatically shows the consistency of our numerical scheme if we adopt the jargon of differential forms as Finite Elements.

T Recall: Consistency: Conti. Prob. $Tu = f$ $\quad T: X \to Y$

discrete Prob. $T_h u_h = f_h$ $\quad T_h : X_h \to Y_h$

Sampling $\mathcal{S}_X : X \to X_h$, $\mathcal{S}_Y : Y \to Y_h$ $\Rightarrow$ $\boxed{T_h \mathcal{S}_X u = \mathcal{S}_Y f}$ consistency

$= \mathcal{S}_Y T u$

§2.3 Discrete Cohomology

§2.3.1 Discrete Potentials

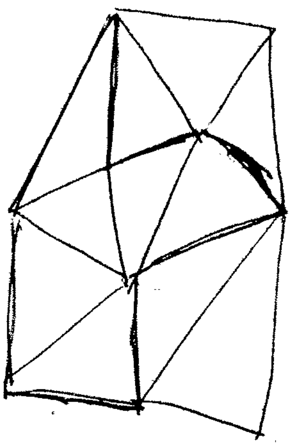
$\mathcal{T}_h : \hat{=}$ mesh of $\Omega \subset \mathbb{A}_3$

Thm 2.3.A. $\left.\begin{array}{l}\vec{w} \in \mathcal{C}^{\ell}(\mathcal{T}_h) \\ d_h \vec{w} = 0\end{array}\right\} \Longrightarrow \begin{array}{l}\exists\, \vec{\eta} \in \mathcal{C}^{\ell-1}(\mathcal{T}_h) \\ \text{s.t. } \vec{w} = d_h \vec{\eta}\end{array}$

Provided that the following assumptions hold:

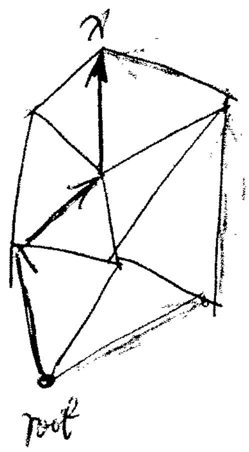
Assumption 2.3.B $\quad Z_\ell(\mathcal{T}_h) = B_\ell(\mathcal{T}_h)$

set of all oriented ℓ- ~~surfaces facets~~ surfaces composed of ℓ-facets of $\mathcal{T}_h$ for which the boundary is empty. $\}$ $\longrightarrow$ //← besides which are the boundary of an $\ell+1$ surface.



$Z_1(\mathcal{T}_h)$: ~~minimum spanning tree~~ closed loops

Proof. In particular $\ell = 1$, $\Sigma \in B_\ell(\mathcal{T}_h)$. $\Sigma = \{S_i$



$\vec{\eta}(x) = \sum_{S \in \gamma_x} \pm \vec{w}(S)$ modulated by relative orientation $\}$ definition independent of choice of path γ_x.

$= \sum_{S \in \gamma_y} \pm \vec{w}(S)$

well-defined since $\gamma_x - \gamma_y \in Z(\mathcal{T}_h)$

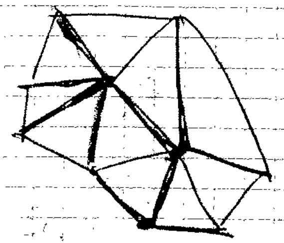
By Assumption 2.3.B $\Rightarrow$ boundary of surface formed a union of 2-facets.

$$\sum_{S \in \gamma_x - \gamma_y} \pm \vec{w}(S) = \sum_{S \in \Sigma} \pm d\vec{w}(S) = 0$$

For $\ell = 2$. Algorithm to construct:

— Build spanning edge tree T of $\mathcal{T}_h$

2D



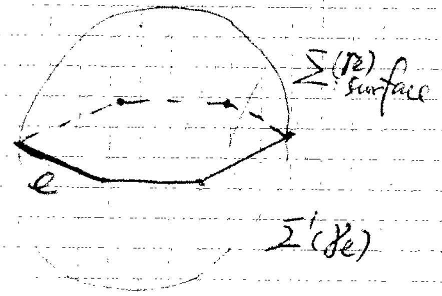
- visit each node
- no cyclic path (· minimum spanning tree)

$\Rightarrow \forall$ edge $e \in \mathcal{S}_1(\mathcal{T}_h) \setminus T$: (cotree)

$\exists !$ edge path $\gamma_e \subset T$ s.t. γ_e connect the end points of e.

By Assump 2.3.B. $\Rightarrow \exists$ 2-surface $\Sigma(\gamma_e [\ldots])$ composed of 2-facets s.t. its boundary is $\gamma_e \cup \{e\} = \partial \Sigma(\gamma_e [\ldots])$.

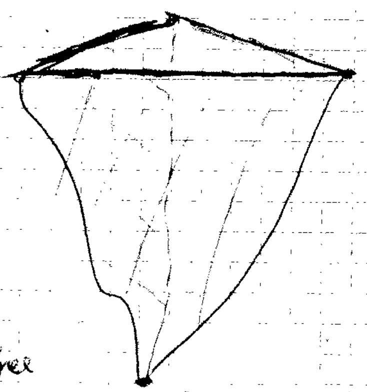
$d_h \vec{\omega} = 0 \triangleright \sum_{S \in \Sigma(\gamma_e [\ldots])} \pm \vec{\omega}(S)$ independent of choice of $\Sigma(\gamma_e \cup \{e\})$



$\triangleright$ define $\vec{\eta}(e) = \sum_{S \in \Sigma(\gamma_e [\ldots])} \pm \vec{\omega}(S)$

& $\vec{\eta}(e) = 0$ whenever $e \in T$.

three cotree edges



$\eta(e_{blue} + e_{red} + e_{green})$

$= \vec{\omega}(Surface\_blue) + \vec{\omega}(S_{red}) + \vec{\omega}(S_{green})$

$= \pm \vec{\omega}$ (Triangle)

Since $Triag \cup S_{blue} \cup S_{red} \cup S_{green} = \partial Volume$

TRS: ① prove for $\ell = 3$ using spanning tree techniques

② implement spanning tree Algo to find 2-cochain potential for 3-cochain