Lecture XII   5-12-2008

4.3. Maxwell Cavity Source Problem (in frequency domain)

↳ V.F. in bounded $\Omega \subset \mathbb{R}^3$ $\underline{e}$ - based   Seek $\underline{e} \in H(\mathrm{curl}, \Omega)$

$$\int_\Omega \underline{\underline{\mu}}^{-1} \mathrm{curl}\,\underline{e} \cdot \mathrm{curl}\,\underline{e}' - \omega^2 \int_\Omega \underline{\underline{\varepsilon}}\,\underline{e}\,e' - i\omega \int_{\partial\Omega} Z^{-1} \underline{e}_t \cdot \underline{e}_t' \, dS = -i\omega \int_\Omega j_s \cdot \underline{e}'\,dx.$$

magnetic energy   angular freq. $(\omega > 0)$ electric energy

$\forall \underline{e}' \in H(\mathrm{curl}, \Omega)$

first order radiation condition on $\partial\Omega$.

$$\underline{e}_t = Z(\underline{h} \times n) \quad . \quad Z = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

Tool for analysis: ~~Riesz - Schauder theory~~ Riesz - Fredholm theory

* Fredholm alternative: for $\underset{\substack{\text{bijective} \\ \text{continuous}}}{A} + \underset{\text{compact}}{K}$

Sesqui-linear form $c : V \times V \to \mathbb{C}$

$\Longleftrightarrow \quad C : V \to V \qquad (CV, V')_V = c(V, V') \quad \forall v' \in V.$

Rellich's Lemma: $Id : H^1(\Omega) \to L^2(\Omega)$ compact.

Ex:

$$m : \begin{cases} H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{C}. \\ (u, u') \longrightarrow \int_\Omega u \cdot \overline{u'}\,dx \quad (L^2\text{-inner prod. mass matrix}) \end{cases}$$

Operator $M : H^1(\Omega) \to H^1(\Omega)$ defined by

$$\int_\Omega \mathrm{grad}\,Mf \cdot \overline{\mathrm{grad}\,v'} + Mf \cdot \overline{v'}\,dx = \int f \overline{v'}\,dx \quad \forall v \in H^1(\Omega)$$

To show M is compact.

Take $(f_n) \subset H^1(\Omega)$ $\|f_n\| \le 1$ By Rellich Lemma

$\longrightarrow \exists$ subsequence $\{f_{n_k}\}_k$ $f_{n_k} \to \tilde{f}$ in $L^2(\Omega)$

Note by def: $\|Mf\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)}$

M is continuous $L^2(\Omega) \to H^1(\Omega)$

$\Rightarrow \underbrace{Mf_{n_k}}_{} \to M\hat{f}$ in $H^1(\Omega)$

↳ convergent subsequence of $(Mf_n)_{n\in\mathbb{N}}$

$M = M_{L^2\to H^1} \circ Id_{H^1\to L^2}$ (composition of continuous/compact operator)

Rule of thumb: if $c: V\times V \to \mathbb{C}$ is still continuous on $V\times W$ or $W\times V$.

where $V \subset W$, $id: V \to W$ is compact.

(L^2-inner product is compact in $H^1(\Omega)$)

Warm-up: Acoustic cavity source problem in freq. domain.

Seek: $u \in H^1(\Omega)$

$$\underbrace{\int_\Omega \operatorname{grad} u \operatorname{grad} u' \, dx}_{a(u,u')} - \omega^2 \underbrace{\int_\Omega \rho u u' \, dx}_{m(u,u')} + i\omega \underbrace{\int_{\partial\Omega} \alpha u u' \, dx}_{b(u,u')}$$

$$= -i\omega \int f u' \, dx \quad \forall u' \in H^1(\Omega)$$

$A: H^1 \to H^1$ pos. semi-def. $\qquad M: H^1 \to H^1 \qquad B: H^1 \to H^1$

Tool: trace inequality $\|w|_{\partial\Omega}\|^2_{L^2(\partial\Omega)} \leq C(\Omega) \|w\|_{L^2(\Omega)} \cdot \|w\|_{H^1(\Omega)}$

Continuity of face fluxes of $H^1(\Omega)$ starlike Ω

$\bar{u} = \frac{1}{|\Omega|} \int_\Omega u \, dx$. $u \to \bar{u}$ is compact (finite rank)

$$a(u,u') = \underbrace{a(u-\bar{u}, u'-\bar{u}')}_{} + \underbrace{a + a - a(\bar{u}, \bar{u}')}_{\text{all are compact operators}}$$

$\geq a(u-\bar{u}, u-\bar{u})$ coercive

$\geq \gamma(\|u\| - \underset{\text{compact}}{\|\bar{u}\|})$

▷ A is bijective after some compact modification.

Structure of variational problem:

$$(\underset{\substack{\uparrow \\ \text{bijective}}}{\hat{A}} + \underset{\substack{\uparrow \\ \text{compact}}}{K})\, u = f$$

thus Fredholm alternat. can be applied.

To show uniqueness of solutions:

$f = 0$ ① test by $u' \in C_0^\infty(\Omega) \Rightarrow -\Delta u - \omega^2 \rho u = 0$

② ~~test by $u' \in C^\infty(\bar{\Omega})$~~ $\Rightarrow u|_{\partial\Omega} = 0$

test by $u' = u$. See imaginary part.

③ test by $u' \in C^\infty(\bar{\Omega}) \Rightarrow \operatorname{grad} u \cdot n|_{\partial\Omega} = 0$.

By analytic continuation $\Longrightarrow u = 0$

Now back to Maxwell case:

Note $\ast$: $\int_\Omega \underline{\varepsilon}\, \underline{e} \cdot \underline{e}' \, dx$ is not compact

$$m: \begin{cases} H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega) \longrightarrow \mathbb{C} \\ (\underline{e}\ ,\ \underline{e}') \longrightarrow \int_\Omega \underline{\varepsilon}\, \underline{e} \cdot \underline{e}' \, dx \end{cases}$$

M:

$$\int_\Omega \mathrm{curl}\, Mf \cdot \mathrm{curl}\, w' + Mf \cdot w' \, dx = \int f\, w' \, dx$$

$H(\mathrm{curl}, \Omega)$ is not compactly embedded in $L_2(\Omega)$

Idea: Choose $f \in H(\mathrm{curl}\, 0, \Omega) \Rightarrow \boxed{Mf = f.}$ (if M is compact $D(M)$ is finite dim)

$\hookrightarrow$ contains bounded sequence without convergent subsequence (ONB) (infinit

Splitting $\underline{e} = \underset{\substack{\uparrow \\ \ker(\mathrm{curl})}}{\underline{e}_0} + \underset{\substack{\uparrow \\ \ker(\mathrm{curl})^\perp}}{\underline{e}_\perp} = Z\underline{e} + R\underline{e}$

with continuous operators: $Z: H(curl,\Omega) \to H(curl\,0,\Omega)$

$R: H(curl,\Omega) \to (H_0^1(\Omega))^3$.

with $Z + R = Id$.

Hence V.F $\Rightarrow$ $\int_\Omega Re \cdot Re' - \int Re \cdot Re'$ → throw this terms as compact garbage

$$\int_\Omega \underline{\mu}^{-1} \operatorname{curl} R\underline{e} \cdot \operatorname{curl} R\underline{e} - \omega^2 \underbrace{\int_\Omega \underline{\underline{\varepsilon}}\, R\underline{e} \cdot R\underline{e}'}_{\text{Ex. (show compactness)}} - \omega^2 \int_\Omega \underbrace{\underline{\underline{\varepsilon}}\, Z\underline{e} \cdot R\underline{e}'}$$

$$-\omega^2 \int_\Omega \underbrace{\underline{\underline{\varepsilon}}\, R\underline{e} \cdot Z\underline{e}'} - \omega^2 \int_\Omega \underline{\underline{\varepsilon}}\, Z\underline{e} \cdot Z\underline{e}' + i\omega \int_{\partial\Omega} Z_0^{-1}\, \underline{e} \cdot \underline{e}'\, dS$$

$$= -i\omega \int_\Omega \underline{j} \cdot \underline{e}'\, dx$$

$\Downarrow$

$R\underline{e} \cdot R\underline{e}'$ — check compactness

$Z\underline{e} \cdot R\underline{e}'$ — not compact

$R\underline{e} \cdot Z\underline{e}'$ — not compact

$Z\underline{e} \cdot Z\underline{e}'$

Step 1. Identify & drop compact terms.

Step 2: Show that remaining sesquilinear form is bijective by removing compact terms.

* for shorthand $d(\underline{e}, \underline{e}') = \int \underline{\mu}^{-1} \operatorname{curl} R\underline{e} \cdot \operatorname{curl} R\underline{e}' + \cdots\cdots$

define: $\tilde{d}(\underline{e}, \underline{e}') = d(\underline{e}, R\underline{e}' - Z\underline{e}')$

Assumption: $R^2 = R$. $Z^2 = Z$ R. Z are projections $R \cdot Z = 0$.

$$\Rightarrow \tilde{d}(\underline{e}, \underline{e}) = \int_\Omega \underline{\mu}^{-1} \operatorname{curl} R\underline{e} \cdot \operatorname{curl} R\underline{e}$$

$$+ \omega^2 \int_\Omega \underline{\underline{\varepsilon}}\, Z\underline{e} \cdot Z\underline{e}'$$

$Z(R\underline{e}' - Z\underline{e}') = Z^2 \underline{e}' = Z\underline{e}'$

cancel

$$+ i\omega \int_{\partial\Omega} Z_0^{-1} \left((Z\underline{e})_t \cdot (R\underline{e})_t - (R\underline{e})_t \cdot (Z\underline{e})_t \right)$$

$$- i\omega \int_{\partial\Omega} Z_0^{-1} (Z\underline{e})_t \cdot (Z\underline{e})_t$$

$$\Rightarrow |\tilde{d}(\underline{e}, \underline{e})| \geq C \left(\|\operatorname{curl} R\underline{e}\|_{L^2}^2 + \|Z\underline{e}\|_{L^2}^2 \right)$$

Assumption: $\|R\underline{e}\|_{L^2} \leq C(\Omega) \|\operatorname{curl} R\underline{e}\|_{L^2}$

$$\geq C \left(\|R\underline{e}\|_{H(curl,\Omega)}^2 + \|Z\underline{e}\|_{H(curl,\Omega)}^2 \right)$$

$$\geq C \|(R+Z)\underline{e}\|_{H(curl,\Omega)}^2$$

$\triangleright$ "$\hat{a}$ is bijective"

Let $X := R - Z \Rightarrow X^2 = Id \Rightarrow X$ is bijective $\Big\} \Rightarrow$ "a is bijective"

By $a = (\underline{e}, \underline{e}') = \hat{a}(\underline{e}, X\underline{e}')$

$\Rightarrow$ sesquilinear form for cavity problem is = "bijective + compact"

Next job: to find R + Z satisfies all assumptions. See Sect 3,

Recall: Lifting operator. L: curl $H(\text{curl}, \Omega) \rightarrow (H^1(\Omega))^3$.

with property: $\text{curl} \circ L = Id$

Define: $R = L \circ \text{curl}$. $\qquad C\|\text{curl} \circ L \circ \text{curl}\, u\| \le C\|\text{curl}\, Ru\|_2$

$\Rightarrow$ continuity $\|Ru\|_{H(\Omega)} \le C\|\text{curl}\, u\|_2 \quad \forall u \in H(\text{curl}, \Omega)$

$\Rightarrow \qquad \text{curl}\, R\underline{u} = \text{curl}\, u$

define $\quad Z := Id - R : \quad H(\text{curl}, \Omega) \longrightarrow H(\text{curl}\, 0, \Omega)$

$\Rightarrow \qquad R^2 = L \circ \underbrace{\text{curl} \circ L}_{Id} \circ \text{curl} = L \circ \text{curl} = R \Rightarrow Z^2 = Z$

$\Rightarrow R \circ Z = Z \circ R = 0.$

Injectivity as in acoustic case: (uniqueness by analytic continuation)

$\Rightarrow$ well-posedness / (existence + uniqueness by Fredholm alternative)

Analysis of Galerkin discretization

Seek in trial/test space $W^1(\mathcal{T}_h)$: $\mathcal{T}_h$: tetrahedral mesh of Ω.

Remark: check inf-sup condition

$L.V.P \quad u \in V : a(u, v) = f(v) \quad \forall v \in V. \rightarrow$ has unique soln.

$u_h \in V_h : a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h$

Assume: $\exists \gamma > 0$: $\sup\limits_{v_h \in V_h} \dfrac{|a(w_h, v_h)|}{\|v_h\|} \geq \gamma \|w_h\| \quad \forall w_h \in V_h$.
(discrete - inf-sup)

$$\gamma \|u_h - w_h\| \leq \sup \frac{|a(u_h - w_h, v_h)|}{\|v_h\|} = \sup \frac{|a(u - w_h, v_h)|}{\|v_h\|}$$

$$\leq C \|u - w_h\|$$

$$\Rightarrow \|u - u_h\| \leq \|u - w_h\| + \|u_h - w_h\|$$

$$\leq (1 + \frac{C}{\gamma}) \|u - w_h\|. \quad \forall w_h \in V_h.$$

Now take $V \hat{=} H(\mathrm{curl}, \Omega)$

$a \hat{=}$ sesquilinear form for Maxwell cavity source problem.

$V_h \hat{=} W^1(\mathcal{T}_h)$

① As shown before $\underline{e} \in V$. $a(\underline{e}, \underline{e}') = -i\omega \int_\Omega \underline{j}\, \underline{e}' dx = f(\underline{e}')$

has a unique solution by Fredholm Alternative $\Rightarrow$ Stability.

viz. $\|\underline{e}\|_V \leq C \|f\|_{V'} = C \sup\limits_{v \in V} \dfrac{|f(v)|}{\|v\|_V} = C \sup\limits_{v \in V} \dfrac{|a(\underline{e}, v)|}{\|v\|_V}$

$$\Rightarrow \|\underline{u}\|_V \leq C \sup \frac{|a(\underline{u}, \underline{v})|}{\|\underline{v}\|} \quad \forall \underline{u} \in V$$

$\Rightarrow$ Well-posedness of problem $\Rightarrow$ continuous inf-sup condition.

Known result: $\exists$ compact $k: V \times V \to R$

$$|\underbrace{a(\underline{e}, X\underline{e}) + k(\underline{e}, \underline{e})}_{\tilde{a}(\underline{e}, \underline{e}')}| \geq \gamma \|\underline{e}\|_V^2 \quad \forall \underline{e} \in V$$

Define $S: V \longrightarrow V$.

$$a(\underline{e}', Sf) = k(f, \underline{e}') \quad \forall \underline{e}' \in V.$$

$\Rightarrow$ S continuous $\quad S := (A^*)^{-1} \circ K$

Hence S is compact.

Take $\underline{v} = (S + X)\underline{u}$ (continuous candidate)

$$|a(\underline{u},\underline{v})| = |a(\underline{u}, X\underline{u}) + a(\underline{u}, S\underline{u})| = |a(\underline{u}, X\underline{u}) + k(\underline{u},\underline{u})|$$

$$\geq \gamma \|\underline{u}\|_V^2 \qquad (*)$$

$$= \sup_{w\in V} \frac{a(u,w)}{\|w\|} \geq \frac{|a(u,v)|}{\|v\|} \geq \frac{|a(\underline{u},\underline{u})|}{\|S\|\,\|\underline{u}\|_V}$$

For $w_h \in V_h$ by ~~$v_h = (S+X)\underline{w}_h$~~ $\notin V_h = W^1(\mathcal{T}_h)$

Try $v_h = P_h S w_h + I_h^1 X w_h$

with $P_h : H(\mathrm{curl},\Omega) \to W^1(\mathcal{T}_h)$ is the $H(\mathrm{curl})$-orthogonal projection)

$I_h^1 \,\hat{=}$ edge interpolation operator

$$a(w_h, v_h) = a\big(w_h, \underbrace{(S+X)w_h}_{\text{by } (*)} - (Id-P_h)Sw_h - (Id-I_h^1)Xw_h\big)$$

$$\geq \gamma\|w_h\|_V^2 - \|(Id-P_h)Sw_h\|_V - \|(Id-I_h^1)Xw_h\|$$

$$- |a(w_h,(Id-P_h)Sw_h)| - |a(w_h,(Id-I_h^1)Xw_h)|$$

Now use continuity of $a(\cdot,\cdot)$

$$|a(w_h,v_h)| \geq \gamma\|w_h\|_V^2 - C_A\|w_h\|_V\big(\|(Id-P_h)Sw_h\|_V + \|(Id-I_h^1)Xw_h\|$$

$$\frac{|a(w_h,v_h)|}{\|w_h\|_V} \geq \gamma - C_A\big(\underbrace{\|(Id-P_h)Sw_h\|_V + \|(Id-I_h^1)Xw_h\|_V}_{\text{below} \ \leq 1/2 \|w_h\|_V}\big)$$

Forced to adopt asymptotic perspective

$\{\mathcal{T}_h\}_{h\in H} \,\hat{=}$ family of uniformly shape-regular meshes with mesh size $h \to 0$. $H = \{h_1, h_2, \dots\}$

- Estimate of $\|(Id - I_h^1) X w_h\|_V$ $\qquad X = R - Z = \underset{(R+Z)}{2R - Id}$

$(Id - I_h^1)(2R - Id) w_h = 2(Id - I_h^1) R w_h$.

Note $(Id - I_h^1) w_h = 0$ for finite element function w_h.

By 3.2.B $\quad \operatorname{curl} \underset{\in (H^1(\Omega))^3}{R w_h} = \underbrace{\operatorname{curl} w_h}_{\text{const curl}} \Rightarrow I_h^1 R w_h$ well-defined

$\|(Id - I_h^1) R w_h\|_{L^2} \le Ch \|\operatorname{curl} R w_h\|_{L^2}$

$|\operatorname{curl}(Id - I_h^1) R w_h| = \operatorname{curl} R w_h - I_h^2 \operatorname{curl} R w_h$
$= (I - I_h^2) \operatorname{curl} w_h$

$\triangleright \quad \|(Id - I_h^1) R w_h\|_V \le Ch \|\operatorname{curl} w_h\|_{L^2(\Omega)}$
$\le Ch \|w_h\|_V$

- Estimate of $\|(Id - P_h) S w_h\|_V$
  - Observe $(C(\bar\Omega))^3$ are dense in $H(\operatorname{curl}, \Omega)$

$\forall \underline{u} \in H(\operatorname{curl}, \Omega) \quad \|\underline{u} - P_h \underline{u}\|_V = \inf_{\underline{v}_h \in V_h} \|u - v_h\| \xrightarrow{\text{as } h \to 0} 0$

Since $\forall \varepsilon > 0. \ \exists u^\infty \in (C^\infty(\bar\Omega))^3$ s.t. $\|u - u^\infty\|_V < \varepsilon$

Further by interpolation error estimate: $u^\infty \in H^2(\Omega)$

$\|u^\infty - I_h^1 u^\infty\|_V \to 0$ as $h \to 0$

$\Rightarrow P_h \underline{u} \to \underline{u}$ in V as $h \to 0$.

To show $\exists \varepsilon \quad R^+ \to R^+ \quad \lim_{h \to 0} \varepsilon(h) = 0$ s.t

$\|(Id - P_h) S w_h\|_V \le \varepsilon(h) \|w_h\|_V$.

⌈ Indirect proof. Assume $\exists \delta > 0$. s.t. $\forall h \quad \exists$ ~~some~~ $w_h \in V_h$

with $\|w_h\|_V = 1$ s.t $\|(Id - P_h) S w_h\|_V \ge \delta . \underset{\shortparallel \, 1}{\|w_h\|_V}$

w_h: bounded $(S w_h)_h$ has a convergent subsequence.
(still denoted by $S w_h$)

$S w_h \longrightarrow w$ (as $h \to 0$)

But $\|(Id - P_h) S w_h\| \le \|S w_h - X + X - P_h X + P_h (X - S w_h)\|$

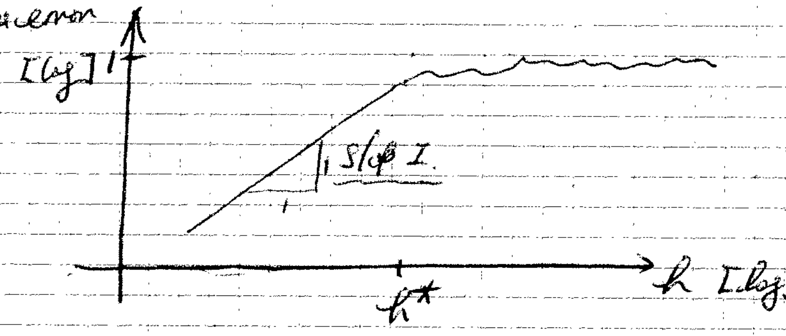
$$\le \|S w_h - X\|_V + \|X - P_h X\|_V + \underbrace{\|P_h\|}_{\le 1 \text{ orth. proj.}} \|X - S w_h\|_V \to 0$$

<u>Summary</u> $\dfrac{|a(w_h, v_h)|}{\|w_h\|_V} \ge (\gamma - C_A \epsilon(h) + C_B h C) \|w_h\|$

Thm 3.3.A. $\exists\, h^* = h^*(\Omega, \underline{\epsilon}, \underline{\mu}, \rho(\mathcal{T}_h)) > 0$. the discrete Galerkin solution $\underline{e}_h \in W^1(\mathcal{T}_h)$ exists if $\boxed{h < h^*}$ and satisfies — Asymptotically quasi-optimal.

$$\|\underline{e} - \underline{e}_h\|_{H(curl, \Omega)} \le C \inf_{v_h \in W^1(\mathcal{T}_h)} \|\underline{e} - \underline{v}_h\|_{H(curl, \Omega)}$$

<u>RK</u>: In computations, we could observe



In contrast, for $\underline{v}$-based model for electrostatics

$$\|v - v_h\|_{H^1} \le C \inf_{w_h \in W^0(\mathcal{T}_h)} \|v - w_h\|_{H^1}$$