Lecture XI 28-11-2008.

Chap. IV. Time-Harmonic Electromagnetic Boundary Value Problem

4.1. Frequency domain.

Setting: local linear material laws.

$$\underline{d} = \underline{\underline{\varepsilon}}\, \underline{e} \qquad \underline{b} = \underline{\underline{\mu}}\, \underline{h}$$

Assume harmonic time-dependence of excitation fields

$$f(x,t) = Re\,(\hat{f}(x)\, e^{i\omega t})$$

$\hat{f}(x)$: complex amplitude; ω: angular frequency

later we don't differentiate between $\hat{f}(x)$ or $f(x)$

Linearity $\Rightarrow$ harmonic time dependence for all fields

Maxwell's equation equivalence in time domain.

$$(4.1.a)\begin{cases} \operatorname{curl} \underline{e}(x) = -i\omega\, \underline{\underline{\mu}}\, \underline{h}(x) \\ \operatorname{curl} \underline{h}(x) = i\omega\, \underline{\underline{\varepsilon}}\, \underline{e}(x) + \underline{\underline{\sigma}}\, \underline{e}(x) + \hat{j}_s(x) \end{cases}$$

[formal: $\partial_t \longrightarrow i\omega$].

Remark: energy / power

$\longrightarrow$ look at energies over one period $T = 2\pi/\omega$.

e.g. electric energy $\displaystyle \frac{1}{2}\int_\Omega \underline{\underline{\varepsilon}}\, \underline{e} \cdot \underline{e}^* \, dx \in \mathbb{R}$ ($\underline{e}^*$: complex conjugate.)

power of source $\displaystyle \frac{1}{2}\int_\Omega \hat{j}_s \cdot \underline{e}^* \, dx \in \mathbb{C}$

$\longrightarrow$ Real part: real power injected into the system. (dissipation)

Imaginary part: reactive power. (fluctuation)

Variational formulation and energy balance:

eliminate $\underline{h}$ in (4.1.a)

$$\operatorname{curl} \underline{\underline{\mu}}^{-1} \operatorname{curl} \underline{e} - \omega^2 \underline{\underline{\varepsilon}}\, \underline{e} + i\omega \underline{\underline{\sigma}}\, \underline{e} = -i\omega \underline{j}_s.$$

test with e' and integate by parts.

$$\int_\Omega \underline{\underline{\mu}}^{-1} \operatorname{curl} \underline{e} \cdot \operatorname{curl} \underline{e}' - \omega^2 \underline{\underline{\varepsilon}}\, \underline{e} \cdot \underline{e}' + \int_{\partial\Omega} \underline{\underline{\mu}}^{-1} \operatorname{curl} \underline{e} \cdot (\underline{e}' \times \underline{n})$$

$$+ i\omega \int_\Omega \underline{\underline{\sigma}}\, \underline{e} \cdot \underline{e}' = -i\omega \int_\Omega \underline{j}_s\, \underline{e}' \quad \forall e' \in H(\operatorname{curl}, \Omega)$$

poynting ~~factor~~

Set $\underline{e}' = \underline{e}^* \; (= \overline{\underline{e}})$

$$\Rightarrow \int_\Omega \underbrace{\underline{\underline{\mu}}^{-1} \operatorname{curl} \underline{e} \cdot \operatorname{curl} \underline{e}^*}_{\text{magnetic energy}} - \omega^2 \underbrace{\underline{\underline{\varepsilon}}\, \underline{e} \cdot \underline{e}^*}_{\text{electric energy}} - \underbrace{i\omega \int_\Omega \underline{h} \cdot (\underline{e}^* \times n)\, dS}_{\text{dissipation energy}}$$

$$+ i\omega \int_\Omega \underbrace{\underline{\underline{\sigma}}\, \underline{e} \cdot \underline{e}^*}_{\text{Ohmic loss}} = -i\omega \int_\Omega \underbrace{\underline{j}_s \cdot \underline{e}^*}_{\text{input power}} dx$$

Take imaginary part,

$\Rightarrow$ { electric / magnetic } fields 互相转换 convert into each other.

4.2. Magnetoquasistatics

↳ a reduced model neglecting energy of the electric field.

Formally. setting $\underline{\underline{\varepsilon}} = 0$ (no charges) charge-free model.

$$\text{(4.1 b)} \quad \begin{aligned} \operatorname{curl} \underline{e}(x) &= -i\omega \underline{\underline{\mu}}\, \underline{h}(x) \\ \operatorname{curl} \underline{h}(x) &= \underline{\underline{\sigma}}\, \underline{e}(x) + \underline{j}_s(x) \end{aligned}$$

$\underline{\underline{\mu}}$ = uniformly positive.

$$\underline{\underline{\sigma}} : \begin{cases} \text{uniformly positive in a bounded conducting domain } \Omega_c \\ = 0 \quad \text{outside} \;\text{------}\; \mathbb{R}^3 \setminus \Omega_c = \Omega_I \end{cases}$$

insulating domain

$\Rightarrow$ $\operatorname{div} \underline{j}_s = 0$ in Ω_I; $\quad \int_{\Gamma_i} \underline{j}_s \cdot \underline{n} \, dS = 0.$ $\Big\}$ suff. & necessary condition to guarantee existence of vector potential

$\Gamma_i \hat{=}$ connected components of $\partial\Omega_c$.

"Far away" $\quad \operatorname{curl} \underline{h}(x) = 0$, $\operatorname{div} \underline{h}(x) = 0$. $\Rightarrow$ "$\underline{h} = \operatorname{grad}\psi$ (? topo. issue) $\Delta\psi = 0$" harmonic $\Rightarrow \psi = 0$ as $r \to \infty$

$\downarrow$
($\underline{\mu} = \underline{\mu}_0$ (vacuum))

Recall: decay condition for electric field potential in electrostatics in $\mathbb{R}^3$.

$$\Rightarrow v(x) = O(|x|^{-1}), \quad \underbrace{\underline{h}(x) = O(|x|^{-2})}_{\text{decay condition for eddy current model.}}$$

! but $\underline{e}$ is not well defined. (non-uniqueness) since only $\operatorname{curl}\underline{e}(x)$ enters the equation

$\underline{e} \longleftrightarrow$ magnetic vector potential $\underline{a}$.

$\left(\underline{a} = -\frac{1}{i\omega}\underline{e}. \quad \underline{a}, \underline{e} \text{ are the same thing (field)} \right)$

$\Rightarrow$ we need gauge condition in Ω_I

$$\underline{e}|_{\Omega_I} \in \left(H_0(\operatorname{curl} 0, \Omega_I)\right)^{\perp_\varepsilon} \Rightarrow \underline{\operatorname{div} \varepsilon \underline{e} = 0 \text{ in } \Omega_I}$$

$$\Rightarrow \int_{\Gamma_i} \varepsilon \underline{e} \cdot n \, dS = 0 \quad \text{(charge-free condition)}$$

w.r.t $\underline{\varepsilon}$-weighted L^2-inner product.

(we reintroduce $\underline{\varepsilon}$ to fix $\underline{e}$-field. not the same as ~~in the assumption setting $\underline{\varepsilon} = 0$~~ has nothing to do with

"Far away" $\quad \operatorname{curl} \underline{e} = \underline{h} = O(|x|^{-2})$, $\Rightarrow \operatorname{div}\underline{e} = 0$ $\Big\} \Rightarrow \underline{e}(x) = O(|x|^{-2})$

$\Rightarrow$ cut-off technique. reduction to simple bounded computational domain Ω. and PEC or PMC on $\partial\Omega$.

4.2.1 Estimation of eddy current modeling error

(compare full matrix solution with eddy current error. truncated error due to cut-off is not investigated)

$$\delta \underline{e} = \underline{e}^m - \underline{e}$$

$$\delta \underline{h} = \underline{h}^m - \underline{h}$$

full Maxwell sol'n. eddy current sol'n.

By linearity. subtract $\overset{(4.1.b)}{\text{one}}$ from $\overset{(4.1.a)}{\text{the other}}$

$$\Rightarrow \begin{cases} \text{curl}\, \delta \underline{e} = -i\omega \underline{\underline{\mu}}\, \delta \underline{h} \\ \text{curl}\, \delta \underline{h} = i\omega\, \underline{\underline{\varepsilon}}\, \underline{e}^m + \underline{\underline{\sigma}}\, \delta \underline{e} \end{cases} \qquad (4.2.c)$$

eddy current equation with source term $i\omega \underline{\underline{\varepsilon}}\, \underline{e}^m$.

(4.2.c) in V.F.

$$\text{curl}\, \underline{\underline{\mu}}^{-1} \text{curl}\, \delta \underline{e} + i\omega \sigma\, \delta \underline{e} = \omega^2 \underline{\underline{\varepsilon}}\, \underline{e}^m \quad \text{in } \Omega$$

(PEC) $\qquad \delta \underline{e} \times \underline{n} = 0 \quad$ on $\partial\Omega$

(Gauge) $\qquad \delta \underline{e}|_{\Omega_I} \perp H_0(\text{curl}\, 0, \Omega_I)$

trial space $\mathcal{X} = \{ \underline{u} \in H(\text{curl}, \Omega) . \ \underline{u}|_{\Omega_I} \perp H_0(\text{curl}\, 0, \Omega_I) \}$

Note that $\underline{e}^m$ of full maxwell solution satisfies the gauge condition

$\Rightarrow \delta \underline{e} \longrightarrow$ " " " " " " " "

Seek $\delta \underline{e} \in \mathcal{X}$ s.t.

$$\underbrace{\int_\Omega \underline{\underline{\mu}}^{-1} \text{curl}\, \delta \underline{e} \cdot \text{curl}\, \delta \underline{e}' + i\omega \sigma\, \delta \underline{e} \cdot \delta \underline{e}'}_{a(\cdot,\cdot)}$$

$$= \int_\Omega \omega^2\, \underline{\underline{\varepsilon}}\, \underline{e}^m \cdot \delta \underline{e}'\, dx \quad \forall\, \delta \underline{e}' \in \mathcal{X}. \qquad (4.2.d)$$

109

Relevant norms:

$$-\frac{1}{i\omega}\underline{\mu}^{-1}\operatorname{curl}\underline{e}$$

"power norm" $\|\underline{e}\|_p^2 = i\omega \int_\Omega \underline{\mu}\, \underline{h}^* \cdot \underline{h}^*\, dx + \int_{\Omega_c} \underline{\sigma}\, \underline{e} \cdot \underline{e}^*\, dx$

norm related to $a(\cdot,\cdot)$ — fluctuation energy — Ohmic loss

$$|a(\delta\underline{e}, \delta\underline{e}^*)| \le \omega^2 \varepsilon_{max} \|e^m\|_{L^2(\Omega)} \|\delta\underline{e}\|_{L^2(\Omega)}$$

$$|a(\delta\underline{e}, \delta\underline{e}^*)| \ge \underbrace{\frac{1}{\sqrt{2}}\,\omega}_{?} \|\delta e\|_p^2 \longrightarrow \text{relate "}L^2\text{-norm" with power norm}$$

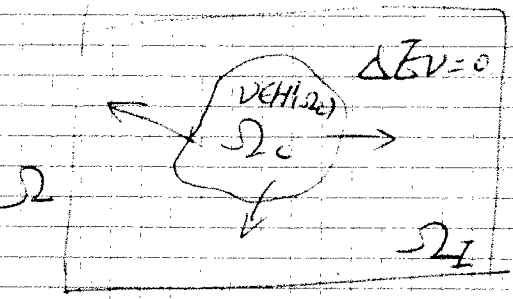
Task: To relate $\|\cdot\|_2$ with $\|\cdot\|_p$ on X

Lemma: 9.2 A. There exists $C = C(\Omega)$ so that ~~$\|u\|_{L^2(\Omega)}$~~

$$\|\underline{u}\|_{L^2(\Omega)}^2 \le C(1+\eta)D^2 \|\operatorname{curl}\underline{u}\|_{L^2(\Omega)}^2 + 2\eta^2 \|\underline{u}\|_{L^2(\Omega)}^2 \quad \forall \underline{u} \in X.$$

$$\eta = \|E\| = \sup \frac{\|Eu\|_{H^1(\Omega)}}{\|u\|_{H^1(\Omega)}} \quad D = \operatorname{diam}(\Omega)$$

E: extension operator : $E: H^1(\Omega_c) \to H_0^1(\Omega)$ is the Harmonic extension



$Ev = v$ in Ω_c

$\Delta Ev = 0$ in Ω_I

$Ev = 0$ on $\partial\Omega$

Sketch of the proof: ① L^2-orthogonal decomposition (Helmholtz decomp.)

$$\underline{u} = \underline{u}_0 + \operatorname{grad}\psi \qquad \operatorname{div}\underline{u}_0 = 0 \qquad \psi \in H^1(\Omega)$$

Ω trivial topology. $\|\underline{u}_0\| \le C(\Omega) D \|\operatorname{curl}\underline{u}_0\|_{L^2(\Omega)}$

Triangle inequality $\|\operatorname{grad}\psi\| \le \|\underline{u}\|_{L^2(\Omega)} + C(\Omega)\, D\, \|\operatorname{curl}\underline{u}_0\|_{L^2(\Omega)}$

② $\underline{w} = \underline{u}_0 + \operatorname{grad} E\psi \rightarrow \underline{w} = \underline{u}$ in Ω_c

$\operatorname{curl}\underline{w} = \operatorname{curl}\underline{u}$ in Ω

$$\|\underline{w}\|_{L^2(\Omega)} \le C(\Omega)\, D\, \|\mathrm{curl}\,\underline{u}\|_{L^2(\Omega)} + \eta\left(\|\underline{u}\|_{L^2(\Omega_c)} + C(\Omega)\, D\, \|\mathrm{curl}\,\underline{u}\|_{L^2(\Omega)}\right)$$

$$\le \eta\, \|\underline{u}\|_{L^2(\Omega)} + (1+\eta)\, C(\Omega) D\, \|\mathrm{curl}\,\underline{u}\|_{L^2(\Omega)}$$

$\underline{u} \in X \Rightarrow \underline{u}$ is the curl-preserving extension of $\underline{u}|_{\Omega_c}$

with min. L^2 norm

$\triangleright \quad \|\underline{u}\|_{L^2(\Omega)} \le \|\underline{w}\|_{L^2(\Omega)}$ #

$$\|\underline{u}\|^2_{L^2(\Omega)} \le \left(c(1+\eta)^2 \omega \mu_{max} D^2 + \frac{2\eta^2}{\sigma_{min}^2} \right) \|u\|_p^2 .$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\, \omega\, \|\delta e\|_p^2 \le a(\delta \underline{e}, \delta \underline{e}^*) \le \omega^2 \varepsilon_{max} \|e^m\|_{L^2(\Omega)} (\ )\cdot \|\delta e\|_p$$

relaxation dielectric ~~…~~ time τ

$\frac{1}{c^2}\ \frac{2\pi}{T} \cdot = \left(\frac{2\pi}{\lambda}\right)^2$

$$\frac{\|\delta e\|_{L^2}}{\|e^m\|_{L^2}},\ \frac{\|\delta e\|_p}{\|e^m\|_p} \le \sqrt{2}\left(\underbrace{C_1}_{c(1+\eta)^2} \varepsilon_{max} \mu_{max} \omega^2 D^2 + \underbrace{C_2}_{2\eta^2} \frac{\omega \varepsilon_{max}}{\sigma_{min}} \right)$$

Small if $D \ll \lambda$

small domain w.r.t wavelength

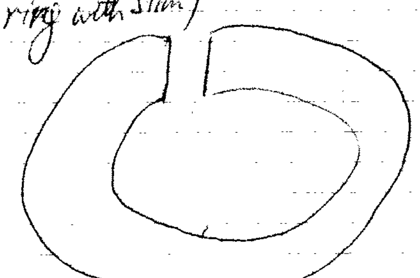
Small if $\tau \ll T$.

("charge distribution is instant")

(ring with slim)

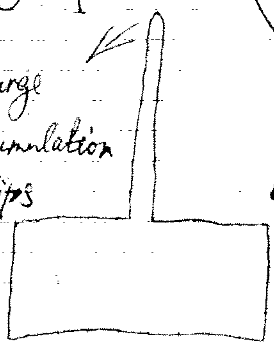
η depends

huge η case!



capacity can not apply eddy current model.

charge accumulation at tips



(slender bar)

6.2.2 Variational Formulation

- $\underline{e}$ - based (Ex: V.F. gauging issue, truncated domain Ω.
  discuss practical F.E. implementation
  ("orthogonal complement")
  PBC on conductor Ω_c.
- $\underline{h}$ - based (Ex. j_s given)
- hybrid V.F. (suppose $j_s \subset \Omega_I$)

Ω_c : eliminate $\underline{e}$

$$\mathrm{curl}\, \underline{\underline{\sigma}}^{-1}\, \mathrm{curl}\, \underline{h} + i\omega \underline{\underline{\mu}}\, h = 0$$

Ω_I : eliminate $\underline{h}$:

$$\mathrm{curl}\, \underline{\underline{\mu}}^{-1}\, \mathrm{curl}\, \underline{e} = -i\omega\, j_s$$

$$e \perp H(\mathrm{curl}\,0, \Omega_I) \qquad e \times n = 0 \text{ on } \partial\Omega$$

V.F. :

$$\boxed{\int_{\Omega_c} \underline{\underline{\sigma}}^{-1} \mathrm{curl}\, \underline{h} \cdot \mathrm{curl}\, \underline{h}' + i\omega \underline{\underline{\mu}}\, \underline{h}\cdot \underline{h}' \quad (1)} \qquad \forall h' \in H(\mathrm{curl}, \Omega)$$

$$+ \int_{\partial\Omega_c} \underbrace{\overset{e_t \text{ tangential}}{\underline{\underline{\sigma}}^{-1} \mathrm{curl}\, \underline{h}} \cdot (\underline{h}' \times \underline{n})}_{\pm(e\times n)\, h'}\, dS = 0$$

$$\boxed{\int_{\Omega_I} \underline{\underline{\mu}}^{-1} \mathrm{curl}\, \underline{e} \cdot \mathrm{curl}\, \underline{e}' \quad (2)} \; - \int_{\partial\Omega_I} \underbrace{\underline{\underline{\mu}}^{-1} \mathrm{curl}\, \underline{e}}_{-i\omega \underline{h}_t} \cdot (\underline{e}' \times n)\, dS$$

n: $\Omega_I \to \Omega_c$

$$= -i\omega \int_{\Omega_I} j_s\, \underline{e}'\, dx \qquad \forall e' \in X|_{\Omega_I}$$

$$\boxed{(1)} + \int_{\partial\Omega_c} e_t \cdot (\underline{h}' \times \underline{n})\, dS = 0$$

cancel if can then

$$\int_{\partial\Omega_c} h_t\, (e' \times n)\, dS + \frac{1}{i\omega} \boxed{(2)} = \int_{\Omega_I} j \cdot e'\, dx$$

$$\Longleftrightarrow \quad a\left(\begin{pmatrix}\underline{h}\\ \underline{e}\end{pmatrix}, \begin{pmatrix}\underline{h}'\\ \underline{e}'\end{pmatrix}\right) = f\begin{pmatrix}\underline{h}'\\ \underline{e}'\end{pmatrix}$$ (112)

$$a\left(\begin{pmatrix}\underline{h}\\ \underline{e}\end{pmatrix}, \begin{pmatrix}\underline{h}^*\\ \underline{e}^*\end{pmatrix}\right) = \int_{\Omega_c} \underline{\sigma}^{-1} \operatorname{curl} \underline{h} \cdot \operatorname{curl} \underline{h}^* + i\omega \underline{\mu}\, \underline{h} \cdot \underline{h}^* \, dx$$

(boundary term cancels $\displaystyle + \int_{\Omega_I} \underline{\mu}^{-1} \operatorname{curl} \underline{e} \cdot \operatorname{curl} \underline{e}^* \, dx.$

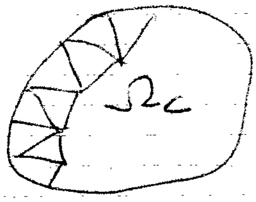
$$\not\geq \gamma \left(\|\underline{h}\|^2_{H(\operatorname{curl}, \Omega)} + \|\underline{e}\|^2_{H(\operatorname{curl}, \Omega)} \right)$$

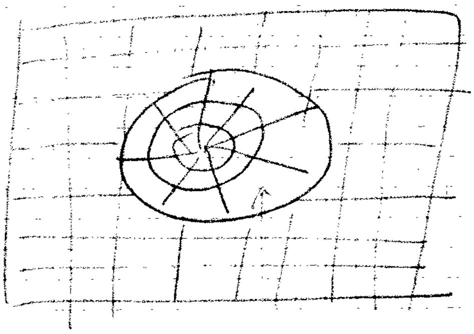
$$X|_{\Omega_I} = H_{\partial\Omega, 0}(\operatorname{curl}, \Omega_I) \cap H_0(\operatorname{curl} 0, \Omega_c)^{\perp}_{\gamma_{t,I}}$$

(" Gauging is not allowed. $\Rightarrow$ change sol^n. Cautious!)

Why hybrid?

- $\underline{e}, \underline{h}$ decouples





- allow uncoupled discretization.

"rotating cylinder"
+ externally magnetic field.

Observe: $\operatorname{curl} \underline{h} = 0$ in $\Omega_I \cap$ neighborhood of $\partial\Omega_c$

$\operatorname{div}_\Gamma (\underline{h} \times n) = 0$ ($\operatorname{div}_\Gamma$: surface divergence)

$\lceil d\omega = 0 \Rightarrow d\underset{\text{trace}}{\widetilde{\omega}} = 0. \rfloor$

$\rightarrow$ constrain $\underline{h}$ to $\{ \underline{h} \in H(\operatorname{curl}, \Omega_c),\ \operatorname{div}_\Gamma (\underline{h} \times \underline{n}) = 0 \} = Y$
(trivial top. of Ω)

$$\rightarrow \quad \int_{\partial\Omega_c} (\underline{h} \times \underline{n}) \cdot \operatorname{grad}_\Gamma v \, dS = 0$$

adding any $\operatorname{grad} H^1(\Omega_I)$ to $\underline{e}$ has no influence (effect) on the $\underline{h}$ solution of variational problem.

New gauging: Let $\underline{e} \perp H_{\partial\Omega}(\mathrm{curl}\, 0, \Omega_I)^{\perp}$

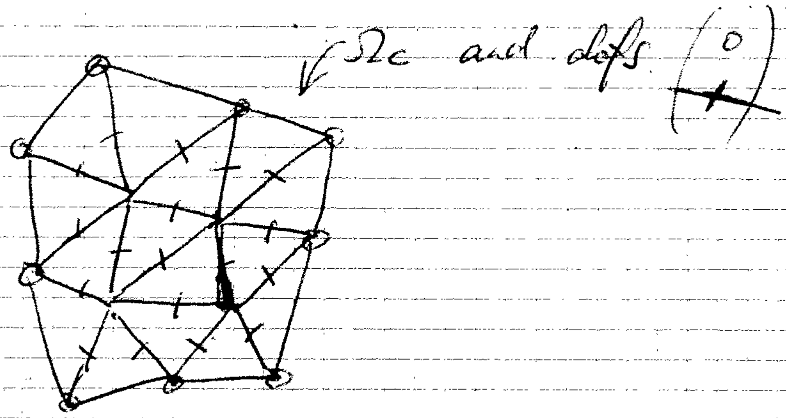
F.E. discretization (Galerkin): (trivial topology)

$$\underline{h}_h \in W^1(\mathring{\mathcal{T}}_{\Omega_C}) \oplus \mathrm{grad}\, W^0(\mathcal{T}_{\Omega_C})$$

Bases $\{b_e\}_{e \in \mathcal{E}(\mathring{\mathcal{T}}_h)} \cup \mathrm{grad}\{b_p\}_{p \in \partial\Omega_C \cap \mathcal{N}(\mathcal{T}_h)}$

interior edge

Ω_C and dofs



$$\underline{e}_h \in W^1(\mathcal{T}_{\Omega_I}) \qquad \underline{e}_t = 0 \text{ on } \partial\Omega$$

$e_{t|_{e}} = 0$ on $\partial\Omega \cap \mathcal{T}_{\Omega_I}$

Constrained equation:

$$\int_{\Omega_I} e_h \cdot \mathrm{grad}\, v_h \, dx = 0 \qquad \forall v_h \in W^0(\mathcal{T}_{\Omega_I})$$

▷: eddy current model with moving body

4.3. Maxwell cavity source problem.

Remark: Silver-Müller radiation conditions for Maxwell equations in frequency domain: ($\varepsilon = \varepsilon_0$, $\mu = \mu_0$ if $|x| > R$)

$$h(\underline{x}) \times \underline{x} - Z^{-1} |\underline{x}| \underline{e}(x) \to 0$$

as $|x| \to 0$ uniformly Z: impedance $\sqrt{\frac{\mu_0}{\varepsilon_0}}$

→ S.M radiation cond. at surface of cut-off domain used as simple absorbing boundary condition.

$$\underline{h}(\underline{x}) \times \underline{n} - \alpha \underline{e}_t = 0 \text{ on } \partial\Omega. \qquad \text{(first order)}$$

D: $\underline{e}$ - based variational problem.

Seek $\underline{e} \in H(\mathrm{curl}, \Omega)$

$$\int_\Omega \underline{\mu}^{-1} \mathrm{curl}\, e \cdot \mathrm{curl}\, \underline{e}' - \omega^2 \underline{\varepsilon}\, \underline{e} \cdot \underline{e}' \qquad (4.3.e)$$

$$- i\omega \int_{\partial\Omega} \underset{(\alpha>0)}{\alpha}\, e_t \cdot e_t'\, dS = -i\omega \int_\Omega \underline{j}_s \cdot \underline{e}'\, dx. \qquad \forall \underline{e}' \in H(\mathrm{curl}, \Omega)$$

!: "indifinite"

Abstract analysis framework:

Observation: take $e' = \underline{e}^*$ and plug in. Let $\underline{j}_s = 0$

$\Rightarrow \quad \underline{e}_t = 0$, on $\partial\Omega$.

Test (4.3.e) with $\underline{e}' \in (C_0^\infty(\Omega))^3 \Rightarrow$

$$\mathrm{curl}\, \underline{\mu}^{-1} \mathrm{curl}\, e - \omega^2 \underline{\varepsilon}\, e = 0.$$

Test (4.3.e) with $\underline{e}' \in (C^\infty(\overline{\Omega}))^3 \Rightarrow$

$$\underline{\mu}^{-1} \mathrm{curl}\, \underline{e} \times n = 0 \quad \text{on } \partial\Omega$$

$\Rightarrow \underline{e}_*$ satisfies both PEC. PMC.

$\Rightarrow \underline{e} = 0$. (by PDE theory)

Def. 4.3.A: $V, W \triangleq$ Hilbert spaces

A linear operator $K: V \to W$ compact.

if $\{v_n\} \subset V$ with $\|v_n\|_V \le 1$

then $\{K v_n\}$ contains a convergent subsequence.

Thm 4.3.B: V, W Hilbert spaces $K \in \mathcal{L}(V, W)$ compact.

$D \in \mathcal{L}(V, W)$. bijective. b.d.d. [Fredholm Alternative]

Then $D+K$ injective $\Longleftrightarrow$ $D+K$ is surjective

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$

$\ker(D+K)=\{0\}\in V$ $\qquad$ Range $(D+K)=W$

uniqueness $\Longleftrightarrow$ solvability.

Application to V.F.

Notation: operator associated with a $\overset{\text{bdd.}}{}$sesquilinear form. $a(V,V)\to\mathbb{C}$.

Define: $A: V\to V$

$$(Av,w)_V = a(v,w) \quad \forall v,w\in V$$

Seek $u\in V$ $\quad a(u,v)+k(u,v)=(f,v) \quad \forall v\in V$.

$$(D+K)u = \vec{f} \quad \text{(operator equation)}$$

Def. 4.3.C $\quad V \hat{=}$ Hilbert space. A sesquilinear form $k: V\times V\to\mathbb{C}$ is compact if the associated operator is compact.

Thm. 4.3.D [Rellich theorem]

$Id: H^1(\Omega)\to L^2(\Omega)$ is compact for bounded $\Omega\subset\mathbb{R}^d$.

" $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ ".