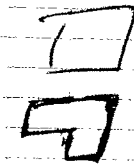
Discrete reg.   2D edge element.

Conti. reg. N_h

$\Omega = \{$ 

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Lecture X 21-11-2008

Problem setting: Magnetostatics in topo. trivial $\Omega \subset \mathbb{R}^3$.

$\longrightarrow$ $\underline{a}$-based formulation

$$\left.\begin{aligned} \operatorname{curl} \underline{\underline{\mu}}^{-1} \operatorname{curl} \underline{a} &= \underline{j} \\ \operatorname{div} \underline{a} &= 0 \\ \underline{a} \times \underline{n} &= 0 \end{aligned}\right\} \text{ in } \Omega$$

where $\operatorname{div} \underline{j} = 0$

Approach 1: discrete variational problem;

Seek $\underline{a}_h \in W^1(\mathring{\mathcal{T}}_h)$

$$\int_\Omega \underline{\underline{\mu}}^{-1} \operatorname{curl} \underline{a}_h \cdot \operatorname{curl} \underline{a}_h' \, dx + \int_\Omega \underline{a}_h' \cdot \operatorname{grad} \varphi_h \, dx = \int_\Omega \underline{j} \cdot \underline{a}_h' \, dx \quad \forall \underline{a}_h' \in W^1(\mathring{\mathcal{T}}_h)$$

$$\int_\Omega a \cdot \operatorname{grad} \Phi_h' \, dx = 0 \qquad \varphi_h = 0 \quad \forall \varphi_h' \in W^0(\mathring{\mathcal{T}}_h)$$

$-(\varphi_h, \varphi_h')_h$: inner prod. on $W^1(\mathcal{T}_h)$

In matrix forms:
$$\underline{C}^T M_{\underline{\mu}^{-1}} \underline{C} \, \vec{a}_h + \underline{M}^1 G \, \vec{\varphi}_h = \vec{j}$$
$$G^T \underline{M}^1 \vec{a}_h - \underline{M}^0 \vec{\varphi}_h = 0$$

$$\Rightarrow \left(\underline{C}^T M_{\mu^{-1}} \underline{C} + \underline{M}^1 \underline{G} \, \underline{M}^{0} \, \underline{G}^T \underline{M}^1\right) \vec{a}_h = \vec{j} \quad (3.9a)$$

Approach II. grad-div regularization.

$$\operatorname{curl} \underline{\underline{\mu}}^{-1} \operatorname{curl} \underline{a} - \operatorname{grad} \operatorname{div} \underline{a} = \underline{j} \text{ in } \Omega.$$

$$\Updownarrow$$

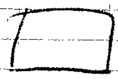
V.F. Seek $\underline{a} \in \underbrace{H_0(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)}_{V}$ s.t.

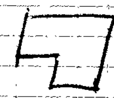
$$\int \underline{\underline{\mu}}^{-1} \operatorname{curl} a \cdot \operatorname{curl} a' + \int_\Omega \operatorname{div} a \cdot \operatorname{div} a' = \int \underline{j} \cdot \underline{a}' \quad \forall a' \in V.$$

Discrete V.F. Seek $\underline{a}_h \in (W^{c}(\mathcal{T}_h))^3 \cap H_0(\mathrm{curl},\Omega) = V_h$

$$\int_\Omega \underline{\underline{\mu}}^{-1}\,\mathrm{curl}\,a\cdot\mathrm{curl}\,a'\,dx + \int_\Omega \mathrm{div}\,a\cdot\mathrm{div}\,a'\,dx = \int \underline{j}\cdot\underline{a}_h'\,dx \quad \forall a_h'\in V_h \qquad (3.9.f)$$

Observations from Numerical tests:

$\Omega =$  — both schemes converge to the same solution.

$\Omega =$  — vastly different solns. for respective mesh.

Rationale behind:

Observ. $V_h \in \underset{\sim}{H}^1_x(\Omega) = (H^1(\Omega))^3 \cap H_0(\mathrm{curl},\Omega)$

$$\underbrace{(3.9.f) \text{ is discretized w.r.t } \underset{\sim}{H}^1_x(\Omega)}_{\text{is this still equivalent to } (3.9.e)\,?}$$

Recall: to show $\mathrm{div}\,\underline{a} = 0$. in (3.9.f) we choose $\underline{a}' = \mathrm{grad}\,\varphi$

for some $\underline{\Delta\varphi \in L^2(\Omega)}$. $\underline{\varphi \in H_0^1(\Omega)}$

For restricted problem, we may only choose $\underline{a}' = \mathrm{grad}\,\varphi$. $\varphi \in H^2(\Omega)\cap H_0^1(\Omega)$

? When $\Delta(H^2(\Omega)\cap H_0^1(\Omega)) \underset{(\text{normally} \subset)}{=} L^2(\Omega)$ (Ex.1) ① "=" $\Leftrightarrow$ $\mathrm{div}\,a = 0$ ② equiv to (3.9.e)

This condition is sufficient for equivalence with (3.9.e)

By elliptic regularity theory

$$\left.\begin{array}{l} \Delta u \in L^2(\Omega) \text{ in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{array}\right\} \Longrightarrow \begin{array}{l} u \in H^2(\Omega) \in H_0^1(\Omega) \\ \text{iff } \Omega \text{ is convex} \end{array}$$

If $\Delta(H^2(\Omega)\cap H_0^1(\Omega))$ is a proper closed subspace of $L^2(\Omega)$

Ex.2. $\Longrightarrow$ equivalence fails. choose a s.t. $\mathrm{div}\,\underline{a} \in (\Delta(\cdot))^\perp$

$\Rightarrow$ a solves V.F. but a does not satisfies (original prob.)

Lemma 3.9.A $\underline{u} \in (C^\infty(\overline{\Omega}))^3$. $\partial\Omega$ p.w. smooth

$$\|\operatorname{curl}\underline{u}\|^2_{L^2(\Omega)} + \|\operatorname{div}\underline{u}\|^2_{L^2(\Omega)} = \|D\underline{u}\|^2_{L^2(\Omega)}$$

$$+ \sum_{i=1}^{p}\int_{\Gamma_i} \operatorname{grad}_{\Gamma}(\underline{u}\cdot n)\,\underline{u}_t - (\operatorname{div}_{\Gamma} \underline{u}_t)(\underline{u}\cdot n)\,dS + \sum_{i=1}^{p}\int_{\Gamma_i}\left(\underline{u}_t^T\, B_t\, \underline{u}_t + tr(B)(\underline{u}\cdot n)^2\right)dS$$

Γ_i: smooth components of $\partial\Omega$.

$B(x)$ $D\underline{n}(x)$: curvature tensor of the boundary.
$\uparrow$ extended normal vector field.

Sketch of proof

$$\|\operatorname{curl}\underline{u}\|^2_{L^2(\Omega)} = \int_\Omega \sum_{j,k=1}^{3} \left|\frac{\partial u_j}{\partial x_k}\right|^2 - \sum_{j,k=1}^{3} \frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_k}\,dx$$

$$\|D\underline{u}\|^2_{L^2(\Omega)} \qquad \overset{I.b.P.}{=} \int_\Omega -\operatorname{grad}(\operatorname{div}\underline{u})\cdot\underline{u}\,dx + \sum_{j,k=1}^{3}\int_{\partial\Omega}\frac{\partial u_k}{\partial x_k} n_k\, u_j\, dS$$

$$\overset{Green's}{=} \int_\Omega |\operatorname{div}\underline{u}|^2 + \int_{\partial\Omega}\sum_{j,k=1}^{3}\frac{\partial u_k}{\partial x_k} n_k u_j - \operatorname{div}\underline{u}\cdot(\underline{u}\cdot\underline{n})\,dS$$

\#

Special case: $\Omega = \{x_1 > 0\}$ $\underline{u}_t = 0$ on $\partial\Omega$

$n = \begin{pmatrix}1\\0\\0\end{pmatrix}$, i.e. $u_2(0,\cdot,\cdot) = 0$ $u_3(0,\cdot,\cdot) = 0$

$\Rightarrow$ boundary value: $\int_{\partial\Omega} \frac{\partial u_1}{\partial x_1}u_1 - \frac{\partial u_1}{\partial x_1}u_1\,dS = 0$

Corollary of Lemma 3.9.A if $\Omega \hat{=}$ polyhedron ($\rightarrow B = 0$ on all Γ)

and $\underline{u}_t = 0$

Then $\|\operatorname{curl}\underline{u}\|^2_{L^2(\Omega)} + \|\operatorname{div}\underline{u}\|^2_{L^2(\Omega)} = \|D\underline{u}\|^2_{L^2(\Omega)}$ (3.9.g)

$\forall\, \underline{u} \in \underset{\sim}{H}^1_t(\Omega)$

If Ω is a polyhedron $\underline{u} = 1$.

(3.9.f) on $\underset{\sim}{H^1_*}(\Omega)$ $\Longleftrightarrow$

$$\int_\Omega \underline{D}\,\underline{a} : \underline{D}\,\underline{a}' = \int_\Omega \underline{j} \cdot a'\,dx \qquad \forall a' \in H^1_*(\Omega)$$

<u>Lemma 3.9.B.</u> $H^1_*(\Omega)$ is a closed subspace of $H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$ for polyhedral domain.

Proof: trivial to show $H^1_*(\Omega) \subset H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$

$\Rightarrow$ closedness: Let (u_n) be a Cauchy Sequence in $H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$

then $(u_n) \in H^1_*(\Omega) \quad \forall n$

$\Rightarrow$ $\{u_n\}$ is also a Cauchy Sequence in $H^1_*(\Omega)$

$\Rightarrow$ u_n converges in $H^1_*(\Omega)$ (the limit is in $H^1_*(\Omega)$ due to completeness of $H^1_*(\Omega)$)

<u>Remark</u>: * If Ω is convex $\Rightarrow$ $H^1_*(\Omega) = H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$

$\Rightarrow$ to show "$\supset$": recall the construction of Lifting operator.

$$L : \mathrm{curl}\,(H_0(\mathrm{curl}, \Omega)) \longrightarrow (H^1_0(\Omega))^3.$$

This shows that for any $\underline{u} \in H_0(\mathrm{curl}, \Omega)$ we can find a splitting

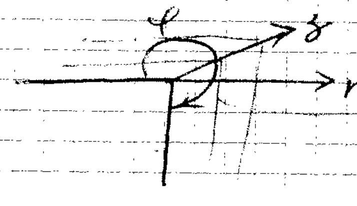
$$\underline{u} = \underline{\Psi} + \mathrm{grad}\,\varphi \qquad (\mathrm{curl}\,\underline{u} = \mathrm{curl}\,\underline{\Psi})$$

where $\underline{\Psi} \in (H^1_0(\Omega))^3$, $\varphi \in H^1_0(\Omega)$

If $\mathrm{div}\,\underline{u} \in L^2(\Omega)$ $\Rightarrow$ $\Delta\varphi \in L^2(\Omega)$

By elliptic regularity $\Rightarrow$ $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$ $\Rightarrow$ $\underline{u} \in H^1_*(\Omega)$

If Ω is non-convex or more specifically has one reentrant corner.



$\Rightarrow$ reentrant angle : α

$v(r,\varphi,z) = r^{\pi/\alpha} \sin(\frac{\pi}{\alpha}\varphi)$

$v|_{\partial\Omega} = 0. \quad \Delta u = 0$ (locally

After multiplication with some smooth cut-off functions $\equiv 1$ (close to edge)

on Ω $\quad v|_{\partial\Omega} = 0, \quad \Delta \in C^\infty(\overline{\Omega})$

Thus $\underline{u} = \operatorname{grad} v \in H_0(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)$

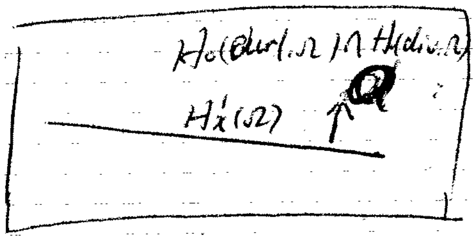
But $\underline{u} \notin (H^1(\Omega))^3$

<u>Ex. 3</u>

because $\int_\Omega |\operatorname{grad} \underline{u}|^2 dx = \infty$.

$\xRightarrow{\text{By 3.9.B}}$ $H^1_x(\Omega)$ is a proper subspace of $H_0(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)$

if Ω has an reentrant edge.



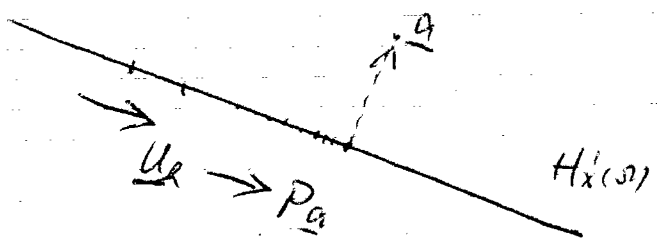
: solution of <u>V.F.</u> 3.9.d of magnetostatics.

if $a \notin H^1_x(\Omega)$ then solution to (3.9.f) $\Rightarrow$ Projection of a onto $H^1_x(\Omega)$

Whenever you have a proper closed subspace of a Hilbert space H.

then $\exists\, x \in H$ with positive distance from the subspace

Since $V_h \subset H^1_x(\Omega)$ Galerkin solutions are imprisoned in $H^1_x(\Omega)$



but the singular part of solution is missing.

"Math justification of "spurious mode""

3.10 Tree Gauging

Magnetostatics: $\underline{a}$ - based formulation,

F.E. Galerkin discretization based on $W^1(\mathcal{T}_h)$
(No gauging)

$$\underline{\underline{C}}^T M_{\mu^{-1}} C \vec{a}_h = \vec{j}_h$$

Note ① $Ker(\underline{\underline{C}}) \neq \{0\}$ ② column of $\underline{\underline{C}} \Longleftrightarrow$ edges of $\mathring{\mathcal{T}}_h$.

Find subspace of $W^1(\mathring{\mathcal{T}})$ but $Ker(\underline{C})$ restricted on this subspace $= \{0\}$

Is there a subset $\mathcal{G}_1(\mathring{\mathcal{T}}_h)$ such that

a subset of columns of $\underline{\underline{C}}$ (to form a matrix $\hat{\underline{\underline{C}}}$)

such that $Ker(\hat{\underline{\underline{C}}}) = 0$ and $Range(\underline{\underline{C}}) = Range(\hat{\underline{\underline{C}}})$

[Select a maximally linearly independent columns of $\underline{\underline{C}}$]

$$\Rightarrow \quad \hat{\underline{\underline{C}}}^T M_{\mu^{-1}} \hat{\underline{\underline{C}}} \hat{\vec{a}}_h = \vec{j}_h$$

($\hat{\vec{a}}_h$ is shorter than $\vec{a}_h$)

coefficients belonging to remaining columns

$$\Rightarrow \quad \begin{pmatrix} \hat{\underline{\underline{C}}} \hat{\vec{a}}_h \\ 0 \end{pmatrix} = \underline{\underline{C}} \vec{a}_h$$

← dependent columns yield "0".

Ex.: describe an algorithm to

all remaining edges except those on spanning tree, $\Rightarrow$
(cotree edges)

Let $\eta_h \in \mathcal{C}^1(\mathcal{T}_h)$

$\downarrow$

$\omega_h = d_h \eta_h$

$\downarrow$

$\exists \eta_h' \in C^1(\mathcal{T}_h)$. $supp\, \eta_h' \subset$ cotree edges $d\eta_h' = \omega_h$

Algorithm:

- Build edge-vertex spanning tree.
- drop all columns contained in the spanning tree to get $\hat{\underline{\underline{C}}}$.

① first to show $\mathrm{kern}(\hat{\underline{\underline{C}}}) = \{0\}$

$\eta_h \in \mathcal{C}^1(\mathcal{T}_h)$ $\eta_h \neq 0$ supported on co-tree.

$\downarrow$

$\exists$ edge $e \in \mathcal{S}^1(\mathcal{T}_h)$ $\eta_h(e) \neq 0$

Spanning tree $\Rightarrow$ $\exists$ tree path γ s.t. $\gamma \cup \{e\}$ is a cycle

Γ trivial topology $\longrightarrow$ $\exists$ F (discrete surface) with $\partial F \subset \gamma \cup \{e\}$

$$\underline{d\eta_h}(F) = \eta_h(\gamma \cup \{e\}) = \eta_h(\{e\}) \neq 0$$

$$\Rightarrow \underbrace{\boxed{\underline{\underline{C}}\,\vec{a}_h \neq 0}}_{\mathrm{curl}\,\underline{a} \neq 0} = \underline{\hat{\underline{\underline{C}}}\,\hat{\vec{\eta}}_h = \underline{\underline{C}}\,\vec{\eta}_h \neq 0}$$

② to show $\mathrm{Range}(\hat{\underline{\underline{C}}}) = \mathrm{Range}(\underline{\underline{C}})$

(By dimension argument) $\subset$ is clear.

$\Rightarrow$ "$\supset$". $\ker(\hat{\underline{\underline{C}}}) = \{0\}$ $\Rightarrow \dim \mathrm{Range}(\hat{\underline{\underline{C}}}) = \dim$ of columns in $\hat{\underline{\underline{C}}}$.

$$\dim \mathrm{Range}(\underline{\underline{C}}) = \# \mathcal{S}^1(\mathcal{T}_h) - \cancel{\#\dim}(\mathrm{kern}(\underline{\underline{C}}))$$

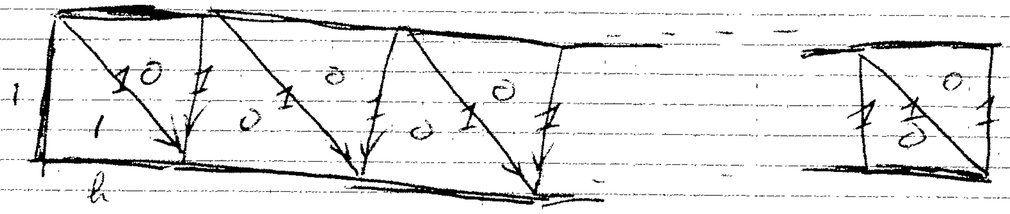
$$\mathrm{Kern}(\underline{\underline{C}}) \underset{\substack{\uparrow \\ \text{trivial topo.}}}{=} \mathrm{Range}(\underline{\underline{G}}) \quad \text{vector edge incidence matrix)}$$

$$\dim \mathrm{kern}(\underline{\underline{C}}) = \dim \mathrm{Range}(\underline{\underline{G}}) = \# \mathcal{S}^0(\mathcal{T}_h) - 1$$

$$\begin{aligned} \dim \mathrm{Rang}(\underline{\underline{C}}) &= \# \mathcal{S}^1(\mathcal{T}_h) - \# \mathcal{S}^0(\mathcal{T}_h) + 1 \\ &= \# \mathcal{S}^1(\mathcal{T}_h) - (\#\text{ of edges in spanning tree}) \\ &= \#\text{ edges in cotree} \\ &= \dim \mathrm{Range}(\hat{\underline{\underline{C}}}) \end{aligned}$$

Drawback of cotree gauging:

Potential instability. [?: 2D may be stable]



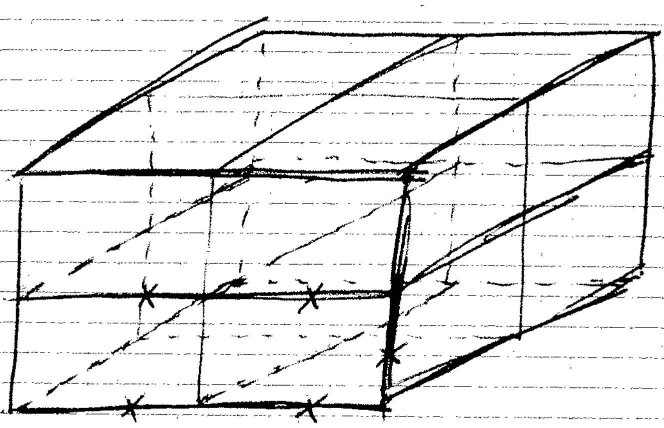
spanning tree: ——

deteriorate

v_h 2-form $\quad \|v_h\|_{L^2(\Omega)} = O(h^{-1}) \quad \Rightarrow \quad \|\eta_h\| \overset{C}{\le} \|v_h\|$

$\eta \Leftrightarrow$ cotree supported 1-form potential $\quad \|\eta_h\|_{L^2(\Omega)} = \frac{1}{h} O(h)$

In 3D.



$\eta_h \in W^1(\mathcal{T}_h)$ defined by

$$\begin{cases} \text{d.o.f.} = 1 & \text{on } *\text{-edges} \\ \text{d.o.f.} = 0 & \text{on all other edges} \end{cases}$$

$\eta_h' \in W^1(\mathcal{T}_h)$ supported on co-tree edges $\quad \operatorname{curl}\eta_h' = \operatorname{curl}\eta_h$

$$\boxed{\|\eta_h'\|_{L^2(\Omega)} \sim h^{-1} \|\underbrace{\eta_h}_{\operatorname{curl}\eta_h}\|_{L^2(\Omega)}}$$

$\Rightarrow$ it may take a "large" (in terms of L^2-norm) η_h supported on co-tree edges to produce a small $\operatorname{curl}\eta_h$ $\Rightarrow$ leads to potential ill-conditioned linear systems.

! Open: How to choose "good" tree

Ex: give a banded of co-tree gauged system matrix

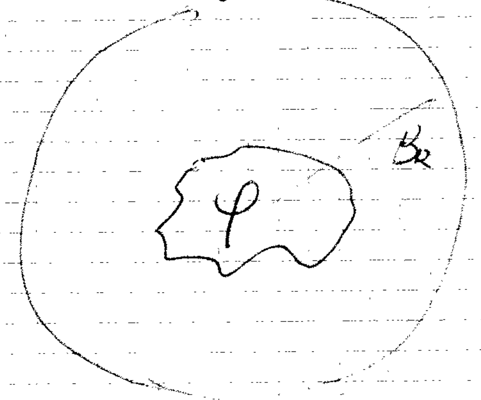
3.11. Unbounded domains.

Example: V-based V.F. for electrostatics on A_3

$$-\Delta v = \varphi \qquad \varphi \in L^2(A_3) \text{ compactly supported!}$$

$\longrightarrow$ uniqueness requires "boundary conditions" at infinity!

(decay and radiation condition)



From far away, φ can be viewed as a point charge

By Gauss Law

$$\int_{\partial B_R} \underline{d}\cdot\underline{n}\, dS = \int_{B_R} \varphi \, dx = Q \text{ (total charge)}$$

So $\underline{d}$ almost radially symmetric. Let $\underline{d} = d(r)\,\underline{e}_r$

$$\int_{\partial B_R} \underline{d}\cdot\underline{n}\, dS = 4\pi r^2 d(r) = Q \quad \forall r$$

$$\Rightarrow d(r) \sim 1/r^2$$

Since $d = \varepsilon \,\mathrm{grad}\, v$, $\quad v = v(r)\underline{e}_r$ for r large.

$$\Rightarrow v(r) \sim 1/r$$

decay condition: $v(x) = O(|x|^{-1})$ for as $|x| \to \infty$

$\rightarrow$ (guarantee uniqueness)

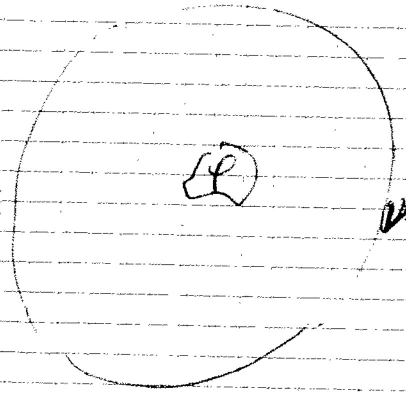
V.F. $\quad v \in H^1_{BL}(A_3) = \left\{ \frac{1}{\sqrt{1+|x|^2}}\, v \in L^2(A_3),\ \mathrm{grad}\, v \in L^2(A_3) \right\}$

$$\int_{A_3} \mathrm{grad}\, v \cdot \mathrm{grad}\, v' \, dx = \int_{A_3} \varphi \cdot v' \, dx \qquad \forall v' \in H^1_{BL}(A_3)$$

Decay condition $\longrightarrow$ v small for $|x|$ large

— $v = 0$ on cut-off boundary

(sufficiently far from support of φ)



effective and robust FE discritization requires:

$v=0$ ① — graded mesh : the farther away from φ

effeciency

the smoother of v

geometrically graded mesh:

$\Gamma \quad v(x) = \frac{1}{|x|}$

$$\| \operatorname{grad}(v - I^0 v) \|_{L^2(\Delta)} \leq C \, h_\Delta \, |v|_{H^2(\Delta)}$$

depends only on $\mathcal{C}_\Delta$

Rule of thumb: equidistribution of the interpolation error.

$$|v|^2_{H^2(\Delta)} = \int_{r_k}^{r_{k+1}} |\partial^2 \tfrac{1}{r^3}|^2 r^2 \, dr = \frac{1}{r_k^5} - \frac{1}{r_{k+1}^5} \qquad h_\Delta = r_{k+1} - r_k$$

Find r_k : st $\quad (r_{k+1} - r_k) \sqrt{\left(\frac{1}{r_k^5} - \frac{1}{r_{k+1}^5} \right)} = \text{Const.}$

② — Check boundary flux

$$\| h \, \underline{\underline{\varepsilon}} \operatorname{grad} v_h \|_{L^2(\partial\Omega)} \leq Tol.$$

h: size of element located on $\partial\Omega$

where h the local meshwidth

If not satisfied $\Rightarrow$ add one more layer of geometric graded mesh.

(A posteriori adapted cut-off mesh