[19-09-2008 Lecture I]

Computational Electromagnetics

(Graduate Course: HS08 R. Hiptmair)

Chap. I Maxwell's Equations

(not just PDE system, but also structures - -)

1.1 Fields

↳ functions on space-time $f = f(x, t)$

$x \in A_3$ (3D affine space) without origin (fix an origin $\Rightarrow R^3$)

$t \in A_1$ (time)

"Meaning" of fields $f \longleftrightarrow$ measurement

1.1.1. Electric fields $\underline{e}$ (captical letters for spaces operator - -)

Measurement through the virtual work it takes to displace a probe charge q

$$\delta W = q\, \underline{e}(x,t) \cdot \underset{\substack{\uparrow \\ \text{small displacement}}}{\delta x} = q\, \underline{e}(x,t)(\delta x)$$

$\cdot \hat{=}$ Euclidean scalar product.

$$\underline{e} \;\hat{=}\; \text{linear mapping} \begin{cases} R^3 \rightarrow R \\ \delta x \rightarrow (\ ) \end{cases}$$

(displacement, (voltage)

▷ Instead of viewing $\underline{e}$ as mapping $A_3 \times A_1 \rightarrow R^3$, it can be regarded as a mapping

diff. form ($\underline{e}(x,t)(\delta x)$)

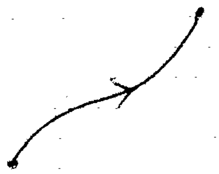
$$\underbrace{\underset{\text{space-time}}{\underline{e} : A_3 \times A_1} \rightarrow \text{linear forms on } \underset{\substack{\uparrow \\ \text{(space of virtual displacement)}}}{R^3}}_{\text{Local Perspective}}$$

Local Perspective
(differential) ?

⇕

nonlocal one.

Realistic measurement, work it takes to move a test charge along a path (with direction)

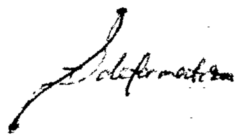


$\rightarrow$ Interpretation of $\underline{e} \;\hat{=}\;$ mapping (integral form)

$$\{\text{directed paths in } \mathbb{R}_3\} \longrightarrow \underset{(\text{voltage})}{\mathbb{R}}$$

with the following properties:

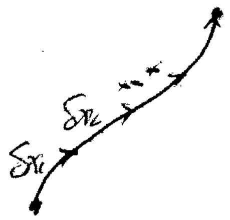
- additive w.r.t. concatenation of paths.
  (implies change of orientation $\Longleftrightarrow$ change of sign)
- continuous w.r.t deformations of paths. Int:
  (see geometric measure theory for details)



Non-local (integral) perspective

Local $\Longleftrightarrow$ Non-Local

Connection: by Riemann Sum



non-local view $\qquad$ local view

$$\underline{e}(\gamma) = \int_\gamma \underline{e}(d\vec{s})$$

Riemann sum

1.1.2 Magnetic Induction $\underline{b}$

Measurement: i) by (virtual) (infinitesimal) work required for a (virtual) transversal shift of a charge moving with velocity $\underline{v}$

$$\delta w = \underbrace{q\,(\underline{b}(x,t) \times \underline{v}) \cdot \delta x}_{\text{Lorenz force}} = q\,\underline{b}(x,t)(\underline{v}, \delta x)$$

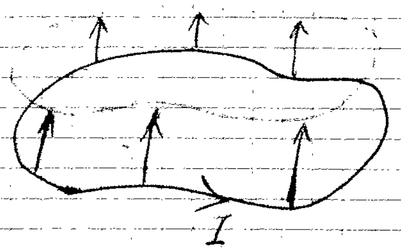
ii) by torque on a magnetic dipole of strength $\underline{m}$

$$\delta w = (\underline{b}(x,t) \times \underline{m}) \cdot \underset{\uparrow \text{ axis vector}}{\delta x} = \underline{b}(x,t)\,(\underline{m}, \delta x)$$

▷ Local point of view:

$$\underline{b}: \underbrace{A_3 \times A_1}_{\text{space - time}} \longrightarrow \{\text{alternating bilinear forms on } \mathbb{R}^3\}$$

alternating $\hat{=}$ $b(x,t)(\underline{v}, \underline{v}) = 0. \quad \forall v \in \mathbb{R}^3 \}$



Measurement: displacement of current carrying Loop.

work required: $\underset{(\text{small})}{W} = I \int_\Sigma \underline{b} \cdot \underline{n} \, dS$

$\Sigma \;\hat{=}$ surface swept by moving Loop. $\quad n \;\hat{=}$ unit outward normal

$\Rightarrow$ Interpretation of $\underline{b}$ $\hat{=}$ mapping

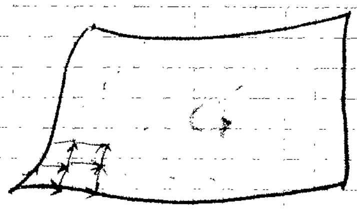
$$\{\text{oriented surfaces in } A_3\} \longrightarrow \underset{(\text{? unit Voltage} \cdot m^2,\ \text{flux})}{\mathbb{R}}$$

with similar properties:

$$\underbrace{\begin{array}{l}\text{— additive w.r.t concatenation of surfaces} \\ \text{— Continuous w.r.t deformations of surfaces}\end{array}}_{\text{Non-local view.}}$$

connection
by Riemann Sum

take into account orientation !!
non-local



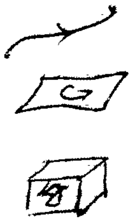
$$\underline{b}(\Sigma) = \int_\Sigma \underline{b} \cdot d\vec{S}$$

small oriented parallelograms.

1.2. Forms

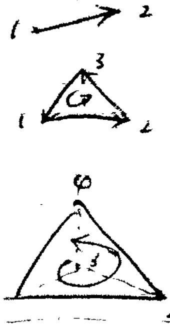
$\mathcal{S}_\ell(A_n) \hat{=} \{$ ~~[illegible]~~ oriented, p.w. smooth ℓ-dim submanifolds of $A_n\}$

$\ell = 0$ point
$\ell = 1$ path
2. surface
3. volume
⋮



$0 \le \ell \le n$

Simplices



cork screw

? cork? screw

Def 1.2A: An (integral) ℓ-form, $0 \le \ell \le n$, on A_n is an additive, continuous mapping ~~from~~ $\mathcal{S}_\ell(A_n) \to \mathbb{R}$.

$\Rightarrow$ vector space $\mathcal{F}^\ell(A_n) : \hat{=} \{\ell\text{-forms}\}$

Notation: $\omega \in \mathcal{F}^\ell(A_n)$, $\Sigma \in \mathcal{S}_\ell(A_n)$: $\omega(\Sigma) =: \int_\Sigma \omega$

RK: $\underline{e}$: 1-form $\underline{b}$: 2-form.

non-local view.

Def 1.2B: A differential ℓ-form on A_n (of class C^m) is a mapping $A_n \longrightarrow \Lambda^\ell(\mathbb{R}^n)$

$\Lambda^\ell(\mathbb{R}^n)$: vector space of alternating ℓ-multilinear forms on $\mathbb{R}^n$.

— Ad ? $\Lambda^\ell(\mathbb{R}^n) :=$

$$\left| \begin{array}{l} \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\ell \text{ times}} \longrightarrow \mathbb{R} \\ (v_1, \cdots, v_\ell) \longrightarrow m(v_1, \cdots v_\ell) \end{array} \right.$$

$\Rightarrow$? vector space $\mathcal{D}\mathcal{F}^\ell_m(A_n)$
$\hat{=} \{C^m$ diff. ℓ-forms$\}$

$m(v_1, \cdots v_\ell) = 0$ if $v_i = v_j$ for $i \neq j$.

Ex: $(\ell = 3, n = 3)$ $(v_1, v_2, v_3) \longmapsto \det(v_1, v_2, v_3)$

Lemma: $\dim \Lambda^\ell(\mathbb{R}^n) = \binom{n}{\ell}$

Local view

— Ad C^m :≙ m-times continuously differentiable (w.r.t. ?)

$$\text{Def } 1.2.A \xleftrightarrow[\text{by Riemann Sum}]{\text{connection}} \text{Def } 1.2B.$$

Euclidean Vector Proxies for $n=3$:

l	Differential forms:	related function / vector fields
0	$x \to \omega(x) \in \mathbb{R}$	$u(x) = \omega(x)$
1	$x \to \{\underline{v} \to \omega(x)(\underline{v})\}$	$\underline{u}(x)\cdot\underline{v} = \omega(x)(\underline{v}) \quad \forall \underline{v}\in\mathbb{R}^3$
2	$x \to \{(\underline{v}_1,\underline{v}_2) \to \omega(x)(\underline{v}_1,\underline{v}_2)\}$	$\underline{u}(x)\cdot(\underline{v}_1\times\underline{v}_2) = \omega(x)(\underline{v}_1,\underline{v}_2) \quad \forall \underline{v}_1,\underline{v}_2\in\mathbb{R}$
3	$x \to \{(\underline{v}_1,\underline{v}_2,\underline{v}_3) \to \omega(x)(\underline{v}_1,\underline{v}_2,\underline{v}_3)\}$	$u(x)\cdot\det(\underline{v}_1,\underline{v}_2,\underline{v}_3) = \omega(x)(\underline{v}_1,\underline{v}_2,\underline{v}_3)$ $\forall \underline{v}_1,\underline{v}_2,\underline{v}_3\in\mathbb{R}^3$.

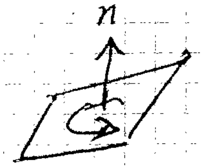
→ establish identification (Isomorphism) $\quad D\mathcal{F}^l(\Omega_3) \cong \{\Omega \to \mathbb{R}^{\binom{3}{l}}\}$

Note: vector field representation depends on inner product, cross product, etc.
But forms don't need these additional structures. $\quad \times \quad \det(\cdot,\cdot,\cdot)$

? Diff. form: measurement instrument

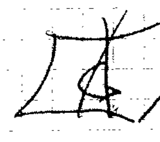
Int. form: measurement result.

Rk: For Euclidean vector proxies



$l=1.\quad \int_\gamma \omega = \int_\gamma \underline{u}\cdot d\vec{s}$

$l=2.\quad \int_\Sigma \omega = \int_\Sigma \underline{u}\cdot d\vec{S} \quad \left(\text{or} = \int_\Sigma \underline{u}\cdot\underline{n}\, dS\right)$



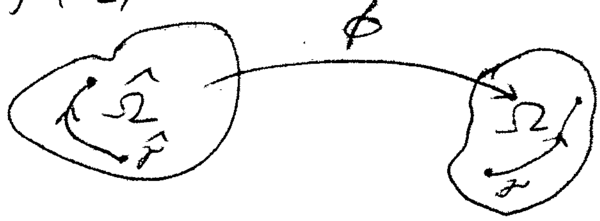
$l=3.\quad \int_V \omega = \int_V u\, dx$

Elementary Calculus on forms

- Transformation w.r.t. diffeomorphism (Smooth. bijective) $\cancel{\phi : A_n \to A}$ $\phi : \hat{\Omega} \to \Omega$

$\hat{\Omega}, \Omega \in A_n$

Def 1.2.C Pullback $\phi^* : \mathcal{F}^l(\Omega) \to \mathcal{F}^l(\hat{\Omega})$



$$\int_{\hat{\Sigma}} \phi^* \omega = \int_{\phi(\hat{\Sigma})} \omega$$

$\forall \hat{\Sigma} \in \mathcal{S}_l(\hat{\Omega})$, $\forall \omega \in \mathcal{F}^l(\Omega)$

$\to$ induces a pullback of differential forms of class C^0.

$\to$ pullback of Euclidean vector proxies for $n=3$.

$x = \phi(\hat{x})$, $\hat{x} \in \hat{\Omega}$

$l = 0$. $\underbrace{VP(\phi^*\omega)(\hat{x})}_{\hat{u}(\hat{x})} = \boxed{VP(\omega)(\phi(\hat{x}))} = \underbrace{VP(\omega)(x)}_{u}$, $x = \phi(\hat{x})$, $\hat{x} \in \hat{\Omega}$

$l = 1$. $\underbrace{VP(\phi^*\omega)(\hat{x})}_{\underline{\hat{u}}(\hat{x})} = D\phi(\hat{x})^T \underbrace{VP(\omega)(x)}_{\underline{u}} = D\phi(\hat{x})^T \underline{u}(x)$

$l = 2$ $\quad \underline{\hat{u}}(\hat{x}) = \det D\phi(\hat{x})\, D\phi(\hat{x})^{-1} \underline{u}(x)$

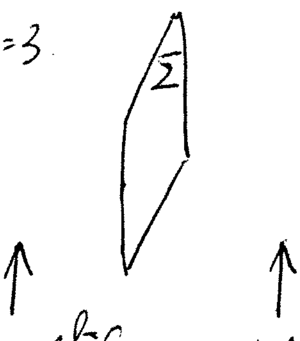
$l = 3$ $\quad \hat{u}(\hat{x}) = \det D\phi(\hat{x})\, u(x)$

covariant transformations

- **Continuity conditions**: (seal local forms to form a global one)

Q: When do we get a global l-form on A_3

$n=3$



Smooth l-forms on both sides of Σ.

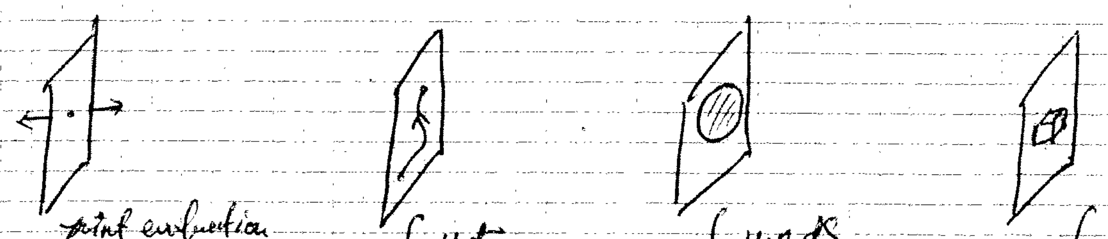
$l = 0$: continuity of vector proxy (VP)

$l = 1$: tangential continuity of VP

$l = 2$: normal continuity of VP

$l = 3$: No continuity required ~~for~~ of VP (but finite fields)

$l=0$ $\quad$ $l=1$ $\quad$ $l=2$ $\quad$ $l=3$ $\quad$ ⑦



point evaluation $\quad$ $\int_\gamma \underline{u}\cdot d\vec{s}$ $\quad$ $\int_\Sigma \underline{u}\cdot\underline{n}\,dS$ $\quad$ $\int_V u\,dx$

- **Exterior product:**

$$\wedge: \text{(wedge product)} \quad \begin{cases} \mathcal{DF}^{l,0}(A_n) \times \mathcal{DF}^{m,0}(A_n) \longrightarrow \mathcal{DF}^{l+m,0}(A_n) \\ (\omega, \eta) \longrightarrow \omega\wedge\eta \end{cases}$$

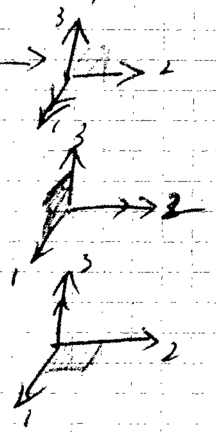
$$(\omega\wedge\eta)(x)(v_1, \dots, v_{l+m}) = \frac{1}{l!}\frac{1}{m!}\sum_{\sigma\in\Pi_{l+m}} \operatorname{sgn}\sigma\, \omega(x)(v_{\sigma(1)}, \dots v_{\sigma(l)}) \cdot \eta(x)(v_{\sigma(l+1)}, \dots v_{\sigma(l+m)})$$

$\Pi_{l+m} \hat{=} \{\text{permutation of } 1 \cdots l+m\}$

"Visualization" $n=3$. $l=1$. $m=2$

$$(\underset{\text{2-form}}{\omega} \times \underset{\text{1-form}}{\eta})(x)(v_1, v_2, v_3) = \omega(x)(v_1, v_2)\cdot\eta(x)(v_3) + \omega(x)(v_2, v_3)\cdot\eta(x)(v_1) + \omega(x)(v_3, v_1)\cdot\eta(x)(v_2) \quad (l=2.\ m=1)$$

$$= \omega(x)(v_1)\cdot\eta(x)(v_2, v_3) + \omega(x)(v_2)\cdot\eta(x)(v_3, v_1) + \omega(x)(v_3)\cdot\eta(x)(v_1, v_2)$$



For Euclidean vector proxies

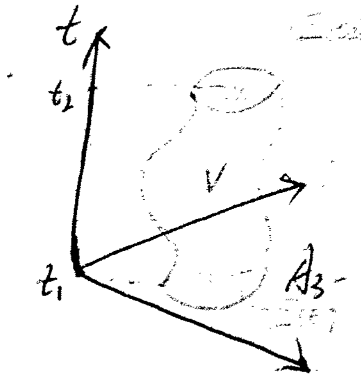
$$VP(\omega\wedge\eta)(x) = \begin{cases} VP(\omega)(x)\cdot VP(\eta)(x) & \text{if } l=0, \text{ or } m=0 \\ VP(\omega)(x) \times VP(\eta)(x) & \text{if } l=m=1 \\ VP(\omega)(x)\cdot VP(\eta)(x) & \text{if } l=1.\ m=2. \end{cases}$$

1.3. Topological electrodynamics Laws

- Faraday's Law (integral form)

$$\int_{t_1}^{t_2} \int_{\underset{\text{boundary}}{\partial \Sigma(t)}} \underline{e}\, \cancel{dt} \; \int_{\Sigma(t_1)} \underline{b} - \int_{\Sigma(t_2)} \underline{b} \qquad \boxed{(1.3.a)}$$

$\{\Sigma(t)\} \hat{=}$ family of p.w. connected compact 2-surfaces on A_3.



(1.3.a)

$\Updownarrow$

$$\int_{\partial V} b + e \wedge dt = 0$$

∂V: 2-manifold. $b + e\wedge dt$: 2-form. dt: 1-form on A_4

$dt(x,t)(V_x, V_t) = V_t$?

- Ampère's Law

Further electromagnetic field quantities:

— 1-form $\underline{h}$ $\hat{=}$ magnetic field.

— 2-form $\underline{d}$ $\hat{=}$ displacement current (electric flux intensity)

— 2-form $\underline{j}$ $\hat{=}$ electric current

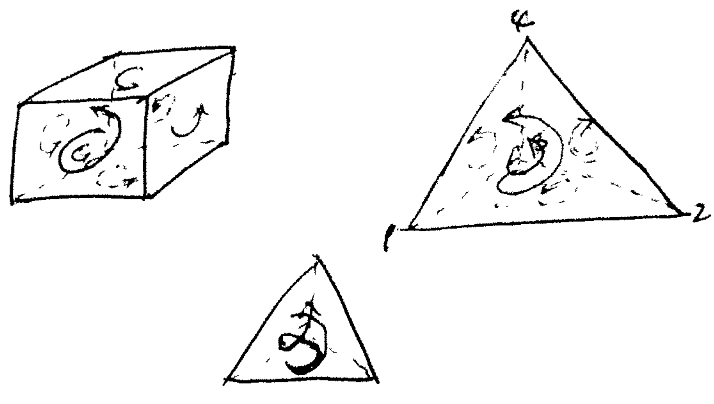
$$\boxed{(1.3.b)}$$

$$\int_{t_1}^{t_2} \int_{\partial \Sigma(t)} \underline{h}\, dt = \int_{\Sigma(t_2)} \underline{d} - \int_{\Sigma(t_1)} \underline{d} + \int_{t_1}^{t_2} \int_{\Sigma(t)} \underline{j}\, dt$$

$$\Leftrightarrow \quad \int_{\partial V} \underline{h} \wedge dt - \underline{d} = \int_V \underline{j} \wedge dt .$$

1.4 Exterior derivatives

induced orientation:



<u>Def. 1.4.A</u>: For $\omega \in \mathcal{F}^l(A_n)$, define its <u>exterior derivative</u> $d\omega \in \mathcal{F}^{l+1}(A_n)$

by
$$\int_\Sigma d\omega = \int_{\partial\Sigma} \omega \quad , \forall \Sigma \in S_{l+1}(A_n)$$

<u>Cor. 1.4.B</u>:
$$\phi^* \circ d = d \circ \phi^*$$

"The boundary of a boundary is empty" $\partial\partial\Sigma = 0$

<u>Cor 1.4.C</u>
$$d \circ d = 0.$$

<u>Thm 1.4.D</u>: For $\omega \in D\mathcal{F}^{l,1}(A_n)$

drop this

$$d\omega(x)(V_1, \cdots, V_{l+1}) = \sum_{i=1}^{l+1} (-1)^{i-1} (D\omega(x)V_i)(V_1, \cdots, \check{V_i}, \cdots V_{l+1})$$

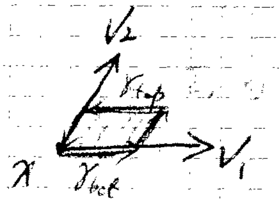
[Recall: $f: V \to W$ $\quad Df: V \to \mathcal{L}(V, W)$]
eg. $f: R^n \to R$ $\quad Df: R^n \to \mathcal{L}(R^n \to R)$
"grad"
$R^n \to R^n$ $\quad Df: R^n \to \mathcal{L}(R^n \to R^n)$
Jacobian

$$\boxed{D\omega : A_n \to \mathcal{L}(A_n, \Lambda^l(R^n))}$$

$D\omega$: directional derivative

Motivation: ω smooth diff. l-form

$$\omega(x)(V_1, \cdots, V_l) \simeq \frac{1}{t^l} \int_{\Sigma_t(x)} \omega \qquad \Sigma_t(x) = \{x + \sum_{k=1}^{l} s_k V_k,\ 0 \le s_k \le 1\}$$



$\lim_{t \to 0}$

$$\triangleright \quad d\omega(x)(V_1, \cdots V_{l+1}) = \lim_{t\to 0} \frac{1}{t^{l+1}} \int_{\Sigma(t)} d\omega$$

$$= \lim_{t\to 0} \frac{1}{t^{l+1}} \int_{\partial\Sigma_t(x)} \omega \qquad \omega(x+\delta x) \doteq \omega(x) + D\omega(x)\,\delta x$$

$$\int_{\gamma_{bot}} \omega + \int_{\gamma_{top}} \omega \doteq \omega(x)\, V_2 - \omega(x+tV_1)\cdot V_2$$

$$\doteq (\, D\omega(x)\, tV_1) \, (V_2)$$

Reisen CS 2008 / 2009 neu

→ 1) Amsterdam OCT 8-10, 08 (2008)

Flug bereits gekauft? Woudschouten annual dutch numerical analysis meeting

→ 2) Warwick, UK, Sept. 29, 08

Flug ab BS Mulhouse SUN, Sept 27, 08, hin.
an London Heathrow
Flug ab LON Heathrow TUE, Sept 30, 08, morgens, rück,
an Zürich Kloten
(falls möglich)

Opening symposium of Warwick's special year in numerical analysis

→ 3) SUN 4. Jan. 09 : FRA → Houston

~~MON~~ SUN 11. Jan. 09 : Houston → FRA

MON 11.1.09 : FRA → London HR

Do (oder) FR 16.1.09 LHR → FRA.

(2009) Texas A&M Univ.

Bitte Rücksprache

Danke, CS.

Di. 2. Nov. 2004

- SAM Report

→ Kopie mit. — 58

$\boxed{9 — 13^{00}}$

$\cancel{11^{00}}$