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Applied Differential Geometry

(A compendium)

The content of these notes is what "compendium" suggests: Not a tutorial, but a list, in logical order, of concepts of differential geometry that can serve in the study of PDE's of classical physics, each with a condensed description[1]. The idea is to guide the reader along a way that can, for one who wants to reach those spots most useful for applications, be faster than conventional courses. But the need for a reference book will probably be felt. Good books about differential geometry, "pure" or "applied", exist in abundance, and the bibliography lists some. Select a few for yourself, and use what follows as a check-list to guide your study. Don't worry too much about mathematical technique as such (there are relatively few subtle or difficult proofs[2] in the subject, in comparison with its richness in concepts and structures), and try to gain geometric intuition.

Italics, besides their use for emphasis, serve here to highlight a word which is being defined, explicitly or implicitly. When a function f maps all or part of X into Y, we'll say that f is "of type $X \to Y$", and denote by $\mathrm{dom}(f)$ the subset of X formed of points for which $f(x)$ makes sense, or *domain* of f. The *codomain* $\{f(x) : x \in X\}$ will be $\mathrm{cod}(f)$. If a function f is defined by some formula or expression, such as $x \to \mathrm{Expr}(x)$, we shall feel authorized to write "$f = x \to \mathrm{Expr}(x)$", thus defining the left side by the right side. Then, both sides of this equality point to the same object, the function f. C^k means "differentiable k times", C^0 means "continuous", C^∞ means "differentiable ad libitum". Its synonym "smooth" can at places be understood as "differentiable as many times as required by the situation" (a cop-up for sure, but so convenient). Symbols may be overloaded, with context-dependent meaning.

0. AFFINE PRELIMINARIES

Affine space. You know what a vector space is. The real n-dimensional vector space will be denoted V_n in what follows. Affine space is "what is left of V_n when the origin is ignored". More seriously, affine space A_n is a set of points on which V_n acts by translations: This means that to each v of V_n is associated a transform $x \to T_v x$ of A_n to itself, the translation by v, with $T_v(T_w x) = T_{v+w}\, x$; moreover, one requires that $T_v x = x$ iff $v = 0$ and that for each y there be a v

[1] Not always proper *definitions*, however. Some fine points are glossed over. There is some risk in this, but the approach to applicable differentiable geometry can too easily be slowed down by excessive fussiness.

[2] Most proofs are terse one- or two-liners, inserted between brackets as a warning that you should rather redo them yourself.

such that $y = T_v\, x$. It's convenient, for obvious reasons, to denote this v by $y - x$, and $T_v x$ by $v + x$.

The other way round, start from an affine space A, select a point o to play the role of origin, and the "translation vectors" $x - o$ form a vector space, *associated with* A. The *dimension* of A is defined as that of V.

Barycenter. Given x, y, and a real θ, this is $(1 - \theta)x + \theta y$, that is, $T_{\theta v}x$, where $v = y - x$. This generalizes to k points, hence the notions of *affine basis, affine subspace, barycentric coordinates* (for $k = n + 1$). We shall indulge in writing $x = \sum_{i = 0, \dots, n} \lambda^i(x)\, x_i$, where what is meant is the *vector* equality $\sum_i \lambda^i(x)\, (x_i - x) = 0$.

Affine space makes a better framework than vector space for elementary classical physics. Unless some point cries out to be taken as origin, as may happen in celestial mechanics for instance, there is really no reason to distinguish one, and to work with vectors when points do just fine. So one should be careful not to say "vector" where "point" is implied, and to distinguish "components" (of vectors) from "coordinates" (of points). Don't confuse $\mathbb{R}^n$ with V_n either.

1. DIFFERENTIABLE MANIFOLDS

Manifolds. These are topological spaces that "look, locally, like A_n": A curve ($n = 1$), a surface ($n = 2$), the configuration space of a solid ($n = 6$), etc. Another way to describe them is as "patchworks of pieces of A_n, smoothly sewn up". With more precision, one is given a set of maps φ_α, $\alpha \in \mathcal{A}$, also called "charts", of type $X \to \mathbb{R}^n$ (the integer n is the *dimension* of X), each of which maps a part of X onto a domain of $\mathbb{R}^n$, bijectively. The set-union of the domains $\mathrm{dom}(\varphi_\alpha)$ should be X, and the $\varphi_\alpha \circ \varphi_\beta^{-1}$ should be C^∞ for all pairs α, β (*compatibility* condition of charts). Each chart maps a part of X onto a piece of $\mathbb{R}^n$, thus providing a system of "local coordinates" on X, and its inverse is a "local parameterization" of X. The idea is to provide sets with a structure that allows to do on them all that can be done in standard differential calculus. Thus, for instance, a function f of type $X \to \mathbb{R}$ is deemed C^∞ at x if the composition $f \circ \varphi^{-1}$ is C^∞, for some chart φ the domain of which covers x. If this property holds for a chart, it holds for any compatible chart: the compatibility condition is there to enforce this.

A set of charts is called an *atlas*. There may be few charts in an atlas (as few as two, for instance, may be enough to cover a sphere, if chosen well), but one may include as many of them as required by convenience in a descriptive atlas. The set of *all* charts compatible two by two with the given ones is the *complete* atlas. "The" manifold should be conceived as the structure formed by X and the complete atlas. (Be aware that the same *set* can conceivably be provided with two different, non-compatible complete atlases. But in that case, of course, one has two different *manifolds.*)

Manifolds with boundary. Same thing, but this time there are points (those of the boundary) where X looks locally like the half-space $\{x \in \mathbb{R}^n : x^1 \geq 0\}$. The boundary forms by itself a manifold (the boundary of which is empty), of dimension $n - 1$, denoted ∂X. From now on, "manifold" should be understood as "manifold with boundary", the boundaryless ones defined above being a subcategory.[3]

Tangent vectors at $x \in X$, called "vectors at x". These are equivalence classes of trajectories passing through x. Such a trajectory is a smooth map $g : \mathbb{R} \to X$ such that $g(0) = x$ *(smooth,* in line with our above conventions, means that $t \to \varphi(g(t))$ is C^∞ for any chart φ covering x). Two trajectories g and g' are equivalent if $|\varphi(g(t)) - \varphi(g'(t))| = o(t)$, which intuitively means they are tangent and velocities at time 0 match. One will check that this equivalence is, like the smoothness of g and g', a chart-independent fact (again, the compatibility relation between charts does this). Tangent vectors can thus be understood as velocity vectors, but beware there is no pre-existing affine space in which the trajectory is inscribed, hence no preexisting vector space the tangent vector is an element of. The tangent vector *is* the equivalence class.

Tangent space at x. Denoted T_xX, it's the set of all tangent vectors at point x, according to the above definition. It's isomorphic to V_n, the n-dimensional real vector space. Its structure of linear space comes from V_n via φ, and this structure does not depend on which chart is considered.

Tangent map at x. A smooth mapping $u \in X \to Y$, with $\mathrm{dom}(u)$ open in X, transforms trajectories at $x \in X$ into trajectories at $y = u(x)$ in Y, and hence vectors of $T_x X$ into vectors of T_yY. The correspondence of type $T_x X \to T_yY$ thus established is linear and is usually denoted $u_*(x)$. If v is a vector at x, the vector $u_*(x)v$ at $y = u(v)$ is dubbed its *push-forward*.

Immersions. Smooth maps, which may not be one-to-one, but the tangent maps of which have maximal rank. For instance, (a) to (d) in Fig. 1.1 are immersions, and (e) is not. The latter may very well be smooth, that is, C^∞ from $\mathbb{R}$ to $\mathbb{R}^2$, in spite of not looking that way (think of the projection of a smooth spatial curve along one of its tangents), but the tangent map at the tip of the cusp is two-to-one. (Another example would be: a trajectory without cross-overs, but with a vanishing speed at some point—but this is harder to render graphically.)

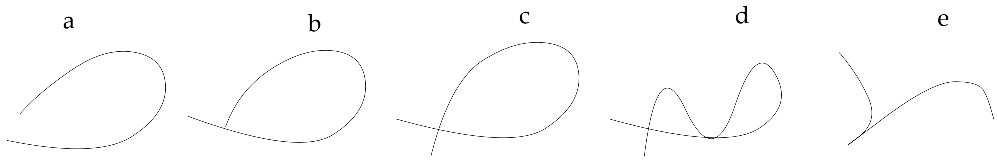

FIGURE 1.1. Five mappings of $\mathbb{R}$ (or as well, the open segment $]0, 1[$) as X, into $\mathbb{R}^2$ as Y. Only (a) qualifies as an embedding, even though (b) is a bijective immersion (end-points are *not* in the image). Maps (b), (c) and (d) are immersions, but not (e). One may legitimately conceive doubts about (d), but observe that a restriction to a small enough subinterval $]\alpha, \beta[$ of $]0, 1[$ is an embedding, in case (d) as well as in cases (b) or (c). Case (d) looks special because it's not a "generic position" immersion: slightly moving the curve will either suppress the first-order contact or create two frank intersections, like the one in (c). The only submanifold of the plane, in this picture, is under (a).

Diffeomorphisms. Bijective C^∞ maps from one manifold onto another, that is, maps which preserve the structure of differentiable manifold. Two diffeomorphic manifolds have the same dimension.

[3] An open disk in $\mathbb{R}^2$, for instance, has the structure of a two-dimensional manifold, but one whose boundary is empty (just like $\mathbb{R}^2$, to which it is diffeomorphic). But this manifold is implicitly described as *embedded* in $\mathbb{R}^2$, where its image *has* a non-empty topological boundary.

Embedding. A one-to-one mapping $u : Y \to X$ which is already an immersion and constitutes a diffeomorphism between Y and $u(Y)$. (Conversely, an immersion is locally an embedding, i.e., reduces to one by restriction to a part of Y.) One then says that $u(Y)$ is a *submanifold* of X (and confuses Y with $u(Y)$ if no hazard ensues). Figure 1.1 gives an example and four counter-examples.[4]

If Y is embedded in X, the tangent space T_xY at $x \in Y$, being abstracted from trajectories inscribed in Y, can be identified with a subspace of T_xX. Tangent vectors at x not contained in T_xY are said *transverse* to Y.

In particular, $T_x\partial X$ is a subspace of codimension 1 inside T_xX. Vectors of T_xX which are transverse to ∂X can be classified, in an obvious way, as *outward* or *inward* transverse vectors.

Remark. Immersion is a useful notion, because some manifolds of dimension 2, quite easily described via charts, cannot be embedded in three-dimensional space, but can still be immersed, which helps visualize them. For instance, Fig. 1.2 shows how to define two (boundaryless) manifolds of dimension 2, the torus and the Klein bottle, together with immersions of them (which as regards the torus is here, but need not be, an embedding). One can *reason* on the upper figures, but the other two are more lively. ◊

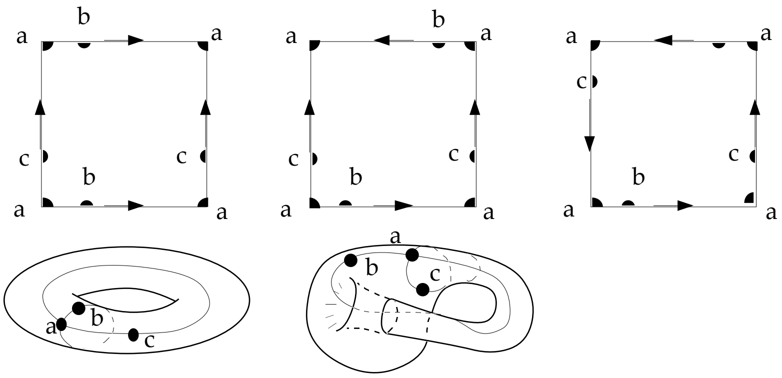

FIGURE 1.2. Torus, Klein bottle and projective plane as "abstract" manifolds and as immersed in 3D space. (Can you supply the missing drawing?) Starting from a manifold with boundary, one identifies the edges as suggested. Charts are equally easy to devise in all cases. (Neighborhoods of a, b and c are suggested.) But finding an immersion for the projective plane is quite a task, a book-size subject [Ap, Br, Co, Fr].

Basis vectors at a point, relative to a chart φ. Consider the trajectories $\gamma_i = t \to \varphi(x) + \{0, ..., t, ...,0\} \in \mathbb{R}^n$, with t in i^{th} position: Thus, γ_i is a straight line, containing $\varphi(x)$, parallel to the i^{th} coordinate axis in $\mathbb{R}^n$, ran along at speed 1. Now consider the trajectories $g_i = t \to \varphi^{-1}(\gamma_i(t))$, which are inverse images under φ of the γ_is. The g_is are the *coordinate lines* on X, containing x. The *basis vectors* at x are, by definition, their equivalence classes. One denotes them by $\partial_i(x)$, $i = 1, ..., n$. (Why "∂", among all symbols, will soon be clear.) They form vector fields (see below) $x \to \partial_i(x)$, denoted ∂_i. These are only locally defined, that is,

[4] Embedding is a delicate notion, a case of "monster-barring" in the sense of Lakatos [La]. The idea is, as always, to avoid pathological situations without sacrificing generality. Close examination of texts by reputable authors like, on the one hand, [KN] or [EDM], on the other hand, [RF] or [Ar], or ['N] (which I follow here, cf. p. 19) will show them in slight disagreement on what an embedding exactly is.

inside the domain of a specific chart, and don't match, as a rule, in the intersection of chart domains.

Remark. Beware, one may very well equip a manifold with basis vectors which do *not* stem from charts. We soon return to this. ◊

Derivatives, relative to a chart-induced basis. If f is a function of $X \to \mathbb{R}$ type, its i^{th} partial derivative in this basis, denoted $\partial_i f$. is by definition the i^{th} partial derivative of the composition $f \circ \varphi^{-1}$ (a function of type $\mathbb{R}^n \to \mathbb{R}$). This is the rate of variation of function f along the trajectory g_i, at x. [Proof: $f(g_i(t)) = f(\varphi^{-1}(\gamma_i(t))) = f(\varphi^{-1}(\varphi(x) + \{0, ..., t, ..., 0\})) = f(x) + t\, \partial_i(f \circ \varphi^{-1})(\varphi(x))$, up to higher-order terms. Divide by t and pass to the limit.]

Tangent bundle. Denoted TX, it's the set of pairs formed by a point x and a vector at x. (This is in physics the notion of "linked" or "bound" vector,[5] as opposed to a "free" vector: not only a direction and magnitude, but a point of application as well.) There is a natural manifold structure on this set (because a chart of X induces one on TX, as we'll see in a moment), whose dimension is $2n$. It's called a *bundle* because all vectors at the same point can be bunched together, hence a map π from TX to X, called *projection*, which sends all the elements of T_xX to point x. This divides TX into kind of slices, or *fibers*, the tangent spaces T_xX, hence the terminology of *fiberedspace* as a synonym for bundle. This applies, more generally, to manifolds equipped with a projection. For example, the manifold of all normals to a spatial curve has a natural structure of fibered space.

Thanks to the existence of a manifold structure on TX, many geometrical notions familiar in ordinary space make sense on manifolds. Take *hodograph*, for instance: one can, given a trajectory in X, assign to each of its points the tangent vector (along the trajectory), thus obtaining a trajectory in TX. More importantly, *vector fields* can be defined: smooth maps v from X into TX, subject to the condition that $\pi v(x) = x$. In plain words, to each point of the domain of the field is attached a vector at this point, with smooth dependence. Note that vector fields over X form a vector space (which we shall denote by $\mathcal{V}_1(X)$ in these notes): $\alpha u + \beta v$, where α and β are real coefficients,[6] is the field $x \to \alpha u(x) + \beta v(x)$.

Not *all* things that can be done in ordinary space make sense on manifolds, however. In particular, vectors at different points cannot be compared, added, etc. (All tangent spaces to X are alike, all being isomorphic to V_n, but there is no canonical identification between any two of them.) Therefore, the notion of "uniform" vector field is meaningless: what would be uniform (i.e., constant) with respect to a chart will not in general be constant under another one.

Back to the tangent maps $u_*(x)$ induced by a smooth map $u : X \to Y$, we see now they can be bundled up into a single map u_*, from TX to TY. If this map, which sends tangent vectors to tangent vectors, is one-to-one, it sends vector fields to vector fields: the image of v by u_* (its push-forward) is the field $y \to u_*(x)\, v(x)$, where $x = u^{-1}(y)$.

[5] As if to make things more exciting, physicists tend to use "bound" and "free" in exactly the opposite way. For many, a "free vector" is one attached to a point, a "bound vector" being an element of V_n.

[6] If α and β are functions, this still makes sense, showing that vector fields have an even richer structure: they form a *modulus* on the ring of functions.

Frame. A *frame* at x is a basis for $T_x X$, that is, a system of n linearly independent vectors. A *local* frame in a neighborhood of x is a locally defined field of frames, that is to say, a system of n smooth vector fields $w_1, \ldots, w_n$, such that the corresponding vectors at x be independent, for all x in the intersection of the domains of each w_i. A frame is *global* if $\mathrm{dom}(w_i)$ is all of X, for all i. One can always find a local frame about a point. For instance, the above basis vectors make one, the basis vector fields, denoted ∂^i, being defined by $\partial_i = x \to \partial_i(x)$. (The domain $\mathrm{dom}(\partial_i)$ is the domain of the defining local chart.) But no global frame exists, in general. On the surface of a sphere, for example, none can be devised.

Components. Given a local frame $\{w_i : i = 1, \ldots, n\}$, a vector field v can be written, still locally, as $v = x \to \sum_{i = 1, \ldots, n} v^i(x)\, w_i(x)$, or in abridged form, $v = \sum_i v^i w_i$. The real numbers v^i are the *components* of v, relative to this frame. If the local frame is made of the chart-induced ∂_is, one has $v = \sum_i v^i \partial_i$, and the v^is are the components of v *relative to this chart*.

Holonomy. This property[7] of a local frame that one can find a local chart the ∂_i's of which are the frame vectors. A given local frame has no reason to be holonomic [Ca], as soon as dimension exceeds two. Most of those encountered in practice, however, are. (See below what makes them so.)

Vector fields as differential operators. A vector field v can be seen as a differentiation: Indeed, if f is a smooth real-valued function on X, its rate of variation along one of the trajectories of class v is $\sum_i v^i \partial_i f$ [an easy computation: $f(g(t)) = f(\varphi^{-1}(\gamma_i(t))) = f(\varphi^{-1}(\varphi(x) + t\{v^1, \ldots, v^i, \ldots, v^n\})) = f(x) + t \sum_i v^i \partial_i(f \circ \varphi^{-1})(\varphi(x))$, up to higher-order terms], and this can be read as $(\sum_i v^i \partial_i)\, f$, that is, as the effect on f of a derivation operator. So why not write that "$v f$"? Replace v by ∂_i, now, and both interpretations of "$\partial_i f$" appear to be in harmony. Hence the use of the differentiation symbol ∂_i to denote the i-th chart-induced basis vector field. The other way round, a *derivation*, that is, a linear operation ∂ of type $C^\infty(X) \to C^\infty(X)$ that maps constants to 0 and obeys *Leibniz's rule*, $\partial(fg) = \partial f\, g + f\, \partial g$, corresponds to the vector field $\sum_i \partial\varphi^i \partial_i$, where φ is the chart (cf. [2], p. 83).

Lie bracket. Given two vector fields u and v, should we expect that $uvf = vuf$? If $u = \partial_i$ and $v = \partial_j$, so is the case ($\partial_i \partial_j f = \partial_j \partial_i f$, denoted $\partial_{ij} f$, by the classical Schwarz equality), but otherwise, $uvf = (\sum_i u^i \partial_i)(\sum_j v^j \partial_j) f = \sum_{ij} u^i v^j \partial_{ij} f + \sum_{ij} u^i \partial_i v^j \partial_j f$, and $(uv - vu)f = \sum_{ij} u^i \partial_i v^j \partial_j f - \sum_{ij} u^i \partial_i v^j \partial_j f = \sum_j (u v^j - v u^j) \partial_j f$, which has no reason to vanish. The vector field $\sum_j (u v^j - v u^j) \partial_j$ is called the *Lie bracket* of u and v, denoted $[u, v]$. The *Jacobi identity*

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0$$

is easily verified. (Whoever knows how to park a car along a kerb has intuitive grasp of the Lie bracket; see [30], p. 234.)

We may now observe[8] that $[\partial_i, \partial_j] = 0$. A given local frame $\{w_i : i = 1, \ldots, n\}$ is

[7] Again, be wary of the frequent use of this word, as criticized, e.g., in [Be], to mean the contrary, that is *an*holonomy — lack of holonomy. The relevant geometric object, a group, which happens to be trivial in case of holonomy, should thus be called "anholonomy group". Needless to say, it's *holonomy group* that is in use, and there is little hope for reconciliation between the two schools.

[8] The concept of Lie bracket applies to vector *fields*, not to vectors.

holonomic, one may prove (cf. [25], p. 47), if all Lie brackets $[w_i, w_j]$ vanish.

Lie algebra. A vector space V equipped with an anti-commutative bilinear operation $[\,,\,] : V \times V \to V$ such that Jacobi's identity holds. Examples: Ordinary 3D Euclidean space (see below), with $[v, w] = v \times w$; The vector space of $n \times n$ matrices, with $[A, B] = AB - BA$.

The vector space $\mathcal{V}_1(X)$ of smooth vector fields over X (or more generally, $\mathcal{V}_1(O)$, where O is an open set of X), is thus a Lie algebra. It's often of interest to know whether a family of vector fields, which make a subspace of $\mathcal{V}_1(X)$, also make a Lie algebra, that is, whether $[u, v]$ always belongs to the family when u and v do. One then has a *Lie subalgebra.* For instance, the fields $\alpha\partial_1 + \beta\partial_2$, with *constant* (beware!) coefficients α and β, form a Lie subalgebra of $\mathcal{V}_1(\mathrm{dom}(\varphi))$.

Covector. An element of the dual (denoted $T_x{}^*X$) of T_xX. The duality product between the covector ω and vector v (i.e., the "effect" of one on the other) is written $<\omega\,;\,v>$. If there is at x a basis $\{\partial_i : i = 1, ..., n\}$ of vectors (x understood), there is also a so-called "dual" basis of covectors: these are the linear maps $d^i : T_x X \to \mathbb{R}$ (x understood) defined[9] by $<d^i, \partial_j> = 1$ if $i = j$, else 0. Set $\omega_i = <\omega\,;\,\partial_i>$. Then $<\omega\,;\,v> = \sum_i \omega_i\, v^i$, so one can expand ω as $\omega = \sum_i \omega_i\, d^i$ in the dual basis, and one has $<\omega\,;\,v> = \sum_i \omega_i\, v^i$.

Cotangent bundle. The manifold of dimension 2n, denoted T*X, composed of the covectors. We just saw how to get a local map.

If u maps X to Y, how covectors behave with respect to u is a natural question. Given η in $T_y{}^*Y$, where $y = u(x)$, its *pull-back* $u^*(x)\eta$ is the covector at x such that $<u^*(x)\eta\,;\,v>_X = <\eta\,;\,u_*(x)v>_Y$ for all v in T_xX.

p-covector at x. Maps of type $T_x X \times ... \times T_x X \to \mathbb{R}$, with p factors on the left of the arrow, which are linear with respect to all factors, and *alternating,* that is, the permutation of two factors reverses the sign of the result. The effect is denoted $<\omega\,;\,v_1, ..., v_p>$. *Determinants* and their minors are a familiar example. If a covector basis $\{d^i : i = 1, ..., n\}$ exists at x, there is also a basis for p-covectors, the elements of which, denoted[10] $d^{\sigma(1)} \wedge d^{\sigma(2)} \wedge ... \wedge d^{\sigma(p)}$, are obtained by taking all *increasing* injections σ from the integer segment $[1, p]$ to $[1, n]$. By definition, the effect of the previous covector on a set of p vectors $\{v_1, v_2, ..., v_p\}$ is

$$<d^{\sigma(1)} \wedge d^{\sigma(2)} \wedge ... \wedge d^{\sigma(p)}\,;\, v_1, v_2, ..., v_p> = \sum_\pi v_1{}^{\sigma\circ\pi(1)}\, v_2{}^{\sigma\circ\pi(2)} \dots v_p{}^{\sigma\circ\pi(p)}\, \mathrm{sign}(\pi) \tag{1.1}$$

where π spans the set of all permutations of $[1, p]$, and $\mathrm{sign}(\pi)$ denotes the signature of π. Thus, for instance, $<d^1 \wedge d^2\,;\,v, w> = v^1 w^2 - v^2 w^1$. There are as

[9] The notation d^i is an innovation. I believe it's more logical than the received one, that would be dx^i, as found in such expressions as $dx^1 \wedge dx^2 \wedge ... \wedge dx^n$, or in low dimension, $dx \wedge dy$, $dx \wedge dy \wedge dz$, etc. See the section on Integration for the relation between "differential elements" like dx, etc., and covectors. The advantages of d^i are so obvious that I hope this will catch. (I've been hoping for now ten years.)

[10] The use of σ (and, below, ς) is also a new proposal. A long familiarity with "old tensor" notation has led to undue indulgence for atrocious displays like $dx^{i1} \wedge dx^{i2} \wedge ... \wedge dx^{in}$, with indexation of unlimited depth, the meaning of which gets clarified only by explicitly introducing the mapping σ (then i_1 is recognized as $\sigma(1)$, etc.).

many p-covectors in the basis as ways to select p objects among a family of n, order being meaningful *(combinations* p by p). So there is a single basis p-covector if $p = 0$ or n. For $p = 0$, it's by convention the real number 1, and for $p = n$, the *determinant* of the matrix of components of the n factors according to the chart φ. We'll denote by $\mathcal{F}_x^p$ the space of p-covectors at x. Each $\mathcal{F}_x^p$, for x fixed, is isomorphic to the linear space, often denoted Λ^p, of alternating p-linear forms on V_n. By convention, a 0-covectro is just a real number, and $\Lambda^0 = R$.

In presence of $u : X \to Y$, the *pull-back of a p-vector* η at $y = u(x)$ is defined by $<u^*(x)\eta \,;\, v_1, ..., v_p>_X = <\eta \,;\, u_*(x)v_1, ..., u_*(x)v_p>_Y$ for all $v_1, ..., v_p$ in T_xX.

For two integers p and q, consider an increasing injection σ from the integer segment $[1, p]$ into $[1, p + q]$. One will denote by ς the increasing injection from $[1, q]$ into $[1, p + q]$ which "complements" σ, uniquely determined by the condition $\varsigma(i) \neq \sigma(j)\ \ \forall\, i, j$. The signature of the permutation $i \to \{$***if*** $i \leq p$ ***then*** $\sigma(i)$ ***else*** $\varsigma(i - p)\}$ will be denoted $\text{sign}(\sigma, \varsigma)$. The set of all possible σ (isomorphic to the set of combinations of p objects chosen among $p + q$) will be denoted[11] $C(p, q)$.

Exercise. Generalize to $p = 0$, or $q = 0$, or both. Generalize to complementary injections from $[k + 1, k + p]$ and $[l + 1, l + q]$ into $[m, m + p + q - 1]$, where k, l, and m can be any relative integers.

Exterior product (aka *wedge* product). The *exterior product* of a p-covector ω and a q-covector η (both at the same point x) is the $(p + q)$-covector defined by

$$<\omega \wedge \eta \,;\, v_1, ..., v_p, v_{p+1}, ..., v_{p+q}> =$$

$$\sum_{\sigma \in C(p,\,q)} <\omega \,;\, v_{\sigma(1)}, ..., v_{\sigma(p)}> <\eta \,;\, v_{\varsigma(1)}, ..., v_{\varsigma(q)}> \text{sign}(\sigma, \varsigma). \tag{1.2}$$

(It *is* a covector: If two v_is are equal, the same term will appear twice in (12) with opposite signatures.) This operation is associative, and one will check that (1.1) above was indeed the exterior product of the basis covectors $d^{\sigma(i)}$, which justifies the use of $\wedge$ in (1.1). Wedge product is not commutative, but *anti*-commutative, in the following sense:

$$\eta \wedge \omega = (-1)^{pq}\, \omega \wedge \eta. \tag{1.3}$$

Wedge and pull-back go along naturally: $u^*(\omega \wedge \eta) = u^*\omega \wedge u^*\eta$, as easily established from (1.2) and the definition of u^*. Note that $\omega \wedge \eta = 0$ if $p + q$ exceeds n, the ambient dimension,

Exercise. Can $\omega \wedge \omega$ be nonzero?

Inner product. To a p-covector ω, $p \geq 1$, and a vector v, this operation associates the $(p - 1)$-covector

$$i_v\omega = \{v_2, ..., v_p\} \to <\omega \,;\, v, v_2, ..., v_p>$$

(also denoted $v \llcorner \omega$ by some, cf. e.g. [Bu]). In particular, when $p = 1$, $i_v\omega$ is just $< \omega \,;\, v>$. About the interaction with pull-back, $i_v(u^*\omega) = u^*[i_{u_*v}\omega]$, that is

$$i_v u^* = u^* i_{u_*v} \tag{1.4}$$

[11] This is in the spirit of H. Cartan, who favored the symmetric notation (p, q) for the binomial coefficient which gives the number of p-by-b combinations of p + q objects (cf. [HP], p. 159).

and $i_v u^* = u^* i_v$ when v is invariant, $u_* v = v$, a remark we'll use later. Wedge and i relate like this, after (1.2):

$$(1.5) \qquad i_v(\omega \wedge \eta) = i_v\omega \wedge \eta + (-1)^p\, \omega \wedge i_v\eta,$$

where p is the degree of ω. [Proof: Let $v_1 = v$. Permutations σ fall into two classes, those for which $\sigma(1) = 1$, those for which $\varsigma(1) = 1$. Observe how this induces complementary injections from $[2, p]$ and $[1, q]$ (in the former case) or from $[1, p]$ and $[2, q]$ (in the latter case) into $[2, n]$, and how the signature of these relates with $\mathrm{sign}(\sigma, \varsigma)$. (They are the same if $\sigma(1) = 1$, differ by the sign of $(-1)^p$ if $\varsigma(1) = 1$.) Split the right-hand side of (1.2) accordingly, into two sums. one of which appears to be $i_v\omega \wedge \eta$ and the other one $(-1)^p\, \omega \wedge i_v\eta$.]

Exercise. Show that $i_v \circ i_v = 0$.

Exercise. When can $i_v\omega \wedge \omega$ be nonzero?

Multivectors. These are the dual objects, a p-vector being an element of the dual of Λ^p, a bound p-vector at x, an element of the dual of $\mathcal{F}_x^{\,p}$. Basis p-vectors $\partial_{\sigma(1)} \vee \partial_{\sigma(2)} \vee \ldots \vee \partial_{\sigma(p)}$ can be defined by the formula

$$<d^{\sigma(1)} \wedge d^{\sigma(2)} \wedge \ldots \wedge d^{\sigma(p)},\, \partial_{\sigma(1)} \vee \partial_{\sigma(2)} \vee \ldots \vee \partial_{\sigma(p)}> = 1$$

(and 0 if indices don't match at they do here, -1 if they do up to an odd permutation). In coordinates, the generic field of p-vectors is

$$x \to \sum_{\sigma \in C(p,\, n-p)} v_\sigma(x)\ \partial_{\sigma(1)} \vee \partial_{\sigma(2)} \vee \ldots \vee \partial_{\sigma(p)}.$$

The "join" $\vee$ thus introduced behaves like[12] the "wedge" $\wedge$: just use (1.2), exchanging vectors and covectors. It goes along with push-forward: $u_*(v \vee w) = u_*v \vee u_*w$.

A "simple" bivector such as $v \vee w$ can be visualized as the oriented parallelogram built on v and w, and since $v \vee w = v \vee (w + \lambda v)$, as any other parallelogram of equal area[13] in this plane, and while we are at it, as any piece of the plane of equal algebraic area.[14] Such a bivector "is" therefore, an equivalence class of figures in a plane, characterized by a common (positive or negative) area, just as a vector is the class of segments of same algebraic length on a given line. Not all bivectors are of this kind, in general, but so is the case in dimension 3: A bivector $\alpha\, \partial_y \vee \partial_z + \beta\, \partial_z \vee \partial_x + \gamma\, \partial_x \vee \partial_y$ can be represented by the join of two vectors. Trivectors (and n-vectors, in dimension n), also have easy geometric interpretation: equivalence classes of figures of equal volume.

Differential form of degree p: A C^∞ field of p-covectors, or p-*form*. One will denote by $\mathcal{F}^p(X)$ the collection of all p-forms on X. (It's more than a vector space, it's a modulus on the ring of functions, cf. Note 6.) Differential forms of degree 0, or 0-forms, are the smooth functions of type $X \to \mathbb{R}$, or *scalar fields*. In

[12] Rota [B&] makes a convincing case for the use of $\vee$ in this context. One more often sees $\wedge$ there.

[13] Area is a metric concept (see below), but *ratio* of areas, in a given plane, is a purely affine notion (take the obvious determinant).

[14] There is a popular interpretation of bivectors as *rotations* in the support plane ([Hs], p. 1020). This requires a metric, however, a fact that fans of this "geometric algebra" approach tend to downplay.

a local chart, a p-form can be written as

$$\omega = x \to \sum_{\sigma \in \mathcal{C}(p,\, n-p)} \omega_\sigma(x)\, d^{\sigma(1)} \wedge d^{\sigma(2)} \wedge \ldots \wedge d^{\sigma(p)},$$

where the coefficients ω_σ are C^∞ functions. The dual concept, a field of p-vectors, is less popular, and has no universally recognized name. (Such fields form a space $\mathcal{V}_p(X)$; we already met with $\mathcal{V}_1(X)$.)

In particular, the generic 1-form is $\omega = x \to \sum_{i=1,\ldots,n} \omega_i(x)\, d^i(x)$. By linearity of ω, one has $<\omega\, ;\, v> = \sum_i \omega_i v^i$ (with x understood), as one would expect. A full development of such coordinate-dependent computational techniques would lead to a brief on *tensor calculus*. Let's just mention the popular *Einstein convention* about implicit summation (not followed here), that consists in omitting the $\sum$ in expressions such as $\sum_i \omega_i\, d^i$, where the summation index appears both as a subscript and a superscript.

In presence of a map $u : X \to Y$, the *pull-back of a p-form* η, living on Y, is the p-form $x \to u^*(x)\eta(u(x))$. In particular, $< u^*\eta\, ;\, v>_X = <\eta\, ; u_* v>_Y$. (One-to-oneness is not required.)

Exterior product of forms. The operation of type $\mathcal{F}^p(X) \times \mathcal{F}^q(X) \to \mathcal{F}^{p+q}(X)$, denoted $\wedge$, defined by $\omega \wedge \eta = x \to \omega_x \wedge \eta_x$.

Inner product of a p-form ω and a vector field v: This is the $(p-1)$-form $i_v\omega = x \to i_{v(x)}\omega(x)$.

Trace of a p-form on the boundary ∂X of X. Vectors tangent to ∂X being also tangent to X, the trace $t\omega$ of ω is simply its restriction to $T\partial X$, and can therefore be denoted by ω if no risk of confusion is incurred. Note that the trace of an n-form is always null (because n vectors tangent to ∂X must be linearly dependent). Trace of a 0-form and of a function are one and the same concept. More generally, the trace of a (smooth, let's remind) p-form makes sense on an embedded q-manifold, and vanishes if $p > q$.

Tensors. Consider at point x the Cartesian product of p copies of $T_x X$ and q copies of $T^*_x X$. A multilinear map (i.e., linear with respect to each factor) from such a product into $\mathbb{R}$ can be imagined, according to a common metaphor, as a machine sitting at x, in which $p + q$ slots wait for the introduction of p vectors and q covectors, ready to deliver a real number in return. A *tensor* of order (p, q) is a field of such machines. (Notice the word "alternating" is absent from this definition.) Particular cases: $p = 1$ and $q = 0$ (the 1-forms), $p = 0$ and $q = 1$ (the vector fields), $p = q = 0$ (the functions). Remark that inner product is an instance of the more general "index contraction" on tensors.

2. ORIENTATION AND TWISTED FORMS

Two different bases in a linear space are linked by a regular transformation matrix. One says that the two bases have same orientation if the determinant of this matrix is positive. This is an equivalence relation, with two classes. To *orient* a vector space consists in selecting one of these classes. The "right-hand rule" is the mnemonic trick by which we recall which orientation we usually choose in three-dimensional vector space. Orienting a manifold consists in orienting all tangent spaces in a consistent way, as we now explain more formally.

Volume. An n-form which doesn't vanish on X. We saw that n-forms can be written in coordinates like this: $\Omega = x \to \alpha(x)\, d^1 \wedge \ldots \wedge d^n$, with α smooth. An n-form is thus a volume if function α (which is only locally defined, beware!) doesn't vanish, in any chart domain.

Orientable manifold. Manifold on which a volume Ω exists. Then $\Omega' = \alpha\, \Omega$, where α is a nowhere vanishing function, is a volume too, and conversely. One says that Ω and Ω' *confer on* X *the same orientation* if the sign of α is positive all over the manifold. An *oriented manifold* is a pair, manifold X and volume Ω, or more correctly, the pair {X, Or}, where Or is the equivalence class of Ω. If X is connected (which one assumes, in general, as part of the definition of manifolds) there are thus two possible orientations, or none.

Local volumes, i.e., defined in a neighborhood of x, always exist: for instance, $\Omega = x \to d^1 \wedge \ldots \wedge d^n$ (within some chart) is a local volume. If $\{w_1, \ldots, w_n\}$ is a local frame, there exists, by the very definition of frames, a neighborhood of x in which the function $x \to \langle \Omega\, ; w_1(x), \ldots, w_n(x)\rangle$ doesn't change sign. Thus there are two classes of local frames in the neighborhood of x. Those for which $\langle \Omega\, ; w_1, \ldots, w_n\rangle$ is positive are the *direct*, or *positively oriented* ones. Those of the other class are the *skew* ones.

Whether the manifold is orientable hinges on the possibility to patch such local volumes together to build a global one. Let thus Ω and Ω' be two local volumes, such that $\mathrm{dom}(\Omega) \cap \mathrm{dom}(\Omega')$ is non-empty and connected. In this intersection, $\Omega' = \alpha\, \Omega$, where α has constant sign. If the sign is minus, changing Ω' to $-\Omega'$ yields a volume "compatible" with Ω, one which defines the same orientation. If, working from one local volume to the next, one can thus get all the signs right, and find a family of compatible local volumes that cover all X, it's then easy (thanks to a technical result about "partitions of unity") to build a global volume from them. But the process can fail (so it does with a Möbius band), and indeed, there are non-orientable manifolds.

Remark. If a global frame exists, X is of course orientable. But this is not a necessary condition (think of the unit sphere in 3D space). ◊

Inner orientation. This refers to a submanifold Y of dimension p, embedded in some n-manifold X (called the *ambient* manifold in what follows). If Y is oriented, one says it has inner orientation. So the word would be redundant, if it were not to mark the opposition with the coming concept of "outer" orientation, which is genuinely novel. If the ambient space is A_3, inner orientation for a point, a line, a surface, and a domain, means respectively, a sign (+ or –), a way to go along, a gyratory sense (e.g., "clockwise"), a screw sense.

Induced orientation. Suppose X is oriented, via a volume Ω. Take on ∂X a field v of outward directed transverse vectors. Then $i_v\Omega$ is an $(n-1)$-volume on ∂X, which orients it. This amounts to saying that a frame $\{v_1, \ldots, v_{n-1}\}$ in $T_x\partial X$ is direct if $\{v, v_1, \ldots, v_{n-1}\}$ is a direct frame in T_xX. Fig. 2.1 gives examples.

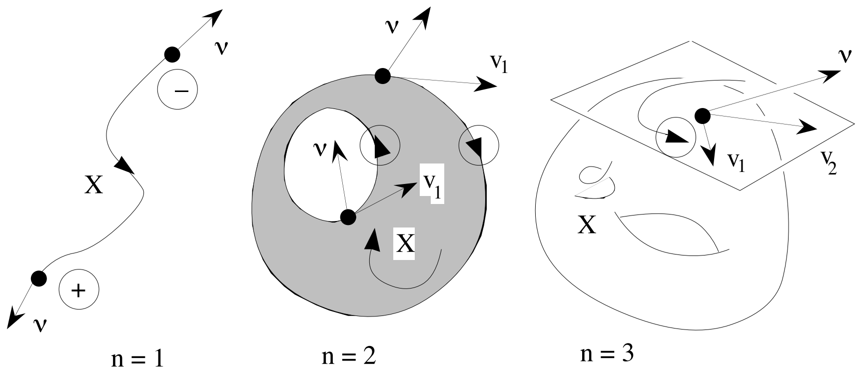

FIGURE 2.1. Orientation induced on ∂X by X, when the latter is, from left to right, a curve, the 2D domain between two closed curves, and the interior of a solid torus. Note, close to the label of each manifold, the conventional *icon* (± sign, arrowhead, curved arrow, helix), by which its inner orientation is depicted. The circles call attention upon icons that correspond to an induced orientation.

Outer orientation. A p-dimensional subspace V of a vector space U is *outer oriented* if a *complementary* subspace W (i.e., such that $u = v + w$, with $v \in V$ and $w \in W$, unique, for all u) has been oriented. No orientation of the *ambient* space U itself is assumed. (But note that, if V is both inner and outer oriented, an orientation of U follows.) The notion extends to submanifolds (of some ambient n-dimensional manifold) the obvious way. Outer-orientation of a *hypersurface*, as one calls (n − 1)-submanifolds in n-dimensional ambient space, consists in providing a *crossing direction* through it. If the ambient space is A_3, outer orientation, for a point, a line, a surface, and a domain, means respectively, a screw sense, a way to turn around the line, a crossing direction, a sign (+ or –).

If the ambient manifold is oriented, this provides a way to outer-orient any orientable submanifold, and vice versa. "Outer orientability", then, is the same as (inner) orientability. But inner and outer orientation make sense independently of any orientation of the ambient manifold. More, the latter need not be orientable at all. In that case, inner- and outer-orientable submanifolds do *not* coincide.

Twisted covectors. Loosely speaking, a *twisted* p-covector[15] at x is a pair, consisting of a covector and an orientation of T_xX, it being understood that the pair formed by the opposite covector and the opposite orientation is the same twisted covector. For those who like to see all mathematical objects as sets, the twisted covector $\tilde{\omega}$ is thus the equivalence class $\{\{\omega, \mathrm{Or}\}, \{-\omega, -\mathrm{Or}\}\}$, composed of the two equivalent pairs just mentioned. A pair $\{\omega, \Omega\}$, where Ω is a volume of the Or class, suffices to define $\tilde{\omega}$. We shall use the notation $\tilde{\omega} \approx \{\omega, \Omega\}$ to mean "the twisted form $\tilde{\omega}$ represented by the pair $\{\omega, \Omega\}$", and abuse language by saying "$\tilde{\omega}$ 'is' the pair $\{\omega, \Omega\}$", etc.

[15] In contrast with other concepts in the list, it may be difficult to find good textbooks that give twisted objects their due. Burke [Bu] does, in the wake of Veblen and Whitehead [VW] (cf. Schouten [24]). See also [Fr] and [Wr].

Twisted p-*vectors*. Same construction, with multivectors. Another name for twisted vector ($p = 1$) is *axial* vector. (*Polar* vectors, thus dubbed for contrast, are just plain vectors. This dubious terminology has done and will do damage beyond reckoning [Bo].)

Twisted (differential) forms. Smooth fields of twisted covectors. The set of twisted p-forms over X will be denoted $\tilde{\mathcal{F}}^p(X)$. Twisted n-forms are called *densities*.

This is a difficult concept, especially in the relatively rare[16] case of non-orientable manifolds, which is also where its interest is fully revealed. If one is only concerned with orientable ambient manifolds, a simpler definition is available: A twisted p-form $\tilde{\omega}$ is a pair $\{\omega, \mathrm{Or}\}$, where ω is a p-form and Or a global orientation (i.e., the class of a global volume), and $\{\omega, \mathrm{Or}\} = \{-\omega, -\mathrm{Or}\}$ by way of definition. (Each twisted form is thus in correspondence with a pair of ordinary forms of opposite signs, and conversely. If X was not orientable, no such correspondence would exist, except locally.)

Operations on twisted forms go as follows, as a rule: Take representative pairs, operate on forms, then do what it takes on orientations. For instance, if $\tilde{\omega} \approx \{\omega, \Omega\}$, the inner product $i_v\tilde{\omega}$ is (the class of) $\{i_v\omega, \Omega\}$. The wedge product by a straight form η is $\tilde{\omega} \wedge \eta \approx \{\omega \wedge \eta, \Omega\}$. If $\tilde{\eta} \approx \{\eta, \Omega'\}$, then $\tilde{\omega} \wedge \tilde{\eta}$ is a *straight* form, equal to $\pm\, \omega \wedge \eta$, the sign being that of the ratio Ω'/Ω. See how this principle applies to the next notion:

Trace of a twisted p-form over ∂X. If $\tilde{\omega}$ is represented by $\{\omega, \Omega\}$, we know about the trace $t\omega$ of ω, but we need a volume on ∂X to complete the pair. For this, let n be a field of *outward* vectors[17] on ∂X. The trace $t\tilde{\omega}$ is then represented by the pair $\{t\omega, i_n\Omega\}$ (or by the pair $\{-t\omega, -i_n\Omega\}$).

3. INTEGRATION AND THE STOKES THEOREM

Let's now review integration, where the notion of density will reveal its worth: For densities are, among all the geometrical objects living on manifolds, those that can be integrated.

We shall call *reference* p-*simplex* the following closed set in $\mathbb{R}^n$:

$$S^p = \{x \in \mathbb{R}^p : x^i \geq 0 \quad \forall\, i \in [1, p],\ \textstyle\sum_{i=1,\dots,p} x^i \leq 1\}, \tag{3.1}$$

and *faces* of S^p the subsets obtained by replacing one or more of the inequalities in (3.1) by equalities. (The empty set and the vertices thus count as faces.) A *simplicial map*[18] is a function of type $S^p \to S^q$ which is affine, one-to-one, and transforms vertices into vertices. (Each face is then transformed into another face of same dimension.) Such a map induces an injection σ from $[0, p]$ into $[0, q]$, obtained by calling $\sigma(i)$ the number of vertex number i of S^p in the larger reference simplex.

[16] But not unheard of. Non-orientable manifolds do intervene in some modellings, about materials with periodic structure, for instance, for which the "fundamental domain" of the repetitive structure, boundary points duly indentified, may form a non-orientable manifold (or worse: not be a manifold at all; see [He] for counter-examples).

[17] What is needed is a *transverse* field, but one could of course choose an *inward* directed one. The choice of an outward one agrees with current use.

[18] This notion, introduced here for convenience, will not serve elsewhere in this document.

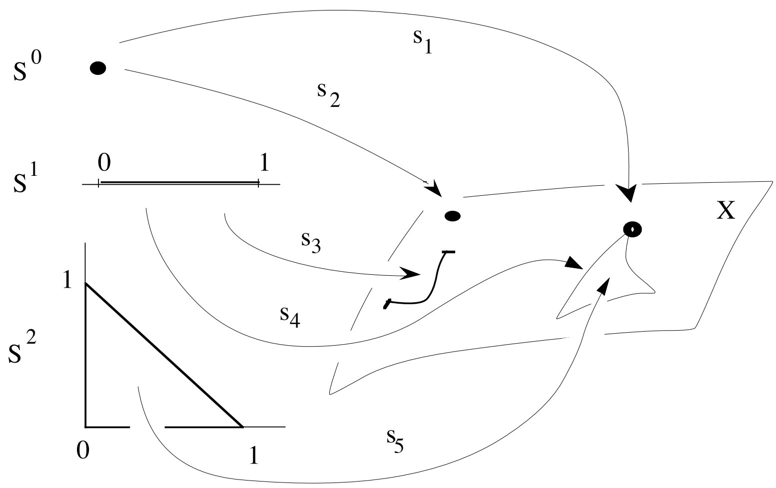

FIGURE 3.1. Examples of simplicial cells, in dimension 2.

Simplicial maps have orientation when $p = q$ or $q - 1$. If $p = q$, the injection σ is actually a permutation, and the map is said *even* or *odd* according to the signature of σ. If $q = p + 1$, the map is even or odd according to the signature[19] of the permutation defined by $\{0, 1, ..., p + 1\} \to \{i, \sigma(0), \sigma(1), ..., \sigma(p)\}$, where i is the vertex, among those of S^{p+1}, which is not in the image of S^p.

Simplicial cell (of dimension p). It's an embedding of S^p into X (Fig. 3.1), or more precisely, an equivalence class (again ...) of such embeddings, the equivalence relation being $s \sim s'$ iff (1) the images of S^p by s and s', denoted $|s|$ and $|s'|$, coincide, (2) $s^{-1} \circ s'$ is an *even* simplicial map. (The image $|s|$ thus corresponds to two simplicial cells, one of each orientation.) The name "simplex" is often meant to refer to the image $|s|$, or to s itself, by a natural abuse of language. The diffeomorphic image of a cell is a cell.

Triangulation, or *simplicial tiling* of an n-dimensional differentiable manifold X. This refers to a family $\mathcal{S}$ of n-dimensional simplicial cells in X, with the following properties:

(a) If s and σ are two cells of $\mathcal{S}$, the mapping $s \circ \sigma^{-1}$ is simplicial,

(b) If $s \neq \sigma$, then $|s| \neq |\sigma|$,

(c) $\cup_{s \in \mathcal{S}} |s| = X$,

(d) Any compact part of X is contained in the set-union of a finite number of images $|s|$.

These axioms almost correspond to the notion of mesh as used in finite element theory, where the manifold X, of dimension 3, is the "computational domain", D say. (In particular, property (a) implies that the intersection of two tetrahedra is a tetrahedron, a facet, an edge, a vertex, or the empty set.) Note that the shape of the elements can be chosen at will, and be made to fit the curved boundary of D, in particular. However, the practice of finite elements requires more than a

[19] Same notion as $\text{sign}(\sigma, \varsigma)$ above, in a slightly changed context.

mesh. One should also be able to identify nodes, edges, etc., all geometric elements that may eventually support a degree of freedom, hence the following refined notion:

Simplicial mesh of an n-dimensional differentiable manifold X. This refers to a family S of (beware!) p-dimensional simplicial cells in X, $0 \le p \le n$, with properties (a) to (d) as above, and in addition,

(e) If $|s| \cap |\sigma|$ is not empty, there exists one and only one cell of S the image of which is $|s| \cap |\sigma|$.

The p-cells, with $p = 0, 1, 2, 3$, are the nodes (or vertices), the edges, facets and tetrahedra (or volumes) of the mesh. The above "only one" restriction corresponds to common usage, but there are cases in which it should be lifted.

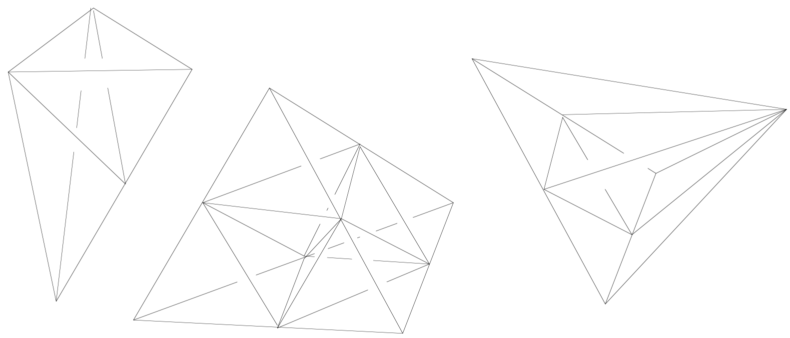

FIGURE 3.2. Local refinement of a mesh. Tetrahedra near the central one, cut into 8 parts, must themselves be cut in 4, like the one on the right, or (for those like the one on the left, which shares an edge with the central one) in 2.

Note that a simplicial tiling or mesh of X induces a simplicial tiling or mesh of the boundary ∂X.

A simplicial mesh S' is *finer* than S if for all $s \in S$, there exists a part S'' of S' which constitutes a simplicial mesh of $|s|$. (Same notion for tilings.) It's of course an order relation. A given mesh can be "refined", that is, made to generate a finer one, by various subdivision procedures (Fig. 3.2 suggests one). One can manage so that the number of cell-shapes[20] generated this way stays bounded [By]. One then has *uniform* refinement. We'll say, rather loosely, that an ordered family of meshes m obtained by uniform refinement "tends to zero", denoted $m \to 0$, if the largest diameter γ_m of all cells (the "grain" of the mesh) tends to 0.[21]

Integral of a density. Let $\tilde{\omega} = \{\omega, \Omega\}$ be a density on X, and S a simplicial tiling. Let us set

$$<\tilde{\omega} \,; s> = (n!)^{-1} <\omega \,; s_*e_1, ..., s_*e_n> \mathrm{sgn}[<\Omega \,; s_*e_1, ..., s_*e_n>],$$

[20] Shape is a metric notion, so we are a bit ahead here. Same remark about the "grain", below.

[21] Just "$\gamma_m \to 0$", without the uniformity, would make sense, but wouldn't do as regards the convergence properties one may need in the practice of finite elements. It is well known that obtuse angles going to π and more generally, uncontrolled "flattening" of the elements, are a bad thing [BA].

where the e_is are the basis vectors in $\mathbb{R}^n$ (i.e., the edges $\{0, i\}$ of the reference simplex). Then, set

$$\mathcal{J}_S(\tilde{\omega}) = \sum_{s \in S} <\tilde{\omega} \; ; s>$$

(a Riemann sum). The integral of $\tilde{\omega}$ on X, denoted $\int_X \tilde{\omega}$, is the limit, when it exists, of $\mathcal{J}_S(\tilde{\omega})$ as S runs along an ordered sequence of meshes for which γ_m tends to zero.[22] The sum $\mathcal{J}_S(\tilde{\omega})$ is indifferent to the orientation of the maps s, and this is why densities, and not n-forms, can be integrated.

If u is a diffeomorphism, it follows from the definition that $\int_{u(X)} \tilde{\omega} = \int_X u^*\tilde{\omega}$. (This may be wrong, or even meaningless, otherwise.)

Twisted p-forms can be integrated on submanifolds of dimension p when these have *outer* orientation. In particular, the integral $\int_{\partial X} \tilde{\omega}$ makes sense if $\tilde{\omega}$ has degree $n - 1$. (It's the integral of the trace, the definition of which was given above.) If u embeds Y, of dimension p, into X, then

$$\int_{u(Y)} \tilde{\omega} = \int_Y u^*\tilde{\omega}.$$

Exterior derivative. If $\omega = x \to \sum_{\sigma \in C(p, n-p)} \omega_\sigma(x)\, d^{\sigma(1)} \wedge d^{\sigma(2)} \wedge \ldots \wedge d^{\sigma(p)}$ is the coordinate description of a p-form, $d\omega$ is the following $(p - 1)$-form:

(3.2) $$d\,\omega = x \to \sum_{\sigma \in C(p, n-p),\ i = 1, \ldots, n} \partial_i \omega_\sigma(x)\, d^i \wedge d^{\sigma(1)} \wedge d^{\sigma(2)} \wedge \ldots \wedge d^{\sigma(p)}.$$

For a twisted form $\tilde{\omega} \approx \{\omega, \Omega\}$, one has $d\tilde{\omega} \approx \{d\omega, \Omega\}$, by definition. If $p = 0$, i.e., if ω is a function f, df is its *differential*, and $<df \; ; v>$ is what was earlier denoted $(\sum_i v^i \partial_i)$ f. A calculation shows that

(3.3) $$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p\, \omega \wedge d\eta,$$

where p is the degree of ω.

Exercise. Can $d\omega \wedge \omega$ be nonzero?

Theorem (Stokes). *If* $\tilde{\omega}$ *is a twisted* $(n - 1)$*-form, then*

(3.4) $$\int_{\partial X} \tilde{\omega} = \int_X d\tilde{\omega}.$$

Proof. Take a tiling of X, fine enough so that each cell be inside the domain of some chart. It's then easy to prove that $\int_{\partial |s|} \tilde{\omega}$ equals $\int_{|s|} d\tilde{\omega}$, by working on expression (3.2) in a chart. When all these contributions are summed up, the integrals on inner $(p - 1)$-faces cancel two by two, and what remains is the contribution of cells contained in ∂X, hence (6). ◊

Note that X need not be orientable. If it is, a corollary is available: $\int_{\partial X} \omega = \int_X d\omega$ for any $(n - 1)$-form. [Proof: First go to $\tilde{\omega} \approx \{\omega, \Omega\}$, where Ω is an orienting volume, and apply (3.4).] This, combined with (3.3), gives the *integration by parts* formula, for a p-form ω and a q-form η,

(3.5) $$\int_X d\omega \wedge \eta = (-1)^{p-1} \int_X \omega \wedge d\eta + \int_{\partial X} \omega \wedge \eta.$$

As another corollary, t being the foregoing trace operator,

[22] The dependence on the metric is only apparent, as one may work within a local chart, by using partitions of unity.

$$dt = td,$$

where the d on the left applies to forms living on ∂X. [Proof: For any p-form ω and any $(p + 1)$-manifold Y embedded in ∂X, one has $\int_Y dt\omega = \int_{\partial Y} t\omega = \int_{\partial Y} \omega = \int_Y d\omega = \int_Y t d\omega$.]

The interaction with pull-back, ruled by

$$u^* d = d u^*, \tag{3.6}$$

also stems from Stokes. [Proof: Notice that $\partial(u(M)) = u(\partial M)$ when u is an embedding. Thus $\int_X du^* \tilde{\omega} = \int_{\partial X} u^* \tilde{\omega} = \int_{\partial(u(X))} \tilde{\omega} = \int_{u(X)} d\tilde{\omega} = \int_X du^* \tilde{\omega}$, for all $\tilde{\omega}$, all X, hence the result.]

Chain. A p-chain over X is a formal sum $c = \sum_{i = 1, \ldots, k} \mu^i M_i$ of oriented submanifolds of X all of dimension p, called here the "chain elements", with "weights" μ^i taken in some ring of coefficients (say $\mathbb{R}$ for definiteness, though such coefficients will most often be signed integers in our examples). We distinguish *straight chains*, for which all components have inner orientation and *twisted chains*, with outer oriented components. Chains of the same class (straight or twisted) can be added and multiplied by a scalar, and thus form a real vector space. To see how, conceive c as a function, real-valued, on the set of all (inner or outer) oriented p-manifolds, with finite support (all but a finite number of manifolds get zero weight). Now, to add or multiply chains, add or multiply these functions. We shall denote by $\mathcal{C}_p(X)$ [resp. $\tilde{\mathcal{C}}_p(X)$] the space of straight [resp. twisted] p-chains. Integration and ∂ extend to *linear* operators on these spaces, as follows:

$$\partial c = \sum_{i = 1, \ldots, k} \mu^i \partial M_i, \quad \int_c \omega = \sum_{i = 1, \ldots, k} \mu^i \int_{M_i} \omega.$$

Remark. The concept of chain is useful in connection with the boundary operator: The boundary ∂M of an oriented submanifold M may be made of several submanifolds B_i which for some reason have been provided with their own orientations, which may agree or not with the one induced by M. Hence the representation of ∂M as the chain $B = \sum_i \beta^i B_i$, with $\beta^i = \pm 1$ according to whether orientations agree or not. ◊

Integration provides a *duality product* between elements of $\mathcal{C}_p$ and $\mathcal{F}^p$ [resp. of $\tilde{\mathcal{C}}_p$ and $\tilde{\mathcal{F}}^p$], which we shall emphasize by using the alternate notation $\langle \omega \,;\, c\rangle$ for $\int_c \omega$. The Stokes theorem, $\langle \omega \,;\, \partial c\rangle = \langle d\omega \,;\, c\rangle$, then makes d appear[23] as the *dual* of ∂.

Lie derivative. Operation of type $\mathcal{F}^p \to \mathcal{F}^p$, defined by $L_v\omega = i_v(d\omega) + d(i_v\omega)$, where v is a vector field. If $p = 0$, then $L_v f = \langle df \,;\, v\rangle$. When f depends on time, $\partial_t f + L_v f$ is the *convective derivative* of fluid dynamics. More on this in Section 5, where a more primitive definition, of which the present one is merely a consequence, will be given.

Divergence of a vector field. Let $p = n$ in what precedes. Then, n-forms making a one-dimensional space, $L_v\omega = d(i_v\omega)$ is proportional to ω. The ratio $\operatorname{div} v$ such

[23] For dual operators, see [Yo], Chap. 7. The concept supposes a topology on $\mathcal{C}_p(X)$, which we have not introduced, but this can be done (in a way which makes ∂ continuous).

that $d(i_v\omega) = (\text{div } v)\ \omega$ is called the *divergence* of v. Like the Lie derivative, we shall meet the divergence again later. The point of mentioning them now is to stress their non-dependence on the structural element that now comes: metric.

4. METRIC STRUCTURES ON MANIFOLDS

Metric. A smooth field of bilinear maps $g \in T_x X \times T_x X$, *symmetric* (i.e., $g_x(v, w) = g_x(w, v)$ $\forall$ v, w, x, and strictly *positive definite:* $g_x(v, v) > 0$ if and only if $v \neq 0$. A *Riemannian manifold* $\{X, g\}$ is a manifold X provided with a metric g. The values $g_{ij} = g(\partial_i, \partial_j)$ obtained for the basis vectors are called the *metric coefficients*. We shall abbreviate the orthogonality relation, $g_x(v, w) = 0$, by $v \perp w$.

Euclidean space. Denoted E_n, this is the affine space A_n equipped with a dot product of its vector associate (which provides a metric in the foregoing sense) and an orientation, which we shall denote by Or in what follows. We'll use the dot product, $v \cdot w$, as an alternative to $g(v, w)$ in the Euclidean framework, and $|v|$ for the *norm* of v, defined as $g(v, v)^{1/2}$.

If $u \in [0, 1] \to X$ is a trajectory, and if $\partial_t u(t)$ is the tangent vector at point $u(t)$, the integral $\int_{[0, 1]} dt\, [g(\partial_t u(t), \partial_t u(t))]^{1/2}$ is the *length* of the image of u. The lower bound of all trajectory lengths, under the condition that $u(0) = x$ and $u(1) = y$, is the *distance* between x and y. The axioms for a distance are indeed satisfied, and providing X with a metric thus transforms it into a metric space. But it gives more: a notion of *angle*, since a scalar product g_x is available in each tangent space, a notion of intrinsic *curvature*, etc., and a quite important structure known as a *connexion*, that makes possible what was not in the general case: to compare two remote vectors, to know what it means for a tensor to be "constant" in the neighborhood of a point, and to measure the rates of variation of such objects, locally (notion of "covariant derivative"). For these notions and their applications to "gauge theories", one may refer to [Bl].

Vector product of two vectors v and w. The vector $v \times w$, orthogonal to both v and w, such that $|v \times w|^2 + |v \cdot w|^2 = |v|^2\, |w|^2$, and such that $\{v, w, v \times w\}$ makes a direct frame.

It's often asserted that "the vector product is an axial vector", which makes no sense ($v \times w$ is just a vector, and nothing more), unless reinterpreted in an appropriate (and different) context, as follows. The mapping $\{v, w\} \to v \times w$ depends on orientation, obviously: change Or to – Or and the vector product, according to the foregoing recipe, is now $-\ v \times w$. On the other hand, the twisted (or axial) vector made of the equivalence class $\{\{v \times w, \text{Or}\}, \{- v \times w, - \text{Or}\}\}$ is an orientation-independent object, well defined as soon as a metric exists. Denoting this object by $v \tilde{\times} w$, we thus define a binary operation, $\tilde{\times}$, metric-dependent but indifferent to orientation, which yields an axial vector from two polar ones. *If* "vector product" refers to *this* operation, then, truly, "the vector product of two vectors is an axial vector". But this implies a context, that of a *non*-oriented space, which is not the one we assume.

The isomorphism between forms and vector fields. On a Riemannian manifold $\{X, g\}$, let a vector field u be given, and consider the linear map $v \to g(u(x), v(x))$

on T_x. This is a covector at x, that we shall denote by $\flat u(x)$, and the field $x \to \flat u(x)$ is a differential form of degree 1, that we shall denote by $\flat u$. Conversely, a 1-form ω generates a vector field $\#\omega$, if one defines $\#\omega(x)$ as the unique vector of T_x such that $g(\#\omega(x), v) = \langle \#\omega(x) ; v \rangle \ \forall\ v \in T_x$.

Now, take a basis, in which $u = \sum_i u^i \partial_i$, and $\omega = \sum_i \omega_i d^i$. Then $g(u, v) = \sum_{i,j} g_{ij} u^j v^i \equiv \sum_i (\flat u)_i v^i$, hence $(\flat u)_i = \sum_i g_{ij} u^j$. No harm done till there, but there is a habit to write u_i instead of $(\flat u)_i$ and to call these numbers the "covariant components" of the *vector* u (instead of "the components of the form $\flat u$", what they truly are), the genuine components u^i then being dubbed "contravariant". Operator $\flat$ is then construed as "lowering the indices", hence the symbol. Why co and contra is not too difficult to explain (although one could make a good case for a reversal of usage [DL]), because of what happens during a change of basis: If a new basis ∂' is given by $\underline{\partial}_j = \sum_j s^i_j \partial_i$, vector $v = \sum_i v^i \partial_i$ becomes $v = \sum_i \underline{v}^i \underline{\partial}_i$ in the new basis, and $v^i = \sum_j s^i_j \underline{v}^j$. So "components change in the other direction" when compared to how basis vectors change. But there is no real parity of status between "contravariant" and "covariant" components: the former are relative to a basis, the latter are relative to a basis *and to* a metric.

Conversely, of course, the components of $\#\omega$, also conventionally denoted by ω^i, are obtained by solving the system $\sum_j g_{ij} \omega^j = \omega_i$, i.e., by $\omega^i = \sum_j g^{ij} \omega_j$, if one denotes by g^{ij} the entries of the inverse of the g_{ij} matrix. This is "raising the indices". Predictably, $\#\flat u = u$, and $\flat\#\omega = \omega$.

Remark. The *norm of a covector* ω is $|\omega| = \sup\{\langle \omega ; v \rangle : |v| = 1\}$, by definition. It's easy to check that this norm stems from a scalar product, whose value for the pair $\{d^i, d^j\}$ is precisely g^{ij}. ◊

All these subtleties evaporate, for the better or the worse, when the basis is *orthonormal*, i.e., $|\partial_i| = 1$ and $\partial_i \perp \partial_j$ for $i \neq j$. Then $u_i = u^i$, etc.

Gradient. The gradient of a function φ is the vector field grad φ such that, using the above isomorphism, grad $\varphi = \#d\varphi$. Since $d\varphi = \sum_i \partial_i\varphi\, d^i$, the "covariant components" of grad φ, in the above sense, are the $\partial_i\varphi$. But its components in the ∂ basis are $\sum_j g^{ij} \partial_j\varphi$. Be wary that "gradient" is often used to refer to the 1-form itself rather than to its "vector representative" grad φ, and with good reason: $d\varphi$ is well defined independently of a metric, and does its job, i.e., encoding the local rate of variation of φ, without it. On the other hand, change the metric and the vector representative will have to change. The latter is therefore a "proxy", metric-dependent, for "the real thing", which is the 1-form $d\varphi$.

Remark. It often happens, as here, that a vector field u corresponds, via $u = \#\omega$, to a form ω the definition of which does not involve any metric. One may refer to u, in that case, as a *proxy field*. One should not jump to the conclusion that all vector fields of physics are proxies of this kind. The one that describes the instant velocity of a fluid mass, for instance, is genuinely a vector field. (Recall the very definition of a tangent vector as a velocity.) ◊

From now on (and hopefully, no confusion with the dimension of X should occur), n denotes the "outward going normal vector field", a vector field, whose domain contains ∂X, such that $|n| = 1$ and $n(x) \perp v \ \forall\ v \in T_x$ at all boundary points. The *normal part* $n\omega$ of a form ω of degree $p \geq 1$ is $t\, i_n\omega$. Note that $n\flat u =$

$g(n, u) \equiv n \cdot u$, i.e., the normal component of u.

Hodge operator. Denoted $*$, and defined as follows. Let ω_x be a p-covector at x and Ω a local volume. Let $\{v_1, ..., v_n\}$ be a direct frame which is orthonormal (in the sense of g), and η_x the unique $(n - p)$-covector such that

(4.1) $$\eta_x\{v_{p+1}, ..., v_n\} = \omega_x(v_1, ..., v_p\}.$$

One denotes by $*\omega$ the twisted form $x \to \{\eta_x, \Omega\}$. If $\tilde{\omega} \approx \{\omega, \Omega\}$, $*\tilde{\omega}$ is the ordinary form $x \to \eta_x$. Observe that $** = \pm 1$, depending on the parities of n and p. More precisely, $** = (-1)^{(n-1)p}$, where p is the degree of the operand.

In dimension 2, where the Hodge of a 1-form is a 1-form (a twisted one), the relation between proxies has some interest. The vector $\#*\flat u$ derives from u by "90° rotation" in the tangent plane. This of course refers to the metric (via the notion of orthogonality) but to the orientation as well: Orientation is what tells us which way to "rotate". Observe that $\#*\flat u$ is a twisted vector field, if u was polar, and the other way round.

Worth mentioning also is this:

(4.2) $$*n = t*,$$

where $*$ on the left is the Hodge operator on ∂X. [Proof: Take an orthonormal basis $\{n, v_2, ..., v_n\}$, with all v_i's tangent to ∂X. Then $<*n\omega\ ; v_{p+1}, ..., v_n> = <n\omega\ ; v_2, ..., v_p> = <\omega\ ; n, v_2, ..., v_p> = <*\omega\ ; v_{p+1}, ..., v_n>$.] As a corollary, $*t = (-1)^p\, n*$, where p is the degree of the form to which this is applied.

Curl. The compact definition is $\text{rot } u = \#*d\flat u$. Only in three dimensions does this make sense: Start from a vector field u, take the d of the corresponding 1-form, hence a 2-form the Hodge dual of which is a twisted 1-form. The latter has a proxy field, which is the curl of u, here denoted $\text{rot } u$.

Note that $\text{rot } u$ depends on orientation, since $*$ does. Changing Or to – Or changes it to $-\text{rot } u$. Just as with the vector product, one may define an axial vector field, the equivalence class $\{\{\text{rot } u, \text{Or}\}, \{-\text{rot } u, -\text{Or}\}\}$, represented by $\text{rot } u$ or $-\text{rot } u$ depending on which orientation is selected. This object only depends on the metric structure. Hence the frequent casual (and possibly confusing) remark that "rot u is actually an axial vector field". In the same spirit, one can define the curl of an axial vector field, which is then polar. Let's stress again that we *do* assume an orientation here, hence a context in which axial and polar vectors do not fit.

One may define the curl in dimension two, by the same process, but the result $*d\flat u$ is a (twisted) scalar field. There is some sense also in introducing the notation "rot φ", where φ is a function, for the vector $-{}^{\perp}\text{grad } \varphi$, i.e., $\text{grad } \varphi$ rotated 90° clockwise. We'll do that in a moment.

In 3D Euclidean space, there is a notation which may help see what's going on. Recall that $g(v, w)$ is now $v \cdot w$, at all points, and that an orientation is selected, which gives sense to the cross product $v \times w$. We'll take the 3-form $\text{vol}(u, v, w) = u \cdot v \times w$ as representative[24] of the direct orientation class, called Or. Now let's proceed.

Start from a vector field u. At point x, the map $v \to u(x) \cdot v$ is a covector. The field these covectors make is a 1-form, that we shall denote by ${}^1u$. But from the same vector field, one may also build a 2-form ${}^2u$, from the 2-covectors $\{v, w\} \to u(x) \cdot (v \times w)$. So u can act as proxy for two different forms, ${}^1u$ and ${}^2u$. Similarly, a scalar field φ generates a 0-form ${}^0\varphi$, which is just φ itself, and a 3-form ${}^3\varphi$, defined as the field of 3-covectors $\{u, v, w\} \to u \cdot (v \times w)\, \varphi(x)$, i.e., φ times the standard volume. Conversely, by the Riesz representation theorem, p-covectors in dimension 3 must be of one of these four kinds, and hence, a form in 3D must be ${}^0\varphi$, ${}^1u$, ${}^2u$, or ${}^3\varphi$, depending on its degree, for some scalar or vector proxy φ or u. (Notice how both metric and orientation play a role in the correspondence.) The notation ${}^1u$ is of course redundant with $\flat u$, but is clearer in the Euclidean 3D context.

Now, it's easy to show that

$$(4.3) \qquad {}^1(\mathrm{grad}\,\varphi) = d\,{}^0\varphi, \quad {}^2(\mathrm{rot}\,u) = d\,{}^1u, \quad {}^3(\mathrm{div}\,u) = d\,{}^2u$$

and to see what the integral of a form means: The integral of ${}^0\varphi$ at point x is just $\pm\,\varphi(x)$, the sign being the orientation of the point (see above). The integral $\int_c {}^1u$ of the 1-form ${}^1u$ along the oriented curve c is the circulation $\int_c \tau \cdot u$, where τ denotes the unit tangent vector, oriented the same way as c itself (Fig. 4.1). The integral $\int_S {}^2u$ of the 2-form ${}^2u$ on the outer oriented surface S is the flux $\int_S n \cdot u$, where n denotes the unit normal vector, oriented in the crossing direction. By applying the Stokes theorem, one finds back all these old vector analysis formulas: $\int_c \tau \cdot \mathrm{grad}\,\varphi = \varphi(y) - \varphi(x)$, where x and y are the end points of c (induced orientation, indeed, is + for y and − for x), $\int_{\partial S} \tau \cdot u = \int_S n \cdot \mathrm{rot}\,u$, and $\int_{\partial D} n \cdot u = \int_D \mathrm{div}\,u$. The inner product is given by

$$(4.4) \qquad i_v\,{}^1u = {}^0(u \cdot v), \quad i_v\,{}^2u = {}^1(u \times v), \quad i_v\,{}^3\varphi = {}^2(\varphi\, v).$$

As for the Hodge operator, the relations are

$$(4.5) \qquad *\,{}^0\varphi = {}^3\varphi, \quad *\,{}^3\varphi = {}^0\varphi, \quad *\,{}^1u = {}^2u, \quad *\,{}^2u = {}^1u.$$

Notice also that

$$(4.6) \qquad {}^2(u \times v) = {}^1u \wedge {}^1v, \quad {}^3(u \cdot v) = {}^2u \wedge {}^1v = {}^1u \wedge {}^2v.$$

One can base a cheap and quick approach to differential forms in E_3 on these formulas: Start from the standard definition of grad, curl, and div in coordinates, then define d via (4.3), $*$ via (4.5), and $\wedge$ via (4.6). Then derive (4.4) from the above non-metric definition of i_v.

Exercise. Use (3.3) and (4.3) to show that,

$$\mathrm{rot}(\varphi\, u) = \varphi\, \mathrm{rot}\, u - u \times \mathrm{grad}\,\varphi, \qquad \mathrm{div}(\varphi\, u) = \varphi\, \mathrm{div}\, u + u \cdot \mathrm{grad}\,\varphi,$$

$$\mathrm{div}(u \times v) = v \cdot \mathrm{rot}\, u - u \cdot \mathrm{rot}\, v.$$

[24] Called the *standard* volume. Note that $\mathrm{vol}(u, v, w) = \det(\mathbf{u}, \mathbf{v}, \mathbf{w})$ where the boldface symbols denote the triples of components of u, v, w in a direct orthonormal basis, and that the volume of the parallelepiped built on u, v, w is thus equal to 1.

[In the same vein, one may derive $(u \times v) \times w = (u \cdot w)\, v - (v \cdot w)\, u$ from (1.5).]

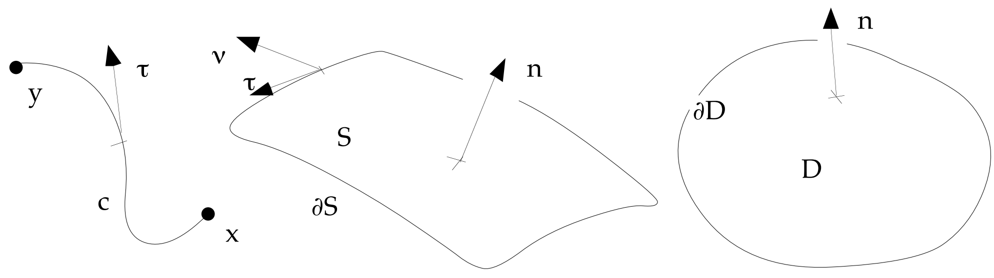

FIGURE 4.1. Notations for the interpretation of thr Stokes theorem: D is a 3D domain, n is unit normal vector, τ is unit tangent with respect to c or to ∂S, ν is in the plane tangent to S, normal to ∂S, unitary, pointing outwards with respect to S.

Remark. The dependence on orientation in all that may be felt as a nuisance. In particular, $<{}^2u\,;\, v, w>$ depends on the sign of $v \times w$, hence ${}^2u$ changes sign if one changes orientation. But the *twisted* form ${}^2\tilde{u}$, represented by the pair $\{{}^2u, \mathrm{vol}\}$, is invariant, and so is the twisted form ${}^3\tilde{\varphi} \approx \{{}^3\varphi, \mathrm{vol}\}$. Thus, in *non*-oriented 3D space with dot product, a function φ and a vector field u give birth to straight forms ${}^0\varphi$, ${}^1u$ and to twisted forms ${}^2\tilde{u}$, ${}^3\tilde{\varphi}$. One will easily see that, symmetrically, a twisted, or axial, φ or u give birth to twisted forms ${}^0\tilde{\varphi}$, ${}^1\tilde{u}$ and straight forms ${}^2u$ and ${}^3\varphi$. It's a useful exercise to edit (4.4) and (4.5) accordingly. (Beware, the dot product of two twisted vectors is a straight scalar.) ◊

Another case of interest is when the Riemannian manifold is a surface S embedded in E_3 (Fig. 4.1, middle), with metric $g_x(v, w) = v \cdot w$ (tangent vectors at x to S on the left of this equality, the same vectors considered as vectors of V_3 on the right). Suppose S orientable (just for convenience — one could easily lift this assumption), orientation being given by a field n of unit normals and the right-hand rule. Let φ_S and u_S be a scalar and a vector field living on S (but which we can also imagine as the traces of 3D fields φ and u whose domain contains S). The forms ${}^0\varphi_S$ and ${}^1u_S$ can then be defined just as previously, and the field of 2-covectors $\{v, w\} \to \varphi_S(x)\, n(x) \cdot (v \times w)$, where v and w are tangent to S at x, makes a 2-form ${}^2\varphi_S$ associated with φ_S. All this is as in dimension 3. But there is an element of novelty: Another 1-form is associated with u_S in a natural way, namely the field of covectors $v \to (n(x) \times u_S(x)) \cdot v$. Let's denote this 1-form, which is just ${}^1(n \times u_S)$, by ${}^{\perp}u_S$. Then (for clarity on first reading, we also subscript surfacic operators by S, but one is advised to drop these subscripts as soon as the mechanism is understood),

$$(4.7) \qquad {}^1(\mathrm{grad}_S\, \varphi_S) = d\, {}^0\varphi_S, \quad {}^2(\mathrm{rot}_S\, u_S) = d\, {}^1u_S, \quad {}^2(\mathrm{div}_S\, u_S) = d\, {}^{\perp}u_S, \quad {}^{\perp}(\mathrm{rot}_S \psi_S) = d\, {}^0\psi_S$$

(recall that $\mathrm{rot}_S\psi_S = -\, n \times \mathrm{grad}_S\psi_S$). The inner products are, for v tangent to S,

$$(4.8) \qquad i_v\, {}^1u_S = {}^0(u_S \cdot v), \quad i_v\, {}^{\perp}u_S = {}^0(n \times u_S \cdot v), \quad i_v\, {}^2\varphi_S = {}^{\perp}(\varphi_S v) \equiv {}^1(\varphi_S\, n \times v).$$

The surface Hodge operators relate like this:

(4.9) $\quad *_S{}^0\varphi_S = {}^2\varphi_S, \quad *_S{}^2\psi_S = {}^0\psi_S, \quad *_S{}^1u_S = {}^\perp u_S, \quad *_S{}^\perp u_S = -{}^1u_S.$

Remark. The normal part of ${}^1u$ is ${}^0(n \cdot u)$, the normal part of ${}^2u$ is $-{}^1(n \times u)$, that is to say, $-{}^\perp u_S$. On the other hand, $t_S\,{}^1u = {}^1u_S$, $t_S\,{}^2u = {}^2(n \cdot u)$. This checks with (4.2), i.e., $*_S n_S = t_S *_S$, as it should. Also, $n \cdot \mathrm{rot}\, u = \mathrm{rot}_S\, u_S$, a case of $td = dt$. ◊

Scalar product of two p-forms. If ω and $\omega' \in \mathcal{F}^p$, the exterior product $\omega \wedge *\omega'$ is a density, so $\int_X \omega \wedge *\omega'$ makes sense. This quantity is defined as the scalar product of ω and ω', here denoted (ω, ω'). Symmetry and positivity stem from (1.2) and (4.1). Remark also that $(*\omega, *\omega') = (\omega, \omega')$. The *norm* of ω is then defined as $(\omega, \omega)^{1/2}$, equal to $[\int_X \omega \wedge *\omega]^{1/2}$. All this turns $\mathcal{F}^p$ into a pre-Hilbertian space. By completion, one gets a Hilbert space of forms, denoted F^p, which is to p-forms what space L^2 was to functions.

The adjoint of d, with respect to this scalar product, is called the *codifferential*, denoted δ. Thanks to (3.5) and (4.2), the domain of δ is $\{\omega' \in F^{p+1} : n\omega' = 0\}$ and $\delta = (-1)^{np+1} *d*$. [Proof: p being the degree of ω, one has $(d\omega, \omega') = \int d\omega \wedge *\omega' = (-1)^{p+1} \int \omega \wedge d*\omega' = (-1)^{p+1+p(n-p)} \int \omega \wedge **d*\omega' = (\omega, \delta\omega')$, since $p^2 = p \bmod 2$.] But be careful: Here, p is not the degree of ω', to which δ is to be applied. The usable formula, where q is the degree of the form to which both sides must apply, is $\delta = (-1)^{n(q-1)+1} *d*$. (We had $p = q - 1$ in the proof.) These minus signs are an unavoidable nuisance. Fortunately, there is not much use for δ, apart from its role in the definition of the *Laplacian* operator on forms: This is $\delta d + d\delta$, up to an optional minus sign. Hence the well known formulas $\Delta = \mathrm{div} \circ \mathrm{grad}$ and $-\Delta = \mathrm{rot} \circ \mathrm{rot} - \mathrm{grad} \circ \mathrm{div}$, when applied to a scalar or a vector proxy.

Let's use L^2 for the space of square summable fields, be they scalar or vector valued. The functional spaces

$$L^2_{grad} = \{\varphi \in L^2 : \mathrm{grad}\, \varphi \in L^2\},\ L^2_{rot} = \{u \in L^2 : \mathrm{rot}\, u \in L^2\},\ L^2_{div} = \{u \in L^2 : \mathrm{div}\, u \in L^2\},$$

with their natural scalar products, such as $((\varphi, \varphi')) = \int_X \varphi\varphi' + \int_X \mathrm{grad}\, \varphi \cdot \mathrm{grad}\, \varphi'$, etc., correspond (for $p = 0, 1, 2$) to the Hilbert space $F^p{}_d$ obtained by completion of $\mathcal{F}^p$ with respect to the norm induced by the scalar product $((\omega, \eta)) = \int_X \omega \wedge *\eta + \int_X d\omega \wedge *d\eta$. When X is the 3D domain of Fig. 4.1, the classical integration by parts formulas,

(4.10) $\quad \int_D v \cdot \mathrm{grad}\, \varphi = -\int_D \varphi\, \mathrm{div}\, v + \int_{\partial D} n \cdot v\ \varphi,$

(4.11) $\quad \int_D u \cdot \mathrm{rot}\, v = \int_D v \cdot \mathrm{rot}\, u - \int_{\partial D} n \times u \cdot v,$

appear as particular cases of a single formula, namely (3.5). In two dimensions, e.g., when X is the surface S of Fig. 4.1, (3.5) yields the 2D version of (4.10), i.e.,

(4.12) $\quad \int_S v_S \cdot \mathrm{grad}_S\, \varphi_S = -\int_S \varphi_S\, \mathrm{div}_S\, v_S + \int_{\partial D} \nu \cdot v_S\ \varphi_S,$

and

(4.13) $\quad \int_S u_S \cdot \mathrm{rot}_S\ \varphi_S = \int_S \varphi_S\, \mathrm{rot}_S\, u_S - \int_{\partial S} n \cdot (\nu \times u_S)\ \varphi_S$

which is nothing else than (4.12) with $v_S = n \times u_S$, since $\mathrm{div}_S(n \times u_S) = -\,\mathrm{rot}_S\, u_S$. (What about dropping all these S's, by the way?)

5. THE LIE DERIVATIVE

In Electromagnetism (our main intended domain of application), differential forms model "fields", in the physical sense of the word. Such fields are probed, measured, with help of devices that are mathematically represented by manifolds, and integrals are the results of such experiments. The case of the electric field (a 1-form, denoted e), measured by a voltmeter, which displays the electromotive force (emf) along the conductive thread (a 1-dimensional, inner oriented manifold c), is typical: The emf is $\int_c e$. As a rule, the rates of variation, as time goes on, of such integral quantities, are physically meaningful. Typical again in this respect is Faraday's law, whose mathematical expression is $d_t \int_S b + \int_{\partial S} e = 0$ for all inner-oriented surfaces S, with b a 2-form, called magnetic induction.

Observers in relative motion, then, get different views of the field, because the rate of variation of what they measure with any given probe will depend on how the latter is moving. The Lie derivative is the mathematical tool by which such rates of variation are handled.

Motion is conveniently described by a smoothly time-dependent family u of diffeomorphisms $u_t : X \rightarrow X$. (One may always assume that $u_0(x) = x$. We shall allow ourselves to omit the t, or on the contrary to use the developed notation $u(t, x)$, as most convenient.) For each x, there is a trajectory $t \rightarrow u_t(x)$, or *orbit*, whose tangent vectors $v_t(y) \equiv v(t, y) = \partial_t u(t, u_t^{-1}(y))$ make the *velocity field* at time t. One may imagine X as filled up with a fluid and construe $u_t(x)$ as the trajectory of the particle that passes at x at time 0, and v as its velocity, although of course no such interpretation is compulsory. The trajectories are *integral curves* of the field v, in the sense that $t \rightarrow u(t, x)$ solves the differential equation $d_t g = v(t, g(t))$ with initial condition $g(0) = x$.

Of particular interest is the case when the family of maps u_t satisfies

$$u_{t+s}(x) = u_s(u_t(x)) \tag{5.1}$$

for all x, s, and t, thus forming a *one-parameter group of diffeomorphisms*. Then v is time-independent[25] [proof: $v(t, x)\, dt \sim u_{t+dt}(u_t^{-1}(x)) - u_t(u_t^{-1}(x)) = u_{dt}(x) - x \sim v(0, x)\, dt$], and u is called its *flow*. Notice that v is invariant under push-forwards of its own flow. [Proof: $(u_t)_*(x)\; v(x)\, dt \sim u_t(u_{dt}(x)) - u_t(x) = u_{dt}(u_t(x)) - u_t(x) \sim v(u_t(x))\, dt$.]

Homotopy. This is the mathematization of the concept of "morphing", i.e., "continuous deformation", as made familiar by many movies. Two continuous maps f and g from a topological space X to a topological space Y are *homotopic* if there exists a continuous map u from $[0, 1] \times X$ (endowed with the product topology) to Y such that $f = x \rightarrow u(0, x)$ and $g = x \rightarrow u(1, x)$. [Imagine X as an object, f as its graphical representation on, say, a computer screen, here modelled by Y. Then $x \rightarrow u(t, x)$ is an image which deforms with t, passing continuously from f to g.] If $X = Y$ and f is the identity, one says that g is a *deformation* of X. If, moreover, g maps X to one of its points, the homotopy is a *contraction*.

[25] Conversely, a smooth vector field generates a flow, if one is content with *local* existence (in space and time).

Space X is then said to be *contractible*. Examples: Affine spaces are contractible. A ball is contractible, a sphere is not. Flow, we see, is an instance of homotopy.

Extrusion of a manifold. Suppose v is transverse to a submanifold M (which must therefore have dimension $p < n$). The union $ext(u_t, M)$ of the images $u_s(M)$ for $0 \le s \le t$ makes an embedded $(p + 1)$-manifold, for t small enough,[26] called the extrusion of M over the lapse of time $[0, t]$.

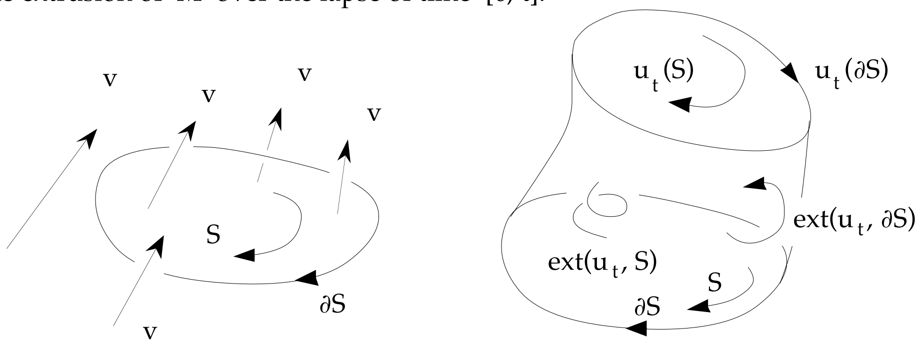

FIGURE 5.1. Extrusion of a surface S by the flow of a vector field v, and the interplay of orientations.

If M is oriented, this orients the extrusion, as follows: If $\{v_1, ..., v_p\}$ is a direct frame at $x \in M$, consider $\{v(x), v_1, ..., v_p\}$ (which *is* a frame, by transversality of v) as direct. We note that, with this convention, the boundary of the extrusion is the p-chain

(5.2) $$\partial[ext(u_t, M)] = u_t(M) - M - ext(u_t, \partial M)$$

and $\partial[ext(u_t, \partial M)] = u_t(\partial M) - \partial M$. (Cf. Fig. 5.1, where M is the surface denoted S.)

Proposition 5.1. *The integral of a* $(p + 1)$-*form* η *over* $ext(u_t, M)$ *is*

$$\int_0^t d\tau \int_{u_\tau(M)} i_v\eta.$$

Proof. Take a simplicial tiling *S* of M, and split the interval $[0, t]$ into segments $[k\delta t, (k + 1)\delta t]$, $k = 0, ..., N$, with $(N + 1)\delta t = t$. Extrude each simplex s of *S* and split the extrusion into slices of the form $ext[u((k + 1)\delta t), s] - ext[u(k\, \delta t), s]$. Each slice can then be cut into simplices (three, if $p = 2$; how many in general?). Form Riemann sums and go to the limit. ◊

When u is a flow, this simplifies a bit:

(5.3) $$<\eta\, ;\, ext(u_t, M)> = t <i_v\eta\, ;\, M>,$$

because all integrals over $u_\tau(M)$ are equal, owing to the invariance of v.

Now the *Lie derivative* of a p-form ω with respect to the flow of a vector field v is defined, at time $t = 0$, by

(5.4) $$L_v\omega = \lim_{t \to 0} {}^1/_t\, [u_t{}^*\omega - \omega].$$

[26] If $\varphi : S \to \mathbb{R}^p$ is a chart for S, $\{s, x\} \to \{s, \varphi(x)\} \in \mathbb{R}^{p+1}$ makes one for the extrusion, provided points $u_s(x)$ for $s \in [0, t]$ are all distinct. In case v depends on time, transversality is understood as holding at time 0. Smoothness in time of v, anyway, warrants it for s small enough.

(The concept makes sense for all kinds of tensors, actually. In particular, the Lie derivative $L_v w$ of a vector field w is the Lie bracket $[v, w]$.) Intuitively, $L_v\omega$ is the rate of variation of ω "as transported by" the flow of v, and "as measured by a fixed observer" (the "fisherman's derivative" of Arnold [Ar]). To express L_v in terms of already known entities, let's consider the integral $\int_M u_t{}^*\omega$, with a fixed M. After (5.2),

$$\int_M u_t{}^*\omega - \int_M \omega = \int_{u_t(M)} \omega - \int_M \omega = <\omega \ ; \ \partial[\mathrm{ext}(u_t, M)]> + <\omega \ ; \ \mathrm{ext}(u_t, \partial M)>$$

$$= <d\omega \ ; \ \mathrm{ext}(u_t, M)> + <\omega \ ; \ \mathrm{ext}(u_t, \partial M) >.$$

Using (5.3) twice (once with $\eta = d\omega$, then again but on ∂M and with $\eta = \omega$), we find $\int_M L_v\omega = <i_v d\omega \ ; M> + <i_v\omega \ ; \partial M>$. Using Stokes, we may conclude:

Proposition 5.2. $L_v = i_v d + d i_v$.

Consequently, $L_v u^* = u^* L_{u_* v}$, after (1.4) and (3.6), and $L_v u^* = u^* L_v$ if v is stationary. If ω is time-dependent, we see that

$$(5.5) \qquad d_t[\int_{u_t(M)} \omega\,]_{\,t=0} = \int_M \,[\partial_t\omega + L_v\omega],$$

which prompts us to define the *convective derivative* $\partial_t\omega + L_v\omega$ of ω with respect to the flow. (That v may depend on time is not a concern here; what is considered is the flow of v as frozen at its value at time 0).) It may be useful also to have the derivative at time t, which is given by the formula

$$(5.6) \qquad \partial_t(u_t{}^*\omega) = u_t{}^*(\partial_t\omega + L_v\omega).$$

Another way to express that, thanks to Leibniz's rule, is

$$(5.7) \qquad \partial_t u_t{}^* = u_t{}^*\, L_{v(t)} = L_{v(0)}\, u_t{}^*.$$

All these formulas have useful applications, in different contexts.

Exercise. Show that $i_v L_v = L_v i_v$, that $d L_v = L_v d$.

Before giving a few examples, let's work out L_v in 3D Euclidean space. Thanks to (4.3) and (4.4), one has:

$$(5.8) \qquad L_v{}^0\varphi = {}^0(v \cdot \mathrm{grad}\ \varphi),$$

$$(5.9) \qquad L_v{}^1u = {}^1(-\, v \times \mathrm{rot}\ u + \mathrm{grad}(v \cdot u)),$$

$$(5.10) \qquad L_v{}^2u = {}^2(v\ \mathrm{div}\ u - \mathrm{rot}(v \times u)),$$

$$(5.11) \qquad L_v{}^3\varphi = {}^3(\mathrm{div}(\varphi\ v)) \equiv {}^3(v \cdot \mathrm{grad}\ \varphi + \varphi\ \mathrm{div}\ v).$$

While we are at it, the analogous formulas on a surface (all right, no S subscripts) are (5.8), unchanged,

$$(5.12) \qquad L_v{}^1u = {}^\perp(v\ \mathrm{rot}\ u) + {}^1(\mathrm{grad}(v \cdot u)) \equiv {}^1(n \times v\ \mathrm{rot}\ u + \mathrm{grad}(v \cdot u)),$$

$$(5.13) \qquad L_v{}^\perp u = {}^\perp(v\ \mathrm{div}\ u) - {}^1(\mathrm{grad}(v \times u)) \equiv {}^\perp(v\ \mathrm{div}\ u - \mathrm{rot}(v \times u)),$$

not so different from (5.9) and (5.10), and

(5.14) $\qquad L_v{}^2\varphi = {}^2(\mathrm{div}(\varphi\, v))$,

the same as (5.11).

Physics provides familiar examples. For instance, if φ is the temperature of a fluid mass, the rate of variation of the temperature of a material particle, as observed at point x and time t as the particle flies by, is $\partial_t\varphi + v \cdot \nabla\varphi$, which is consistent with (5.8). If the fluid has density q, conservation of its mass is expressed by $\partial_t{}^3 q + L_v{}^3 q = 0$, that is $\partial_t q + \mathrm{div}(q\ v) = 0$, as can be read off from (5.11). And (5.10) explains why the effective electric field in a moving conductor (the field to which the current density is proportional) is $e + v \times b$, where e and b (divergence-free) are the laboratory-frame electric field and magnetic induction. For indeed, $d(e + v \times b) = de - L_v b$, and $\partial_t b + L_v b + d(e + v \times b) = 0$ expresses Faraday's law in the comoving frame.

Lie and wedge. Contrary to the d, the Lie derivative behaves like a genuine derivation, i.e., it respects Leibniz's rule:

(5.15) $\qquad L_v(\omega \wedge \eta) = L_v\omega \wedge \eta + \omega \wedge L_v\eta.$

Proof. Since $u^*(\omega \wedge \eta) = u^*\omega \wedge u^*\eta$, one has $u^*(\omega \wedge \eta) - \omega \wedge \eta = u^*\omega \wedge u^*\eta - \omega \wedge \eta = (\omega + u^*\omega - \omega) \wedge (\eta + u^*\eta - \eta) - \omega \wedge \eta = (u^*\omega - \omega) \wedge \eta + \omega \wedge (u^*\eta - \eta) + (u^*\omega - \omega) \wedge (u^*\eta - \eta)$. Now take $u = u_t$, divide by t, and let t go to 0. The result stems from (5.4). ◊

Proposition 2 suggests another proof, which avoids the passage to the limit, but is more delicate. By (1.5) and (3.3), one has, with $p = \mathrm{degree}(\omega)$,

$$\begin{aligned} i_v d(\omega \wedge \eta) &= i_v(d\omega \wedge \eta + (-1)^p\, \omega \wedge d\eta)) \\ &= i_v d\omega \wedge \eta - (-1)^p\, d\omega \wedge i_v\eta + (-1)^p\, i_v\omega \wedge d\eta + \omega \wedge i_v d\eta \end{aligned}$$

and

$$\begin{aligned} d i_v(\omega \wedge \eta) &= d(i_v\omega \wedge \eta + (-1)^p\, \omega \wedge i_v\eta) \\ &= d i_v\omega \wedge \eta - (-1)^p\, i_v\omega \wedge d\eta + (-1)^p\, d\omega \wedge i_v\eta + \omega \wedge d i_v\eta, \end{aligned}$$

and extra terms cancel nicely.

The Poincaré Lemma. In A_n, and in more general conditions as we shall see in a moment, $d\omega = 0$ implies the existence of a $(p - 1)$-form α such that $\omega = d\alpha$. This is a particular case of the following representation formula, derived from (5.6):

$$u_1{}^*\omega = u_0{}^*\omega + \int_0^1 u_t{}^*\, L_v\omega \ \ dt,$$

provided the u_ts are defined over the semi-open interval $[0, 1[$. If things can be arranged so that u is a contraction $u_1{}^*$ is the identity and $u_0{}^* = 0$, so $\omega = \int_0^1 u_t{}^*\, L_v\omega\ dt$, that is to say, thanks to the commutation relations $u^* L_v = L_v u^*$,

(5.16) $\qquad \omega = d[\int_0^1\ i_v u_t{}^*\,\omega\ dt] + i_v[\int_0^1 u_t{}^*\ d\omega\ dt] = d\alpha + \beta,$

If $d\omega = 0$, then $\beta = 0$, hence the Lemma: $\omega = d\alpha$, which we see is valid for all contractible manifolds.

For instance, o being an arbitrary origin in E_n, set $u_t(x) = o + t(x - o)$, which is the flow of the vector field $x \to x - o$, which from now on we denote by x. This is a contraction. We note that $(u_t)_* w = u_t(x + w) - u_t(x) = tw$, so $(u_t)_*$ is the homothetic transformation $w \to tw$. Hence $(u_t{}^*\omega)(x) = t^p\ \omega(tx))$ if ω has degree p, and

$$\alpha(x) = \int_0^1 t^{p-1}\, i_x\omega(tx)\ dt, \quad \beta(x) = \int_0^1 t^p\, i_x d\omega(tx)\ dt,$$

If $\omega = {}^1e$, with $\mathrm{rot}\ e = 0$, we thus find that $u = \mathrm{grad}\ \psi$ with

$$\psi(x) = x \cdot \int_0^1 e(tx)\ dt,$$

a not so obvious fact. Even less obvious, if $\omega = {}^2b$ with $\mathrm{div}\ b = 0$, then $u = \mathrm{rot}\ a$ with

$$a(x) = -\, x \times \int_0^1 t\ \ b(tx)\ dt, \tag{5.17}$$

and any scalar field q (since, automatically, $d\,{}^3q = 0$) is of the form $q = \mathrm{div}\ d$, with

$$d(x) = x \int_0^1 t^2\, q(tx)\ dt.$$

If $d\omega = 0$, then $\beta = 0$, hence the Lemma: $\omega = d\alpha$. With the present choice of u_t, one has more, by working out (5.16).

Let us note (with [Ko] and [Sk]) that $x \cdot a(x) = 0$ in (5.17): this constraint, which selects one of the vector potentials a such that $\mathrm{rot}\ a = b$, differs from the more common "gauges" of Coulomb and Lorenz.[27] It is called *Poincaré's gauge* [Br, Cr], and *axial gauge* by some.

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