

Lecture 3: Linear systems

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Linear systems

- **Linear systems:**
 - **Exponential** of a matrix;
 - Linear systems with **constant** coefficients;
 - Linear system with **non-constant** real coefficients;
 - **Second order** linear equations;
 - **Linearization** and **stability** for **autonomous systems**.

Linear systems

- Exponential of a matrix:
 - $\mathbb{M}_d(\mathbb{C})$: vector space of $d \times d$ matrices with entries in \mathbb{C} .
 - $GL_d(\mathbb{C}) \subset \mathbb{M}_d(\mathbb{C})$: group of invertible matrices.
 - DEFINITION: Matrix norm

$$\|A\| = \max_{|y|=1} |Ay|.$$

Linear systems

- **LEMMA: Properties of the norm**
 - $|Ay| \leq \|A\| |y|$ for all $y \in \mathbb{C}^d$;
 - $\|A + B\| \leq \|A\| + \|B\|$ for all $A, B \in \mathbb{M}_d(\mathbb{C})$;
 - $\|AB\| \leq \|A\| \|B\|$ for all $A, B \in \mathbb{M}_d(\mathbb{C})$.

Linear systems

- **LEMMA: Jordan-Chevalley decomposition**
 - $A \in \mathbb{M}_d(\mathbb{C})$.
 - There exists $C \in GL_d(\mathbb{C})$ s.t. A has a **unique decomposition**

$$C^{-1}AC = D + N;$$

- D : **Diagonal**; N : **Nilpotent** (i.e., $N^d = 0$);

$$ND = DN.$$

Linear systems

- Exponential of a matrix.
- DEFINITION:
 - For $A \in \mathbb{M}_d(\mathbb{C})$,

$$e^A = \sum_{n \geq 0} \frac{A^n}{n!}.$$

Linear systems

- **Properties:**

- Exponential of the **sum**: $A, B \in \mathbb{M}_d(\mathbb{C})$,

$$\text{If } AB = BA \Rightarrow e^{A+B} = e^A e^B.$$

- **Conjugation** and exponentiation:

- $A, B \in \mathbb{M}_d(\mathbb{C})$ and $C \in GL_d(\mathbb{C})$ s.t. $A = C^{-1}BC$.

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$$e^A = C^{-1}e^B C.$$

- **PROOF:**

$$e^A = \sum_{n \geq 0} \frac{A^n}{n!} = \sum_{n \geq 0} \frac{(C^{-1}BC)^n}{n!} = \sum_{n \geq 0} \frac{C^{-1}B^n C}{n!} = C^{-1}e^B C;$$

Linear systems

- Exponential of a **diagonalizable matrix**:
 - A : **diagonalizable**

$$A = C^{-1} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix} C;$$

- $\lambda_1, \dots, \lambda_d \in \mathbb{C}$ and $C \in GL_d(\mathbb{C})$.
- \Rightarrow

$$e^A = C^{-1} \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_d} \end{pmatrix} C.$$

Linear systems

- **Exponential of a block matrix:**
 - $A_j \in \mathbb{M}_{h_j}(\mathbb{C})$ for $j = 1, \dots, p$; A : **block matrix** of the form

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix}.$$

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$$e^A = \begin{pmatrix} e^{A_1} & & 0 \\ & \ddots & \\ 0 & & e^{A_p} \end{pmatrix}.$$

Linear systems

- **Derivative:** $A \in \mathbb{M}_d(\mathbb{C})$,

$$\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A.$$

Linear systems

- **Linear systems with constant coefficients**

- $A \in \mathbb{M}_d(\mathbb{C})$: **independent** of t .
- $f \in \mathcal{C}^0([0, T])$.
- **Linear ODE with constant coefficients:**

$$(*) \quad \begin{cases} \frac{dx}{dt} = Ax(t) + f(t), & t \in [0, T], \\ x(0) = x_0 \in \mathbb{R}^d. \end{cases}$$

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$$|A(x - y)| \leq \|A\| |x - y| \quad \text{for all } x, y \in \mathbb{C}^d,$$

- **Cauchy-Lipschitz theorem** \Rightarrow there **exists a unique solution** x to $(*)$.
- If $f = 0$: $(*)$ **autonomous** system of equations.

Linear systems

- If $d = 1$ (i.e., $A = a \in \mathbb{C}$), then by the method of integrating factors,

$$x(t) = e^{at} x_0 + \int_0^t e^{a(t-s)} f(s) ds.$$

- General case ($d \geq 1$), if $f = 0$,

$$x(t) = e^{tA} x_0.$$

Linear systems

- For an arbitrary f ,

$$\frac{d}{dt}(e^{-tA}x) = e^{-tA}f(t),$$

- \Rightarrow

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}f(s)ds.$$

Linear systems

- Linear system with non-constant real coefficients
 - Homogeneous case;
 - Inhomogeneous case.
- Homogeneous case:
 - $M_d(\mathbb{R})$: vector space of $d \times d$ matrices with entries in \mathbb{R} .
 - PROPOSITION:
 - $A : [0, T] \rightarrow M_d(\mathbb{R})$: continuous.
 - S : linear subspace of $C^1([0, T]; \mathbb{R}^d)$ of dimension d :

$$S = \left\{ x \in C^1([0, T]; \mathbb{R}^d) : x \text{ satisfies } \frac{dx}{dt} = A(t)x \right\}$$

Linear systems

- **PROOF:**
 - $x, y \in S \Rightarrow$ for any $\alpha, \beta \in \mathbb{R}$, $\alpha x + \beta y \in C^1([0, T]; \mathbb{R}^d)$: also a solution.
 - $\Rightarrow S$: linear subspace of $C^1([0, T]; \mathbb{R}^d)$.

Linear systems

- **Dimension of $S = d$:**
 - Define $F : S \rightarrow \mathbb{R}^d$ by $F[x] = x(t_0)$ for some $t_0 \in [0, T]$.
 - F : **linear**:

$$F[\alpha x + \beta y] = \alpha x(t_0) + \beta y(t_0) = \alpha F[x] + \beta F[y].$$

- F : **injective**,

$$F[x] = 0 \quad \Rightarrow \quad x = 0;$$

- x solves

$$\frac{dx}{dt} = A(t)x(t)$$

with the initial condition $x(t_0) = 0$.

- **Cauchy-Lipschitz theorem** $\Rightarrow x = 0$.

Linear systems

- F : **surjective**: for any $x_0 \in \mathbb{R}^d$,

$$\begin{cases} \frac{dx}{dt} = A(t)x(t), & t \in [0, T], \\ x(t_0) = x_0, \end{cases}$$

has a solution $x \in \mathcal{C}^1([0, T]; \mathbb{R}^d)$.

Linear systems

- **PROPOSITION:**

- $x_1, \dots, x_d \in S$;
- $[x_1, \dots, x_d]$: $d \times d$ matrix with columns $x_1, \dots, x_d \in \mathbb{R}^d$;
- det: **determinant** of a matrix;
- **Equivalent statements:**
 - (i) $\{x_1, \dots, x_d\}$: **basis** of S ;
 - (ii) **det** $[x_1(t), \dots, x_d(t)] \neq 0$ for all $t \in [0, T]$.
 - (iii) **det** $[x_1(t_0), \dots, x_d(t_0)] \neq 0$ for **some** $t_0 \in [0, T]$.

Linear systems

- **PROOF:**
 - (i) \Leftrightarrow (ii).
 - (i) \Rightarrow (iii): $\{x_1, \dots, x_d\}$: basis of $S \Rightarrow \{F[x_1], \dots, F[x_d]\}$: basis of \mathbb{R}^d .
 - (iii) \Rightarrow (i): t_0 s.t. (iii) holds; $F : S \rightarrow \mathbb{R}^d$: **isomorphism** relative to t_0 .
 - $F^{-1} : \mathbb{R}^d \rightarrow S$: **isomorphism** \Rightarrow
 $x_1 = F^{-1}[x_1(t_0)], \dots, x_d = F^{-1}[x_d(t_0)]$: basis of S .

Linear systems

- **DEFINITION: Fundamental matrix**
 - If (i), (ii) or (iii): holds $\Rightarrow x_1, \dots, x_d$: **fundamental system** of solutions of the differential equation $\frac{dx}{dt} = A(t)x$.
 - $X = [x_1, \dots, x_d]$: **fundamental matrix** of the equation.

Linear systems

- **DEFINITION: Wronskian determinant**
 - $x_1, \dots, x_d \in S$.
 - Wronskian determinant $w \in \mathcal{C}^1([0, T]; \mathbb{R})$ of x_1, \dots, x_d :

$$w(t) = \det[x_1(t), \dots, x_d(t)].$$

Linear systems

- **THEOREM:**

- $x_1, \dots, x_d \in S$; $w \in C^1([0, T]; \mathbb{R}^d)$: **Wronskian determinant** of x_1, \dots, x_d .
- w solves the differential equation

$$(**) \quad \frac{dw}{dt} = (\operatorname{tr}A(t))w \quad \text{for } t \in [0, T].$$

- tr : trace of a matrix.

Linear systems

- **PROOF:**

- If x_1, \dots, x_d : **linearly dependent** $\rightarrow w = 0$ and **(**)** trivially **holds**.
- Suppose that x_1, \dots, x_d : **linearly independent**, i.e., $w(t) \neq 0$ for all $t \in [0, T]$.
- $X : [0, T] \rightarrow \mathbb{M}_d(\mathbb{R})$: fundamental matrix having as columns the solutions x_1, \dots, x_d , i.e.,

$$X(t) = (x_{ij}(t))_{i,j=1,\dots,d}, \quad t \in [0, T],$$

$$x_j = (x_{1j}, \dots, x_{dj})^\top \text{ for } j = 1, \dots, d.$$

Linear systems

- z_j : solution of

$$\begin{cases} \frac{dz_j}{dt} = A(t)z_j(t), \\ z_j(t_0) = e_j, \end{cases}$$

$\{e_j\}_{j=1,\dots,d}$: standard unit orthonormal basis in \mathbb{R}^d .

Linear systems

- $\Rightarrow \{z_1, \dots, z_d\}$: basis of the space of solutions to $dz/dt = Az$.
- There exists $C \in GL_d(\mathbb{R}^d)$ s.t.

$$X(t) = Z(t)C, \quad t \in [0, T],$$

$$Z = [z_1, \dots, z_d].$$

- $v(t) := \det Z(t)$ solves

$$\frac{dv}{dt}(t_0) = \text{tr}A(t_0).$$

- $Z(t_0) = I \Rightarrow v(t_0) = 1$.

Linear systems

- Definition of the **determinant** of a matrix \Rightarrow

$$\begin{aligned}\frac{dv}{dt}(t) &= \frac{d}{dt} \sum_{\sigma \in S_d} (-1)^{\text{sgn } \sigma} \prod_{i=1}^d z_{i\sigma(i)}(t) \\ &= \sum_{\sigma \in S_d} (-1)^{\text{sgn } \sigma} \sum_{j=1}^d \frac{d}{dt} z_{j\sigma(j)}(t) \prod_{i \neq j} z_{i\sigma(i)}(t);\end{aligned}$$

S_d : set of all permutations of the d elements $\{1, 2, \dots, d\}$; $\text{sgn } \sigma$: signature of the permutation σ .

Linear systems

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$$\prod_{i \neq j} z_{i\sigma(i)}(t_0) = 0 \quad \text{unless } \sigma = \text{identity};$$

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$$\begin{aligned} \frac{dz_{jj}}{dt}(t_0) &= (A(t_0)z_j(t_0))_j \\ &= \sum_{h=1}^d a_{jh}(t_0)z_{hj}(t_0) = \sum_{h=1}^d a_{jh}(t_0)\delta_{hj}(t_0) \\ &= a_{jj}(t_0). \end{aligned}$$

- \Rightarrow

$$\frac{dv}{dt}(t_0) = \sum_{j=1}^d a_{jj}(t_0) = \text{tr}A(t_0).$$

Linear systems

- Differentiation of

$$w = \det X = \det(ZC) = (\det C) \det Z = (\det C)v;$$

- \Rightarrow

$$\frac{dw}{dt}(t_0) = (\det C) \frac{dv}{dt}(t_0) = (\det C) \operatorname{tr} A(t_0).$$

- $v(t_0) = 1 \Rightarrow$

$$\frac{dw}{dt}(t_0) = \operatorname{tr} A(t_0) w(t_0).$$

Linear systems

- **REMARK:**

- $t_0 \in [0, T]$.
- **Abel's** identity or **Liouville's** formula:

$$w(t) = w(t_0) e^{\int_{t_0}^t \text{tr} A(s) ds} \quad \text{for } t \in [0, T].$$

- It suffices to check that the determinant of the fundamental matrix: **nonzero for one** $t_0 \in [0, T]$.

Linear systems

- Inhomogeneous case
- Inhomogeneous linear differential equation:

$$(***) \quad \begin{cases} \frac{dx}{dt} = A(t)x + f(t); \end{cases}$$

- $A(t) \in C^0([0, T]; \mathbb{M}_d(\mathbb{R}))$ and $f \in C^0([0, T]; \mathbb{R}^d)$.
- X : **fundamental matrix** for the homogeneous equation $dx(t)/dt = A(t)x(t)$,

$$\frac{dX}{dt} = AX \quad \text{and} \quad \det X \neq 0 \quad \text{for all } t \in [0, T].$$

- Any solution x to the **homogeneous** equation:

$$x(t) = X(t)c, \quad t \in [0, T],$$

for some (column) vector $c \in \mathbb{R}^d$.

Linear systems

- Method of integrating factors: $c \in C^1([0, T]; \mathbb{R}^d)$.

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$$\frac{dx}{dt} = \frac{dX}{dt}c + X\frac{dc}{dt} = AXc + X\frac{dc}{dt} = Ax + X\frac{dc}{dt}.$$

- $\Rightarrow X\frac{dc}{dt} = f(t)$.

- X : invertible \Rightarrow

$$\frac{dc}{dt} = X^{-1}f(t).$$

- \Rightarrow

$$c(t) = c_0 + \int_0^t X(s)^{-1}f(s)ds,$$

for some $c_0 \in \mathbb{R}^d$.

Linear systems

- **THEOREM:**

- X : **fundamental matrix** for the homogeneous equation $dx/dt = Ax$.
- For all $c_0 \in \mathbb{R}^d$,

$$(***) \quad x(t) = X(t) \left(c_0 + \int_0^t X(s)^{-1} f(s) ds \right)$$

solution to (***) .

- Any solution to (***) : **of the form (***)** for some $c_0 \in \mathbb{R}^d$.
- Formula (***) : **Duhamel's formula**.

Linear systems

- **PROOF:**
 - First statement: already proved.
 - Second statement:

- x_2 : solution to (***) .

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$$\frac{d}{dt}(x_2 - x(t)) = A(x_2 - x),$$

- $\Rightarrow x_2 - x = Xc_1$ for some $c_1 \in \mathbb{R}^d$.

Linear systems

- **Second order linear equations**

- $d = 1$:

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right),$$

for a given scalar function f .

- Linear ODE if f : linear in x and dx/dt ,

$$f\left(t, x, \frac{dx}{dt}\right) = g(t) - p(t)\frac{dx}{dt} - q(t)x,$$

g, p, q : functions of t but not of x .

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$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x = g(t).$$

- **Initial conditions:**

$$x(t_0) = x_0, \quad \frac{dx}{dt}(t_0) = x'_0, \quad x_0, x'_0 \in \mathbb{R}.$$

Linear systems

- **Homogeneous** if $g = 0$ and **inhomogeneous** otherwise.
- ODE with constant coefficients: $p(t)$ and $q(t)$: constant.
- Suppose $p, q \in C^0([0, T])$.
 - If **NOT**: points at which either p or q fail to be continuous: **singular points**.
 - **EXAMPLES**:

Bessel's equation: $p(t) = \frac{1}{t}, q(t) = 1 - \frac{\nu}{t^2},$ (at $t = 0$);

Legendre's equation: $p(t) = \frac{2t}{1-t^2}, q(t) = \frac{n(n+1)}{1-t^2}, n \in \mathbb{N}$
(at $t = \pm 1$).

Linear systems

- **THEOREM:**
 - Suppose that $p, q, g \in \mathcal{C}^0([0, T], \mathbb{R}^d)$.
 - There **exists a unique solution** $x(t)$ on $[0, T]$.

Linear systems

- Structure of the general solution.
- DEFINITION:
 - Two functions x_1 and x_2 on $[0, T]$: **linearly independent** if neither of them is a multiple of the other.
 - Otherwise, x_1 and x_2 on $[0, T]$: **linearly dependent**.
- PROPOSITION:
 - w : **Wronskian determinant**

$$w(t) := x_1(t) \frac{dx_2}{dt}(t) - x_2(t) \frac{dx_1}{dt}(t) = \det \begin{pmatrix} x_1 & x_2 \\ \frac{dx_1}{dt} & \frac{dx_2}{dt} \end{pmatrix}.$$

- $w(t)$: **not zero at some $t_0 \in [0, T]$ \Rightarrow x_1 and x_2 : linearly independent.**

Linear systems

- **PROOF:**

- Prove: x_1 and x_2 : linearly dependent $\Rightarrow w(t) = 0$ for all $t \in [0, T]$.
- Suppose x_1 and x_2 : linearly dependent.
- **Nontrivial solution** (α_1, α_2) :

$$\begin{cases} \alpha_1 x_1 + \alpha_2 x_2 = 0, \\ \alpha_1 \frac{dx_1}{dt} + \alpha_2 \frac{dx_2}{dt} = 0, \end{cases} \quad \text{for all } t \in [0, T],$$

- \Rightarrow

$$w(t) = \det \begin{pmatrix} x_1 & x_2 \\ \frac{dx_1}{dt} & \frac{dx_2}{dt} \end{pmatrix} = 0, \quad \text{for all } t \in [0, T].$$

Linear systems

- **PROPOSITION:**

- If x_1 and x_2 : solutions on $[0, T]$.
- $w(t)$: either identically zero or not equal to zero at any point of $[0, T]$.

- **PROOF:**

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$$\frac{dw}{dt}(t) = x_1 \frac{d^2 x_2}{dt^2} - x_2 \frac{d^2 x_1}{dt^2}.$$

- x_1, x_2 : solutions \Rightarrow

$$\frac{d^2 x_i}{dt^2} = -p(t) \frac{dx_i}{dt} - q(t)x_i, \quad i = 1, 2.$$

- \Rightarrow

$$\frac{dw}{dt} = -p(t) \left(x_1 \frac{dx_2}{dt} - \frac{dx_1}{dt} x_2 \right) = -p(t)w(t).$$

- $w(t) = w(t_0)e^{-\int_{t_0}^t p(s)ds}$: either identically zero or never vanishes depending on $w(t_0)$.

Linear systems

- Structure of the general solution to the homogeneous system.
- **THEOREM:**
 - Suppose that x_1 and x_2 : solutions for $g = 0$.
 - Suppose that x_1 and x_2 : linearly independent.
 - **General solution:** of the form $c_1x_1 + c_2x_2$; c_1 and c_2 : constant coefficients.

Linear systems

- **PROOF:**
 - \tilde{x} : arbitrary solution with the initial condition $\tilde{x}(t_0) = \tilde{x}_0, d\tilde{x}/dt(t_0) = \tilde{x}'_0$.
 - Consider the system of equations for (c_1, c_2)

$$\begin{cases} c_1 x_1(t_0) + c_2 x_2(t_0) = \tilde{x}_0, \\ c_1 \frac{dx_1}{dt}(t_0) + c_2 \frac{dx_2}{dt}(t_0) = \tilde{x}'_0. \end{cases}$$

- $x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} \neq 0$ at $t = t_0 \Rightarrow$ there exists a unique nontrivial solution $(c_1, c_2) = (\tilde{c}_1, \tilde{c}_2)$.
- **Existence and uniqueness theorem** for the initial value problem of the second order ODE $\Rightarrow \tilde{c}_1 x_1 + \tilde{c}_2 x_2 = \tilde{x}$.

Linear systems

- **Linear n -th order ODE with constant coefficients**
- Solve a linear n -th order ODE with constant coefficients.
- Consider

$$\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_1 \frac{dx}{dt} + a_0 x = 0,$$

$a_i \in \mathbb{R}$ for $i = 0, \dots, n - 1$.

- General solution:

$$x(t) = c_1 x_1 + \dots + c_n x_n;$$

- $\{x_i\}_{i=1}^n$: set of linearly independent solutions (a fundamental set of solutions) and c_i : constant coefficients.

Linear systems

- $w(t)$: **Wronskian determinant** of the set $\{x_1, \dots, x_n\}$,

$$w(t) = \det \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ \frac{dx_1}{dt} & & & \\ \vdots & & & \\ \frac{d^{n-1}}{dt^{n-1}} x_1 & \frac{d^{n-1}}{dt^{n-1}} x_2 & \dots & \frac{d^{n-1}}{dt^{n-1}} x_n \end{bmatrix}.$$

- If $w(t_0) \neq 0$ for some $t_0 \Rightarrow (x_1, \dots, x_n)$ forms a **fundamental set of solution**.
- Solve the equation through the **characteristic equation**

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0.$$

- Equation: derived by guessing a solution $x(t) = e^{\lambda t}$ with $\lambda \in \mathbb{C}$.

Linear systems

- Characteristic equation: n complex roots $\hat{\lambda}_j$ counted with their multiplicities l_j .
- Rewritten in the form

$$\prod_{j=1}^m (\lambda - \hat{\lambda}_j)^{l_j} = 0$$

with $\sum_{j=1}^m l_j = n$.

- General solution $x(t)$: linear combination of $t^k e^{\hat{\lambda}_j t}$ for $0 \leq k < l_j$ and $j = 1, \dots, m$.
- In particular, if $m = n$, then $x(t)$: linear combination of $e^{\hat{\lambda}_j t}$.

Linear systems

- **THEOREM:**

- $\hat{\lambda}_j, 1 \leq j \leq m$: zeros of the characteristic polynomial;
- l_j : corresponding multiplicities.
- n linearly independent solutions:

$$x_{j,k}(t) = t^k e^{\hat{\lambda}_j t}, \quad 0 \leq k < l_j, \quad 1 \leq j \leq m.$$

- Any other solution can be written as a linear combination of these solutions.

Linear systems

- Reduction of order
- Method for finding a second solution to the homogeneous second order ODE when a first solution: known by reducing the order.
- x_1 : a solution.

$$x(t) = v(t)x_1(t).$$

$$\frac{dx}{dt}(t) = \frac{dv}{dt}x_1 + v\frac{dx_1}{dt}$$

$$\frac{d^2x}{dt^2}(t) = \frac{d^2v}{dt^2}x_1 + 2\frac{dv}{dt}\frac{dx_1}{dt} + v\frac{d^2x_1}{dt^2}.$$

Linear systems

- \Rightarrow

$$\frac{d^2 v}{dt^2} + \left(p + 2 \frac{dx_1/dt}{x_1} \right) \frac{dv}{dt} = 0.$$

- $u = dv/dt \Rightarrow$ first order ODE

$$\frac{du}{dt} + \left(p + 2 \frac{dx_1/dt}{x_1} \right) u = 0.$$

- \Rightarrow

$$u(t) = ce^{-\int^t (p + 2 \frac{dx_1/dt}{x_1}) ds} = \frac{c}{(x_1(t))^2} e^{-\int^t p(s) ds}.$$

- $v = \int^t u(s) ds \Rightarrow$

$$x(t) = x_1(t) \int^t u(s) ds.$$

Linear systems

- **Example:**

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$$\frac{d^2x}{dt^2} - 2t \frac{dx}{dt} - 2x = 0.$$

- $x_1(t) = e^{t^2}$: solution.

- $x(t) = e^{t^2} v(t)$:

$$\frac{d^2v}{dt^2} + 2t \frac{dv}{dt} = 0.$$

- Solution:

$$\frac{dv}{dt} = e^{-t^2},$$

- \Rightarrow

$$v(t) = \int_0^t e^{-s^2} ds = \frac{\sqrt{\pi}}{2} \operatorname{erf}(t),$$

- erf: **Gauss** error function.

- \Rightarrow

$$x_2(t) = e^{t^2} \operatorname{erf}(t).$$

Linear systems

- Linearization and stability for autonomous systems:
 - Linear systems;
 - Nonlinear systems.

Linear systems

- **Linear systems**
- $A \in \mathbb{M}_d(\mathbb{R})$: independent of t .
- Linear system of ODEs:

$$\begin{cases} \frac{dx}{dt} = Ax(t), & t \in [0, +\infty[, \\ x(0) = x_0 \in \mathbb{R}^d. \end{cases}$$

- There exists $C \in GL_d(\mathbb{C})$ s.t.

$$C^{-1}AC = D + N,$$

where D is diagonal, N is nilpotent, and $ND = DN$.

- $\lambda_j, j = 1, \dots, J$: (distinct) **eigenvalues** of A .
- m_j : multiplicity of λ_j ; $E_j = \ker(A - \lambda_j I)^{m_j}$: **characteristic subspace** associated with λ_j .
- $\bigoplus E_j = \mathbb{C}^d$.

Linear systems

- **DEFINITION:**
- Linear system: **stable** if there exists a positive constant C_0 s.t.

$$|x(t)| \leq C_0|x_0| \quad \text{for all } t \in [0, +\infty[.$$

- **LEMMA:**
- Linear system: **stable** iff $\Re\lambda_j < 0$ or $\Re\lambda_j = 0$ and $N|_{E_j} = 0$ for $j = 1, \dots, J$.

Linear systems

- **PROOF:**

- $\tilde{x}(t) = C^{-1}x(t)$ and $\tilde{x}_0 = C^{-1}x_0$.
- $\tilde{x}(t) = e^{tD+tN}\tilde{x}_0, \quad t \in [0, +\infty[.$
- $DN = ND \Rightarrow$

$$\tilde{x}(t) = \left(\sum_{i=0}^{d-1} \frac{(tN)^i}{i!} \right) e^{tD}\tilde{x}_0, \quad t \in [0, +\infty[.$$

- \tilde{x}_0 belongs to the vector eigenspace associated with the eigenvalue $\lambda_j \Rightarrow$

$$\tilde{x}(t) = e^{t\lambda_j} \left(\sum_{i=0}^{d-1} \frac{(tN)^i}{i!} \right) \tilde{x}_0, \quad t \in [0, +\infty[.$$

- $\Rightarrow x(t)$ satisfies the **stability estimate** for some positive constant C_0 iff $\Re\lambda_j < 0$ or $\Re\lambda_j = 0$ and $N|_{E_j} = 0$.

Linear systems

- **Nonlinear systems**
- **Autonomous** system: $f \in \mathcal{C}^1$,

$$\begin{cases} \frac{dx}{dt} = f(x), \\ x(0) = x_0 \in \mathbb{R}^d, \end{cases}$$

- $f(x^*) = 0$: x^* : an **equilibrium point**.

Linear systems

- **THEOREM: Local stability**
 - Suppose that **all the eigenvalues** λ of the **Jacobian** $f'(x^*)$ of f **at x^* : negative real parts.**
 - There exists $\delta > 0$ s.t. if $|x_0 - x^*| \leq \delta$, then

$$|x(t) - x^*| \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Linear systems

- **PROOF:**
 - **Linearized system:** $A = f'(x^*)$

$$\begin{cases} \frac{dy(t)}{dt} = Ay(t), & t \geq 0, \\ y(0) = x_0 - x^*. \end{cases}$$

- **Explicit** solution $y(t) = e^{tA}(x_0 - x^*)$ for $t \geq 0$.
- $\Re \lambda < 0$ for any eigenvalue λ of $f'(x^*)$: negative real parts.
- There exists $r > 0$ s.t.

$$|e^{tA}z| \leq C_0 e^{-rt}|z| \quad \text{for all } z \in \mathbb{R}^d,$$

C_0 : depends only on f .

Linear systems

- Small perturbation of the linearized system:

$$\begin{cases} \frac{dx}{dt} = A(x - x^*) + g(x), \\ x(0) = x^*, \end{cases}$$

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$$g(x) = |x - x^*|\epsilon(x), \quad \text{with } \epsilon \in C^0 \quad \text{and } \epsilon(x^*) = 0.$$

Linear systems

- There exists $\delta_0 > 0$ s.t. for all $\delta \in]0, \delta_0[$,

$$\sup\{|g(x)| : |x - x^*| \leq \delta\} < \frac{r\delta}{C_0}.$$

- It suffices to prove that if $|x_0 - x^*| < \min(\delta, \delta/C_0)$, then

$$|x(t) - x^*| \leq \delta \quad \text{for all } t \geq 0.$$

-

$$x(t) - x^* = e^{tA}(x_0 - x^*) + \int_0^t e^{(t-s)A} g(x(s)) ds,$$

- \Rightarrow

$$\begin{aligned} |x(t) - x^*| &\leq e^{-rt} C_0 |x_0 - x^*| + \int_0^t e^{-r(t-s)} C_0 |g(x(s))| ds \\ &\leq e^{-rt} C_0 |x_0 - x^*| + (1 - e^{-rt}) \frac{C_0}{r} \sup\{|g(x(s))| : 0 \leq s \leq t\}. \end{aligned}$$

Linear systems

- For all $t \geq 0$,

$$|x(t) - x^*| \leq \max \left(C_0 |x_0 - x^*|, \frac{C_0}{r} \sup\{|g(x(s))| : 0 \leq s \leq t\} \right).$$

- Introduce

$$T := \inf\{t > 0 : |x(t) - x^*| \geq \delta\}.$$

- Assume that T : finite \Rightarrow

$$|x(t) - x^*| \leq \delta \quad \text{for all } t \in [0, T], \quad |x(T) - x^*| = \delta.$$

- \Rightarrow Contradiction.

Linear systems

- **DEFINITION:**

- A function $V \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$: **Lyapunov function** for the ODE if

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$$V(x^*) < V(x) \quad \text{for any } x \neq x^*;$$

-

$$f(x) \cdot V'(x) \leq 0 \quad \text{for any } x \in \mathbb{R}^d.$$

Linear systems

- **EXAMPLE:**

(i) Consider

$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = -2x_1 - x_2. \end{cases}$$

- $x^* = (0, 0)$: **equilibrium point** and

$$V(x) = x_1^2 + \frac{1}{2}x_2^2 : \quad \text{Lyapunov function.}$$

Linear systems

- (ii) Suppose that $f(x) = -\nabla\Phi(x)$. Suppose that the potential Φ : smooth and there exists x^* s.t. $\Phi(x^*) < \Phi(x)$ for any $x \neq x^*$. Then

$V = \Phi$: Lyapunov function.

Linear systems

- **THEOREM:**
 - Suppose that there exists a **Lyapunov function** V .
 - \Rightarrow For any $\epsilon > 0$, there exists $\delta > 0$, s.t.

$$\sup_{t \geq 0} |x(t) - x^*| \leq \epsilon$$

provided that $|x_0 - x^*| \leq \delta$.

Linear systems

- **PROOF:**

- Condition on V implies that for fixed $\epsilon > 0$, there exists $\gamma > 0$ (sufficiently small) s.t.

$$\{x : |x - x^*| \leq 2\epsilon, V(x) \leq V(x^*) + \gamma\} \subset \{x : |x - x^*| \leq \epsilon\}.$$

- Choose δ ($0 < \delta < \epsilon$) s.t.

$$\{x : |x - x^*| \leq \delta\} \subset \{x : |x - x^*| \leq 2\epsilon, V(x) \leq V(x^*) + \gamma\}.$$

Linear systems

- Fundamental property of a Lyapunov function V :

$$\frac{d}{dt} V(x(t)) = f(x(t)) \cdot V'(x(t)) \leq 0, \quad t \geq 0;$$

- \Rightarrow

$$V(x(t)) \leq V(x_0) \leq V(x^*) + \gamma \quad \text{if } |x_0 - x^*| \leq \delta.$$

-

$$|x(s) - x^*| \leq 2\epsilon \quad \text{for all } s \geq 0,$$

since otherwise, there would exist $t > 0$ s.t. $|x(t) - x^*| = 2\epsilon$.

- $V(x(t)) \leq V(x^*) + \gamma \Rightarrow$ contradiction.

Linear systems

- **THEOREM: Global stability**

- Suppose that there exists $V \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ satisfying

$$V(x^*) < V(x) \quad \text{for any } x \neq x^*$$

s.t.

$$f(x) \cdot V'(x) < 0 \quad \text{for any } x \neq x^*.$$

- Suppose that the set $\{x : V(x) \leq V(x_0)\}$: bounded.
- \Rightarrow The solution $x(t)$ converges to x^* as $t \rightarrow +\infty$.

Linear systems

- **PROOF:**

- $V(x(t)) \leq V(x_0) \Rightarrow \{x(t) : t \geq 0\}$: bounded.

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$$\begin{aligned} & \int_0^{+\infty} |f(x(t)) \cdot V'(x(t))| dt \\ &= \int_0^{+\infty} -f(x(t)) \cdot V'(x(t)) dt \leq V(x_0) - V^*; \end{aligned}$$

$$V^* := \lim_{t \rightarrow +\infty} V(x(t)).$$

- $(x(t))_{t \geq 0}$: bounded $\Rightarrow V^* > -\infty$.

Linear systems

- $(t_n)_{n \in \mathbb{N}}$ s.t. $x(t_n) \rightarrow \tilde{x}$ and $f(x(t_n)) \cdot V'(x(t_n)) \rightarrow 0$ as $n \rightarrow +\infty$.

- \Rightarrow

$$f(\tilde{x}) \cdot V'(\tilde{x}) = 0;$$

- $\Rightarrow \tilde{x} = x^*$.