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Numerical Analysis II

ETH Zürich
D-MATH

Exam Summer 2022

Last Name		Note
First Name		
Degree Programme		
Legi Number		
Date	16.08.2022	

1	2	Marks

- First fill out the cover sheet and place your Legi on the edge of the desk.
- Begin each problem on a separate sheet of paper. Please write out the problem ID in a striking font.
- Every sheet must bear your name and Legi number.
- Write with neither red nor green pens nor with a pencil.
- Please write out your ideas clearly and show your reasoning rigorously.
- You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Good luck!

Problem 1 **[60 pts.]**

Consider

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), \ \ t \in [0,T], \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

with $f \in C^\infty([0,T] \times \mathbb{R})$ satisfying the Lipschitz condition

$$\Big| f(t,x) - f(t,y) \Big| \leq C_f |x-y|, \ \forall x,y \in \mathbb{R}, \ \forall t \in [0,T],$$

for some positive constant C_f.

(1a)

(i) Does (1.1) have a unique solution $x(t) \in C^\infty([0,T])$? You should justify your answer.

(ii) If we also regard $x(t)$ as a function of the initial value x_0, what is the equation satisfied by the derivative with respect to t of $\frac{\partial x(t)}{\partial x_0}$? Is it a linear differential equation? You should justify your answer. Check that its solution in terms of $\frac{\partial f}{\partial x}(t, x(t))$ is given by $\frac{\partial x}{\partial x_0}(t) = e^{\int_0^t \frac{\partial f}{\partial x}(y, x(y))\mathrm{d}y}$.

(1b) Consider the numerical scheme

$$x^{k+1} = x^k + \frac{\Delta t}{2}\Big[f(t_k, x^k) + f\big(t_{k+1}, x^k + \theta \Delta t f(t_k, x^k)\big)\Big], \tag{1.2}$$

where $\Delta t > 0$ is small enough, $\frac{T}{\Delta t}$ is an integer and $t_k = k\Delta t$, for $k \in \mathbb{N}$. Here, θ is a positive fixed real parameter.

(i) The scheme (1.2) is an

explicit one-step method. ☐ explicit two-step method. ☐

implicit one-step method. ☐ implicit two-step method. ☐

(ii) Let $\phi(t,x,h)$ be defined by

$$\phi(t,x,h) = \frac{1}{2}\Big[f(t,x) + f(t+h, x + \theta h f(t,x))\Big]$$

so that (1.2) can be rewritten in the form

$$x^{k+1} = x^k + \Delta t \phi(t_k, x^k, \Delta t).$$

Prove, from the definition of consistency, that (1.2) is consistent with (1.1).

(iii) Define the truncation error by

$$T_k(\Delta t) = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} - \phi(t_k, x(t_k), \Delta t),$$

where $x(t)$ is the solution of (1.1). Prove, using Taylor's theorem, that (1.2) is of order two as $\Delta t \to 0$ if and only if $\theta = 1$.

(iv) Prove that (1.2) is stable, i.e., there exist positive constants h_0 and C_ϕ, such that

$$\left|\phi(t, x, h) - \phi(t, y, h)\right| \le C_\phi |x - y|,$$

for all $t \in [0, T]$ and for all $x, y \in \mathbb{R}$ and $h \in [0, h_0]$.

(v) Is (1.2) for solving (1.1) convergent? What is the order in Δt of the global error $e_k = x^k - x(t_k)$ in terms of θ? You should justify your answer.

(1c) Consider the numerical scheme

$$x^{k+1} = x^k + \Delta t f(t_k + \theta \Delta t, (1 - \theta)x^k + \theta x^{k+1}), \qquad (1.3)$$

where θ is a positive fixed real parameter.

(i) The scheme (1.3) is an

explicit one-step method. ☐	explicit two-step method. ☐
implicit one-step method. ☐	implicit two-step method. ☐

(ii) Let $\phi(t, x, h)$ be defined implicitly by

$$\phi(t, x, h) = f(t + \theta h, x + \theta h \phi(t, x, h))$$

so that (1.3) can be rewritten in the form

$$x^{k+1} = x^k + \Delta t \phi(t_k, x^k, \Delta t).$$

Prove that (1.3) is consistent with (1.1).

(iii) Prove that (1.3) is of order two if and only if $\theta = \frac{1}{2}$.

(iv) Prove that (1.3) is convergent provided that

$$\Delta t < \frac{1}{C_f \theta}.$$

(1d) Suppose that (1.2) with $\theta = 1$ and (1.3) with $\theta = \frac{1}{2}$ are applied to the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \sin(t) + \lambda x(t), \quad t \in [0, 1], \\ x(0) = 0, \end{cases} \tag{1.4}$$

where λ is a positive real parameter.

(i) Implement (1.2) with $\theta = 1$ and (1.3) with $\theta = \frac{1}{2}$ for solving (1.4) for different values of Δt and λ. You should use the templates provided in `1(d).py`. For the (1.3) part, you should use the provided functions `EulerStep`, `ImpEulerStep` and `ImpMidPointSolveWithEuler`.

(ii) Illustrate the convergence properties.

Problem 2 **[40 pts.]**

Consider the system of equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x,y), \quad t \in [0,T], \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x,y), \quad t \in [0,T], \end{cases} \tag{2.1}$$

with the initial conditions $x(0) = x_0$ and $y(0) = y_0$. Here, $f, g : \mathbb{R}^2 \to \mathbb{R}$ are C^1 functions. Throughout this problem, we assume that there exists a unique pair (x, y) solution of (2.1) for given initial conditions.

(2a)

(i) Use the integrability lemma (Lemma 1.27 in the lecture notes) to prove that (2.1) is locally Hamiltonian if and only if

$$\frac{\partial g}{\partial y} = -\frac{\partial f}{\partial x}.$$

(ii) Is the system given by

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u(v-2) \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v(1-u) \end{cases} \tag{2.2}$$

Hamiltonian? You should justify your answer.

(iii) Use the transformation $x = \ln(u)$ and $y = \ln(v)$ in (2.2) to verify that the resulting system in x, y is Hamiltonian with the Hamiltonian function H given by

$$H(x,y) = x - e^x + 2y - e^y.$$

(iv) Deduce from (iii) that

$$F(u,v) := \ln(u) - u + 2\ln(v) - v$$

is an invariant of (2.2).

(2b) Consider the numerical scheme, for solving (2.1),

$$\begin{cases} x^{k+\frac{1}{2}} = x^k + \dfrac{\Delta t}{2} f(x^k, y^k), \\ x^{k+1} = x^{k+\frac{1}{2}} + \dfrac{\Delta t}{2} f(x^{k+\frac{1}{2}}, y^k), \\ y^{k+1} = y^k + \Delta t g(x^k, y^k), \end{cases} \tag{2.3}$$

where Δt is small enough, $\frac{T}{\Delta t}$ is an integer and $t_k = k\Delta t$ for $k \in \mathbb{N}$.

(i) Verify that (2.3) reduces to

$$\begin{cases} x^{k+1} = x^k + \dfrac{\Delta t}{2} f(x^k, y^k) + \dfrac{\Delta t}{2} f\Big(x^k + \dfrac{\Delta t}{2} f(x^k, y^k), y^k\Big), \\ y^{k+1} = y^k + \Delta t g(x^k, y^k). \end{cases}$$

(ii) Compute the order of the truncation errors $T_k^{(x)}$ and $T_k^{(y)}$ as $\Delta t \to 0$, where $T_k^{(x)}$ and $T_k^{(y)}$ are given by:

$$T_k^{(x)}(\Delta t) = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} - \frac{1}{2}\Big[f\Big(x(t_k), y(t_k)\Big) + \\ + f\Big(x(t_k) + \frac{\Delta t}{2} f(x(t_k), y(t_k)), y(t_k)\Big)\Big]$$

and

$$T_k^{(y)}(\Delta t) = \frac{y(t_{k+1}) - y(t_k)}{\Delta t} - g\Big(x(t_k), y(t_k)\Big).$$

Here, the pair $(x(t), y(t))$ is the solution to (2.1).

(iii) Consider (2.1) with

$$f(x, y) = -x + y \quad \text{and} \quad g(x, y) = -y.$$

Prove that

$$\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = \begin{bmatrix} \Big(1 - \frac{\Delta t}{2}\Big)^2 & \Delta t\Big(1 - \frac{\Delta t}{4}\Big) \\ 0 & 1 - \Delta t \end{bmatrix} \begin{pmatrix} x^k \\ y^k \end{pmatrix}.$$

Is (2.3) for solving (2.1) convergent in this case? You should justify your answer.

(2c) Consider (2.1) with

$$f(x,y) = -y \quad \text{and} \quad g(x,y) = x.$$

(i) Is (2.1) a Hamiltonian system in this case? What is the associated Hamiltonian function?

(ii) Prove that in this case

$$\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = \begin{bmatrix} 1 & -\Delta t \\ \Delta t & 1 \end{bmatrix} \begin{pmatrix} x^k \\ y^k \end{pmatrix}. \tag{2.4}$$

Is the energy preserved by the scheme (2.4)? You should justify your answer.
Is (2.4) symplectic? You should justify your answer.
Is (2.4) symmetric? You should justify your answer.

(2d) Now, we return to the Hamiltonian system obtained in (2a)(iii). Implement (2.3), using the templates in `2(d).py`, for solving this system. Plot the graph $H(x^k, y^k)$. Is this energy preserved by (2.3)? You may use different initial data to conclude.