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Numerical Analysis II

ETH Zürich
D-MATH

Summer 2021

Last Name		Note
First Name		
Degree Programme		
Legi Number		
Date	13.08.2021	

1	2	Marks

- First fill out the cover sheet and place your Legi on the edge of the desk.
- Begin each problem on a separate sheet of paper. Please write out the problem ID in a striking font.
- Every sheet must bear your name and Legi number.
- Write with neither red nor green pens nor with a pencil.
- Please write out your ideas clearly and show your reasoning rigorously.
- You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Good luck!

Problem 1 **[52 points]**

Consider

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), \qquad t \in [0,T], \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

with $f \in C^\infty([0,T] \times \mathbb{R})$ satisfying the Lipschitz condition

$$|f(t,x) - f(t,y)| \leq C_f |x-y|, \quad \forall x, y \in \mathbb{R}, \forall t \in [0,T],$$

for some positive constant C_f.

(1a) **[2 points]** Does (1.1) have a unique solution $x(t) \in C^\infty([0,T])$? Justify.

(1b) Consider the following numerical scheme:

$$\begin{cases} x^{k+\frac{1}{2}} = x^k + \Delta t f(t_k, x^k), \\ x^{k+1} = x^k + \frac{\Delta t}{2}(f(t_k, x^k) + f(t_{k+1}, x^{k+\frac{1}{2}})), \end{cases} \tag{1.2}$$

where $\Delta t > 0$ is small enough, $N = \frac{T}{\Delta t}$ is an integer and $t_k = k\Delta t$ for $k = 0, ..., N$.

Let $\phi(t, x, \Delta t)$ be defined by

$$\phi(t,x,\Delta t) = \frac{1}{2} f(t,x) + \frac{1}{2} f(t + \Delta t, x + \Delta t f(t,x)),$$

so that (1.2) can be rewritten in the form

$$x^{k+1} = x^k + \Delta t \phi(t_k, x^k, \Delta t).$$

(i) **[2 points]** Prove, from the definition of consistency, that (1.2) is consistent with (1.1).

(ii) **[6 points]** Define the truncation error by

$$T_k(\Delta t) = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} - \phi(t_k, x(t_k), \Delta t).$$

Prove, using Taylor's theorem, that (1.2) is of order two as $\Delta t \to 0$.

(iii) **[4 points]** Prove that (1.2) is stable, i.e., there exist positive constants h_0 and C_ϕ such that

$$|\phi(t, x, \Delta t) - \phi(t, y, \Delta t)| \leq C_\phi |x - y|,$$

for all $t \in [0, T]$ and for all $x, y \in \mathbb{R}$ and $\Delta t \in [0, h_0]$.

(iv) **[2 points]** Is (1.2) for solving (1.1) convergent?

(1c) Consider the numerical scheme

$$\begin{cases} x^{k+1} = x^k + \alpha_1 \kappa_1 + \alpha_2 \kappa_2, \\ \kappa_1 = \Delta t f(t_k, x^k), \\ \kappa_2 = \Delta t f(t_k + \lambda \Delta t, x^k + \lambda \kappa_1), \end{cases} \tag{1.3}$$

where $\alpha_1, \alpha_2, \lambda \in \mathbb{R}$.

(i) **[6 points]** Denote by

$$g(h) = f(t_k + \lambda h, x(t_k) + \lambda h x'(t_k))$$

with $x(t)$ being the solution of (1.1) and $x'(t_k)$ being the derivative of x with respect to t calculated at t_k.

Prove that

$$g(0) = x'(t_k), \quad g'(0) = \lambda x''(t_k).$$

(ii) **[6 points]** Prove that if $\alpha_1 + \alpha_2 = 1$ and $\alpha_2 \lambda = \frac{1}{2}$, then (1.3) is at least of order two.

HINT: Use formulas for $g(0), g'(0)$ stated in (i).

(iii) **[2 points]** Explicit (1.3) for $\lambda = 1$.

(iv) **[8 points]** Prove that (1.3) is consistent and stable and deduce that it is convergent.

(1d) Consider the differential equation

$$\begin{cases} \dfrac{\mathrm{d}^2 x}{\mathrm{d}t} + \alpha x(t) + \beta t x^2(t) + \gamma \dfrac{\mathrm{d}x}{\mathrm{d}t} = 0, \quad t \in [0, T], \\ x(0) = 0, \quad \dfrac{\mathrm{d}x}{\mathrm{d}t}(0) = 1. \end{cases} \tag{1.4}$$

(i) **[4 points]** Explicit schemes (1.2) and (1.3) for solving (1.4).

We explicit scheme (1.3):

$$Z^{(k+1)} = Z^{(k)} + \Delta t(\alpha_1 F(t_k, Z^{(k)}) + \alpha_2 F(t_k + \lambda \Delta t, Z^{(k)}) + \lambda \Delta t F(t_k, Z^{(k)})).$$

(ii) **[10 points]** Let $\alpha = 1$, $\beta = \gamma = 0$. Write down the explicit solution.

Implement the two schemes (1.2) and (1.3) for solving (1.4). Choose in (1.3) $\alpha_1 = \frac{3}{4}$, $\alpha_2 = \frac{1}{4}$, $\lambda = 2$.

Compare the numerical solutions with the exact solution.

Problem 2 **[64 points]**

Let $\begin{pmatrix} x_0 \\ p_0 \end{pmatrix} \in \mathbb{R}^2$. Consider the second-order differential equation:

$$\begin{cases} \frac{d^2x}{dt^2}(t) + x(t) = 0, \quad t \geq 0, \\ \frac{dx}{dt}(0) = p_0, \\ x(0) = x_0. \end{cases} \tag{2.1}$$

(2a) Introduce

$$\begin{aligned} H : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,p) &\longmapsto \frac{1}{2}(x^2 + p^2). \end{aligned}$$

(i) **[2 points]** Show that (2.1) can be rewritten as

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix}(t) = \begin{pmatrix} \frac{\partial H}{\partial p}(x,p) \\ -\frac{\partial H}{\partial q}(x,p) \end{pmatrix}, \quad t \geq 0, \\ \begin{pmatrix} x \\ p \end{pmatrix}(0) = \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}. \end{cases} \tag{2.2}$$

(ii) **[2 points]** Prove that if $\begin{pmatrix} x \\ p \end{pmatrix}$ solves (2.2) then

$$H(x(t), p(t)) = H(x_0, p_0), \forall t \geq 0.$$

(2b) Let $\Delta t > 0$ be small enough and let $t_k = k\Delta t$, $k \in \mathbb{N}$.

Consider the scheme

$$\begin{pmatrix} x^{k+1} \\ p^{k+1} \end{pmatrix} = \begin{pmatrix} x^k \\ p^k \end{pmatrix} + \Delta t \begin{pmatrix} p^k \\ -x^k \end{pmatrix}. \tag{2.3}$$

(i) **[6 points]** Find explicitly $H(x^k, p^k)$.

(ii) **[4 points]** Find the limit of the norm of the vector $\begin{pmatrix} x^k \\ p^k \end{pmatrix}$ as $k \to +\infty$. Consider separately the cases $\begin{pmatrix} x^0 \\ p^0 \end{pmatrix} \neq 0$ and $\begin{pmatrix} x^0 \\ p^0 \end{pmatrix} = 0$.

(iii) **[4 points]** Consider the scheme

$$\begin{pmatrix} x^{k+1} \\ p^{k+1} \end{pmatrix} = \begin{pmatrix} x^k \\ p^k \end{pmatrix} + \Delta t \begin{pmatrix} p^{k+1} \\ -x^{k+1} \end{pmatrix}. \tag{2.4}$$

Are (2.3) and (2.4) symplectic? Justify.

(2c) Let

$$\begin{aligned} H_{\text{num}} : \mathbb{R}^2 \times \mathbb{R}_+ &\longrightarrow \mathbb{R} \\ (x, p, \Delta t) &\longmapsto \frac{1}{2}(x^2 + p^2 + \Delta t x p). \end{aligned}$$

(i) **[6 points]** Prove that

$$\left(1 - \frac{\Delta t}{2}\right) H(x,p) \leq H_{\text{num}}(x, p, \Delta t) \leq \left(1 + \frac{\Delta t}{2}\right) H(x,p).$$

(ii) **[4 points]** Justify that the scheme

$$\begin{pmatrix} x^{k+1} \\ p^{k+1} \end{pmatrix} = \begin{pmatrix} x^k \\ p^k \end{pmatrix} + \Delta t \begin{pmatrix} p^k \\ -x^{k+1} \end{pmatrix}, \quad \begin{pmatrix} x^0 \\ p^0 \end{pmatrix} = \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}, \tag{2.5}$$

is well-defined. Is (2.5) symplectic? Justify.

(iii) **[4 points]** Consider $(x^k)_{k\in\mathbb{N}}, (p^k)_{k\in\mathbb{N}}$ the sequences generated from (2.5). Prove that $\forall k \in \mathbb{N}$,

$$H_{\text{num}}(x^k, p^k, \Delta t) = H_{\text{num}}(x_0, p_0, \Delta t).$$

(iv) **[8 points]** Prove that (2.5) is convergent and is at least of order one.

(2d)

(i) **[4 points]** Write down the equations of the adjoint of (2.5).

(ii) **[6 points]** Write down the composition of (2.5) with its adjoint. Is the obtained scheme symplectic? Justify. Is the obtained scheme of order at least two? Justify.

(2e) [14 points] Implement (2.5) and the composition with its adjoint. Verify their order and that they approximately preserve the energy.