Lecture 2: Existence, uniqueness, and regularity in the Lipschitz case

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Existence, uniqueness, and regularity

- Banach fixed point theorem
- DEFINITION: Contraction Let
  - (X, d): metric space.
  - $F : X \to X$: **contraction** if there exists $0 < \lambda < 1$ s.t. for all $x, y \in X$
    $$d(F(x), F(y)) \leq \lambda d(x, y).$$
- THEOREM: Banach fixed point theorem
  - (X, d): **complete** metric space (i.e., every Cauchy sequence of elements of X: convergent);
  - $F : X \to X$: **contraction**.
  - There exists a unique $x \in X$ s.t.
    $$F(x) = x.$$

Existence, uniqueness, and regularity

- Gronwall's lemma

  LEMMA: Gronwall's lemma

  - $I = [0, T]$; $\phi \in \mathcal{C}^0(I)$.
  - There exist two constants $\alpha, \beta \in \mathbb{R}$, $\beta \geq 0$, s.t.

$$(*) \quad \phi(t) \leq \alpha + \beta \int_0^t \phi(s)\mathrm{d}s \quad \text{for all } t \in I.$$

  - $\Rightarrow$

$$\phi(t) \leq \alpha e^{\beta t} \quad \text{for all } t \in I.$$

Existence, uniqueness, and regularity

- PROOF:
  - $\varphi : I \to \mathbb{R}$

$$\varphi(t) := \alpha + \beta \int_0^t \phi(s)\mathrm{d}s.$$

  - $\phi \in \mathcal{C}^0 \Rightarrow \varphi \in \mathcal{C}^1$,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \beta\phi(t) \qquad \text{for all } t \in I.$$

  - $(*) \Rightarrow$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} \leq \beta\varphi.$$

Existence, uniqueness, and regularity

- $\psi(t) := \exp(-\beta t)\varphi(t)$ for $t \in I$,

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\beta e^{-\beta t}\varphi(t) + e^{-\beta t}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = e^{-\beta t}\left(-\beta\varphi(t) + \frac{\mathrm{d}\varphi}{\mathrm{d}t}\right) \leq 0.$$

- $\psi(0) = \varphi(0) = \alpha \Rightarrow \psi(t) \leq \alpha$ for $t \in I$,

$$\varphi(t) \leq \alpha e^{\beta t};$$

- $\Rightarrow \phi(t) \leq \varphi(t) \leq \alpha e^{\beta t}$ for all $t \in I$.

Existence, uniqueness, and regularity

- Cauchy-Lipschitz theorem
- $I = [0, T]$; d: positive integer; $f : I \times \mathbb{R}^d \to \mathbb{R}^d$.
- Suppose that $f \in \mathcal{C}^0(I \times \mathbb{R}^d)$.
- DEFINITION: Lipschitz condition
  - There exists a constant $C_f \geq 0$ s.t., for any $x_1, x_2 \in \mathbb{R}^d$ and any $t \in I$,

$$(**) \quad |f(t, x_1) - f(t, x_2)| \leq C_f |x_1 - x_2|.$$

  - f satisfies a **Lipschitz condition** on I.
  - C_f: **Lipschitz constant** for f.

Existence, uniqueness, and regularity

- THEORM: Cauchy-Lipschitz theorem
  - Consider
$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x), \quad t \in [0, T], \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^d. \end{cases}$$
  - If $f \in \mathcal{C}^0(I \times \mathbb{R}^d)$ satisfies the Lipschitz condition $(**)$ on $[0, T]$, then there exists a unique solution $x \in \mathcal{C}^1(I)$ on $[0, T]$.

Existence, uniqueness, and regularity

- PROOF:
  - $$x(t) = x_0 + \int_0^t f(s, x(s)) \mathrm{d}s, \quad \forall t \in [0, T].$$
  - Define $F : \mathcal{C}^0([0, T]; \mathbb{R}^d) \to \mathcal{C}^0([0, T]; \mathbb{R}^d)$ by
  $$F(y) := x_0 + \int_0^t f(s, y(s)) \mathrm{d}s.$$
  - For $y \in \mathcal{C}^0([0, T]; \mathbb{R}^d)$, norm of y:
  $$\|y\| := \sup_{t \in [0, T]} \{|y(t)| e^{-C_f t}\};$$
  - C_f: Lipschitz constant for f.
  - Equivalent to the usual norm $\sup_{t \in [0, T]} |y(t)| \Rightarrow \mathcal{C}^0([0, T]; \mathbb{R}^d)$ equipped with the new norm: complete.

Existence, uniqueness, and regularity

- Compute

$$\begin{aligned}
\|F[y_1] - F[y_2]\| &= \sup_{t\in[0,T]} |F[y_1](t) - F[y_2](t)| e^{-C_f t} \\
&\leq \sup_{t\in[0,T]} e^{-C_f t} \int_0^t |f(s, y_1(s)) - f(s, y_2(s))| \mathrm{d}s \\
&\leq \sup_{t\in[0,T]} e^{-C_f t} C_f \int_0^t |y_1(s) - y_2(s)| \mathrm{d}s \\
&\leq \sup_{t\in[0,T]} e^{-C_f t} C_f \int_0^t e^{C_f s} e^{-C_f s} |y_1(s) - y_2(s)| \mathrm{d}s \\
&\leq \sup_{t\in[0,T]} \{e^{-C_f t} C_f \int_0^t e^{C_f s} \mathrm{d}s\} \|y_1 - y_2\| \\
&\leq (1 - e^{-C_f T}) \|y_1 - y_2\|.
\end{aligned}$$

Existence, uniqueness, and regularity

- Banach fixed point theorem in a complete metric space $\Rightarrow$ there exists a unique $y \in \mathcal{C}^0([0,T];\mathbb{R}^d)$ s.t. $F(y) = y$.
- $\Rightarrow$ Existence and uniqueness of a solution.
- Picard iteration $y^{(n+1)} = F[y^{(n)}]$: Cauchy sequence and converges to the unique fixed point y.

Existence, uniqueness, and regularity

- REMARK:
  - Existence and uniqueness theorem: holds true if $\mathbb{R}^d$: replaced with a Banach space (a complete normed vector space).
  - Same proof.

Existence, uniqueness, and regularity

- REMARK:
  - If f: continuous, there is no guarantee that the initial value problem possesses a unique solution.
  - EXAMPLE:
    - Consider
$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^{\frac{2}{3}}, \quad x(0) = 0.$$
    - There are two solutions given by $x_1(t) = \frac{t^3}{27}$ and $x_2(t) = 0$.

Existence, uniqueness, and regularity

- THEOREM: Cauchy-Peano existence theorem
  - f: continuous.
  - There exists a solution $x(t)$: at least defined for small t.
- PROOF: Use Arzela-Ascoli theorem.

Existence, uniqueness, and regularity

- DEFINITION: Equicontinuity
  - A family of functions $\mathcal{F}$: equicontinuous on $[a, b]$ if for any given $\epsilon > 0$, there exists $\delta > 0$ s.t.

$$|f(t) - f(s)| < \epsilon$$

    whenever $|t - s| < \delta$ for every function $f \in \mathcal{F}$ and $t, s \in [a, b]$.

- DEFINITION: Uniform boundedness
  - A family of continuous functions $\mathcal{F}$ on $[a, b]$: uniformly bounded if there exists a positive number M s.t. $|f(t)| \leq M$ for every function $f \in \mathcal{F}$ and $t \in [a, b]$.

Existence, uniqueness, and regularity

- THEOREM: Arzela-Ascoli
  - Suppose that the sequence of functions $\{f_n(t)\}_{n\in\mathbb{N}}$ on $[a, b]$: uniformly bounded and equicontinuous.
  - There exists a subsequence $\{f_{n_k}(t)\}_{k\in\mathbb{N}}$: uniformly convergent on $[a, b]$.

Existence, uniqueness, and regularity

- EXAMPLE:
  - Consider
  $$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2, \quad x(0) = x_0 \neq 0.$$
  - Separation of variables $\Rightarrow$
  $$\frac{\mathrm{d}x}{x^2} = \mathrm{d}t.$$
  - $\Rightarrow$
  $$-\frac{1}{x} = \int \frac{\mathrm{d}x}{x^2} = t + C,$$
  - $\Rightarrow$
  $$x = -\frac{1}{t + C}.$$
  - $x(0) = x_0 \Rightarrow$
  $$x(t) = \frac{x_0}{1 - x_0 t}.$$

Existence, uniqueness, and regularity

- If $x_0 > 0$, $x(t)$ blows up when $t \to \frac{1}{x_0}$ from below.
- If $x_0 < 0$, the singularity: in the past $(t < 0)$.
- Only solution defined for all positive and negative t: constant solution $x(t) = 0$, corresponding to $x_0 = 0$.

Existence, uniqueness, and regularity

- Local existence and uniqueness theorem: If $f(t,x)$ satisfies a Lipschitz condition in a bounded domain, then a unique solution exists in a limited region.
- THEOREM:
  - Assume that f is continuous and satisfies the Lipschitz condition in the closed domain $K := \{|x| \leq k\}$ and $t \in [0,T]$,

    $$|f(t,x_1) - f(t,x_2)| \leq C_f |x_1 - x_2|, \quad \text{for all } x, y \in K, t \in [0,T].$$

  - Then

    $$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), \quad t \in [0,T], \\ x(0) = x_0, \quad x_0 \in K, \end{cases}$$

    has a unique solution in $t \in [0, \min\{T, \frac{k}{M}\}]$, where

    $$M := \sup_{x \in K, t \in [0,T]} |f(t,x)|.$$

Existence, uniqueness, and regularity

- Continuity of the solution.
- THEOREM:
  - f satisfies the Lipschitz condition.
  - $x_1(t)$ and $x_2(t)$: two solutions of corresponding to the initial data $x_1(0)$ and $x_2(0)$, respectively.
  - Continuity with respect to the initial data:

$$|x_1(t) - x_2(t)| \leq e^{C_f t} |x_1(0) - x_2(0)| \quad \text{for all } t \in [0, T].$$

Existence, uniqueness, and regularity

- PROOF:
  - $$\frac{\mathrm{d}}{\mathrm{d}t}|x_1(t) - x_2(t)|^2 = 2(f(t, x_1(t)) - f(t, x_2(t)))(x_1(t) - x_2(t)) \leq 2C_f|x_1(t) - x_2(t)|^2, \quad t \in [0, T],$$
  - $\Rightarrow$ $$\frac{\mathrm{d}}{\mathrm{d}t}\left(|x_1(t) - x_2(t)|^2 e^{-2C_f t}\right) \leq 0.$$
  - Integration from 0 to t: $$|x_1(t) - x_2(t)|^2 e^{-2C_f t} \leq |x_1(0) - x_2(0)|^2.$$
  - $\Rightarrow$ $$|x_1(t) - x_2(t)| \leq |x_1(0) - x_2(0)| e^{C_f t}.$$

Existence, uniqueness, and regularity

- Differentiability with respect to the initial data.
- Formal: differentiate the solution x with respect to the initial data $\Rightarrow$

$$(***) \quad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \dfrac{\partial x(t)}{\partial x_0} = \dfrac{\partial f}{\partial x}(t, x(t)) \dfrac{\partial x(t)}{\partial x_0}, \\ \dfrac{\partial x(t)}{\partial x_0} = 1. \end{cases}$$

- THEOREM:
  - $f \in \mathcal{C}^1$.
  - $x_0 \mapsto x(t)$: differentiable and $\partial x(t)/\partial x_0$: unique solution of the linear equation $(***)$.

Existence, uniqueness, and regularity

- PROOF:
  - $\Delta x(t, x_0, h) := x(t, x_0 + h) - x(t, x_0)$: difference quotient.
  - Mean-value theorem $\Rightarrow$

$$\Delta x(t, x_0, h) = h + \int_0^t (f(s, x(s, x_0 + h)) - f(s, x(s, x_0)))\mathrm{d}s$$

$$= h + \int_0^t (f(s, x(t, x_0) + \Delta x(s, x_0, h)) - f(s, x(s, x_0)))\mathrm{d}s$$

$$= h + \int_0^t \frac{\partial f}{\partial x}(s, x(s, x_0) + \tau \Delta x)\Delta x \mathrm{d}s.$$

  - $\tau = \tau(s, x_0, h) \in [0, 1]$.

Existence, uniqueness, and regularity

- There exists a positive constant M s.t. $|\frac{\partial f}{\partial x}| < M \Rightarrow$

$$|\Delta x| \leq |h| + M \int_0^t |\Delta x(s, x_0, h)| \mathrm{d}s.$$

- Gronwall's lemma $\Rightarrow$

$$|\Delta x(t, x_0, h)| \leq |h| e^{Mt}.$$

Existence, uniqueness, and regularity

- $v(t)$: unique solution of $(***)$.
- Compute

$$
\begin{aligned}
&\frac{\Delta x(t, x_0, h)}{h} - v(t) \\
&= \int_0^t \left(\frac{f(s, x(s, x_0 + h)) - f(s, x(s, x_0))}{h} - \frac{\partial f}{\partial x}(s, x(s, x_0)) v(s) \right) \mathrm{d}s \\
&= \int_0^t \frac{\Delta x(s, x_0, h)}{h} \left[\frac{\partial f}{\partial x}(s, x(s, x_0) + \tau \Delta x(s, x_0, h)) - \frac{\partial f}{\partial x}(s, x(s, x_0)) \right] \mathrm{d}s \\
&\quad + \int_0^t \frac{\partial f}{\partial x}(s, x(s, x_0)) \left(\frac{\Delta x(s, x_0, h)}{h} - v(s) \right) \mathrm{d}s.
\end{aligned}
$$

Existence, uniqueness, and regularity

- Uniform continuity of $\frac{\partial f}{\partial x}$ $\Rightarrow$ For any $\epsilon > 0$ there exists $h_0 > 0$ s.t., for any $|h| \leq h_0$, the first term on the right-hand side: of order $O(\epsilon)$.
- Gronwall's lemma $\Rightarrow$ for $|h|$ small enough,

$$|\frac{\Delta x(t, x_0, h)}{h} - v| \leq \epsilon M T e^{MT}.$$

- $\Rightarrow x_0 \mapsto x(t)$: differentiable and its derivative given by

$$\frac{\partial x}{\partial x_0} = v.$$

Existence, uniqueness, and regularity

- Stability

  THEOREM:

  - Two ODEs on $[0, T]$:

  $$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t,x) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g(t,y).$$

  - f satisfies the Lipschitz condition on $[0, T]$ and there exists $\epsilon > 0$ s.t., for any $x \in \mathbb{R}^d$, $t \in [0, T]$,

  $$|f(t,x) - g(t,x)| \leq \epsilon.$$

  - Strong continuity:

  $$|x(t) - y(t)| \leq |x(0) - y(0)|e^{C_f t} + \frac{\epsilon}{C_f}(e^{C_f t} - 1), \quad t \in [0, T].$$

Existence, uniqueness, and regularity

- REMARK: g may not satisfy a Lipschitz condition.
- PROOF:
  - Compute:

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t) - y(t)|^2 = 2(f(t, x(t)) - g(t, y(t)))(x(t) - y(t))$$

$$= 2(f(t, x(t)) - f(t, y(t)))(x(t) - y(t))$$

$$+2(f(t, y(t)) - g(t, y(t)))(x(t) - y(t)).$$

Existence, uniqueness, and regularity

- $\Rightarrow$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|x(t)-y(t)|^2 &\leq \left|\frac{\mathrm{d}}{\mathrm{d}t}|x(t)-y(t)|^2\right| \\
&\leq 2|f(t,x(t))-f(t,y(t))|\,|x(t)-y(t)| \\
&\quad +2|f(t,y(t))-g(t,y(t))|\,|x(t)-y(t)| \\
&\leq 2C_f\,|x(t)-y(t)|^2+2\epsilon|x(t)-y(t)| \\
&\leq 2C_f\,|x(t)-y(t)|^2+2\epsilon\sqrt{|x(t)-y(t)|^2}.
\end{aligned}
$$

Existence, uniqueness, and regularity

- $h(t) := |x(t) - y(t)|^2$:

$$\frac{\mathrm{d}h}{\mathrm{d}t} \leq 2C_f h + 2\epsilon\sqrt{h}.$$

- Consider

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = 2C_f u + 2\epsilon\sqrt{u}, \\ u(0) = |x(0) - y(0)|^2. \end{cases}$$

- $C_f > 0$, $u(0) \geq 0 \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}t}$: always non-negative when $t \geq 0$;
- $\Rightarrow u$: increasing.

Existence, uniqueness, and regularity

- Let $z(t) := \sqrt{u(t)}$ and suppose that $h(0) > 0$.
- $$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} - C_f z = \epsilon, \quad t \in [0, T], \\ z(0) = \sqrt{u(0)}. \end{cases}$$
- $\Rightarrow$
$$\sqrt{u(t)} = z(t) = \sqrt{u(0)}e^{C_f t} + \frac{\epsilon}{C_f}(e^{C_f t} - 1).$$

Existence, uniqueness, and regularity

- Contradiction: There exists $t_1 \in [0, T]$ s.t. $h(t_1) > u(t_1)$.
- $t_0 := \sup\{t : h(t) \leq u(t)\}$.
- Continuity: $h(t_0) = u(t_0)$.

Existence, uniqueness, and regularity

- Prove:
$$\frac{\mathrm{d}}{\mathrm{d}t}\left((h(t)-u(t))e^{-(2C_f+\frac{2\epsilon}{\sqrt{u(0)}})t}\right) \leq 0.$$

- $\Rightarrow$ Integration from t_0 to t $\Rightarrow$
$$(h(t)-u(t))e^{-(2C_f+\frac{2\epsilon}{\sqrt{u(0)}})t} \leq (h(t_0)-u(t_0))e^{-(2C_f+\frac{2\epsilon}{\sqrt{u(0)}})t_0}.$$

- $u(t_0)=h(t_0) \Rightarrow$
$$h(t) \leq u(t) \quad \text{for } t \in [t_0, t_1].$$

- Contradiction.

Existence, uniqueness, and regularity

- $$\frac{\mathrm{d}}{\mathrm{d}t}(h(t)-u(t)) \leq 2C_f(h(t)-u(t)) + 2\epsilon(\sqrt{h(t)}-\sqrt{u(t)})$$
$$= 2C_f(h(t)-u(t)) + 2\epsilon\frac{h(t)-u(t)}{\sqrt{h(t)}+\sqrt{u(t)}}.$$

- $t \geq t_0 \Rightarrow$
$$\frac{\mathrm{d}}{\mathrm{d}t}(h(t)-u(t)) \leq 2(h(t)-u(t))\big(C_f + \frac{\epsilon}{\sqrt{u(0)}}\big).$$

- $\Rightarrow$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left((h(t)-u(t))e^{-(2C_f+\frac{2\epsilon}{\sqrt{u(0)}})t}\right)$$
$$\leq 0.$$

Existence, uniqueness, and regularity

- $h(t) \leq u(t)$ for $t \in [0, T] \Rightarrow$

$$\begin{aligned} |x(t) - y(t)| &\leq \sqrt{u(t)} \\ &= \sqrt{u(0)} e^{C_f t} + \frac{\epsilon}{C_f}(e^{C_f t} - 1) \\ &= \sqrt{h(0)} e^{C_f t} + \frac{\epsilon}{C_f}(e^{C_f t} - 1). \end{aligned}$$

- $\Rightarrow$

$$|x(t) - y(t)| \leq |x(0) - y(0)| e^{C_f t} + \frac{\epsilon}{C_f}(e^{C_f t} - 1).$$

Existence, uniqueness, and regularity

- If $h(0) = 0$:

$$\begin{cases} \dfrac{\mathrm{d}u_n}{\mathrm{d}t} = 2C_f u_n + 2\epsilon\sqrt{u_n}, \quad t \in [0, T], \\ u_n(0) = \dfrac{1}{n}, \end{cases}$$

- Explicit solution:

$$u_n(t) = \left[\frac{1}{\sqrt{n}} e^{C_f t} + \frac{\epsilon}{C_f}(e^{C_f t} - 1)\right]^2.$$

Existence, uniqueness, and regularity

- Only need to prove that for each $n \in \mathbb{N}$,

$$h(t) \leq u_n(t) \quad \text{for all } t \in [0, T].$$

- Letting $n \to +\infty$, $u_n \to u \Rightarrow h(t) \leq u(t)$.

Existence, uniqueness, and regularity

- Proof by contradiction:
  - Suppose that there exists $t_1 > 0$ s.t. $h(t_1) > u_n(t_1)$.
  - t_0: the largest t in $0 < t \leq t_1$ s.t. $h(t_0) \leq u_n(t_0)$.
  - Continuity of $h(t)$ and $u_n(t) \Rightarrow$

$$h(t_0) = u_n(t_0) > 0,$$

    and $h(t) > u_n(t)$ on $(t_0, t_0 + \epsilon)$, a small right-neighborhood of t_0.
  - Impossible according to the discussion in the case $h(0) > 0$.

Existence, uniqueness, and regularity

- Regularity

  THEOREM:

  - $f \in \mathcal{C}^n$ for $n \geq 0$.
  - $\Rightarrow x \in \mathcal{C}^{n+1}$.

- PROOF:

  - Proof by induction.
  - Case $n = 0$: clear.
  - If $f \in \mathcal{C}^n$ then x: at least of class $\mathcal{C}^n$, by the inductive assumption.
  - The function $t \mapsto f(t, x(t)) = dx(t)/dt \in \mathcal{C}^n$.
  - $\Rightarrow x(t) \in \mathcal{C}^{n+1}$.

- REMARK:

  - f: real analytic function $\Rightarrow x$: real analytic.