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Numerical Analysis II

ETH Zürich
D-MATH

Problem Sheet 9

Problem 9.1 Extrapolation of the Implicit Mid-Point Rule.

Starting with the implicit mid-point rule, we develop another single step method for the autonomous initial value problem $\dot{\mathbf{y}} = f(\mathbf{y}), \mathbf{y}(0) = \mathbf{y}_0$ by extrapolation based on two micro-steps. The right hand side f is assumed to be "sufficiently smooth".

(9.1a) Denote by $\mathbf{y}_h$, $\mathbf{y}_{h/2}$ the approximations for $\mathbf{y}(T)$ obtained by the application of the implicit mid-point rule with step sizes h and $h/2$, respectively.

Consider the extrapolated method, given by

$$\mathbf{y}_h^{ex} = -\frac{1}{3}\mathbf{y}_h + \frac{4}{3}\mathbf{y}_{h/2}.$$

Show that this method is consistent and find its order.

HINT: In this (more general) setting, we say that a method is consistent if $\mathbf{y}_h^{ex} = \mathbf{y}(h) + O(h)$ and has order $n \in \mathbb{N}$ if $\mathbf{y}_h^{ex} = \mathbf{y}(h) + O(h^n)$.

HINT: Use the following theorem concerning the asymptotic expansion of the discretization error of single-step methods:

Theorem. *Let $\mathbf{y}_h$ be the approximate value of $\mathbf{y}(T)$ obtained by an application of $N := T/h$ steps of a single-step method of order p with step size $h > 0$. Here, $\mathbf{y}(t)$ denotes the solution of the initial value problem $\dot{\mathbf{y}} = f(\mathbf{y})$, $\mathbf{y}(t_0) = \mathbf{y}_0$. Then there exist smooth functions $\mathbf{e}_i : [t_0, T] \mapsto \mathbb{R}^d$, $i = p, p+1, \ldots, p+k$, with $k \in \mathbb{N}$ determined by regularity of f, and a (for sufficiently small h) uniformly bounded function $(T, h) \mapsto \mathbf{r}_{k+p+1}(T, h)$, such that*

$$\mathbf{y}_h - \mathbf{y}(T) = \sum_{l=0}^{k} \mathbf{e}_{l+p}(T) h^{l+p} + \mathbf{r}_{k+p+1}(T, h) h^{k+p+1} \quad \text{for } h \to 0 \ .$$

with $\mathbf{r}_{k+p+1}(T, h) = \mathcal{O}(T - t_0)$ for $T - t_0 \to 0$ uniformly in $h < T$, where additionally $\mathbf{e}_l(T) = \mathcal{O}(T - t_0)$ for $T \to t_0$.

(9.1b) Give the discrete evolution of the implicit mid-point rule for the logistic differential equation

$$\dot{y} = \lambda y(1 - y), \quad \lambda > 0,$$

with initial value $y(0) = y_0 > 0$ in closed form.

HINT: The discrete evolution of the implicit mid-point rule leads to a quadratic equation which can be solved explicitely. Try to solve y_{k+1} as function of y_k out of implicit step function $y_{k+1} = g(y_k, y_{k+1})$.

(9.1c) Which of the two solutions from subproblem (9.1b) is admissible? Justify your answer. HINT: Since $y(0) = y_0 > 0$, one of the discrete evolutions from subproblem (9.1b) is illegal. Please state the reason.

(9.1d) Complete the Python template `IMPExtrapLog.py` to implement the extrapolated mid-point rule for the initial value problem from subproblem (9.1b).

Set the step size to be $h = 0.1$. Use template `Plotlogistic.py` to plot both your numerical result for the time interval $[0, 1]$ for the parameter value $\lambda = 10$ and initial value $y(0) = 0.2$ as well as the exact solution

$$y(t) = \frac{1}{1 + (y_0^{-1} - 1)e^{-\lambda t}}.$$

(9.1e) Complete the Python template `IMPExtrapLogConv.py`, to determine the convergence order of the extrapolation method for the logistic differential equation empirically. For this, use the initial value problem from subproblem (9.1d) and the step size $h = 2^{-n}$, $n = 4, 5, \ldots, 9$. Save the relevant plot in the file `conv.pdf`.

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