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# Numerical Analysis II

ETH Zürich
D-MATH

## Problem Sheet 2

### Problem 2.1 Exact differential equation

**(2.1a)** Is the following differential equation exact in $\mathbb{R}^2$?

$$2tx + 1 = -(t^2 - 2x)\frac{\mathrm{d}x}{\mathrm{d}t}$$

**(2.1b)** Solve the differential equation in (2.1a) implicitly.

**(2.1c)** Is the following differential equation exact?

$$-x\mathrm{d}t + t\mathrm{d}x = 0.$$

### Problem 2.2 Hamiltonian of the Lotka-Volterra Differential Equation

The Lotka-Volterra differential equation is defined as

$$\begin{aligned}\dot{u} &= u(v-2)\\ \dot{v} &= v(1-u).\end{aligned}$$

It is an example for a differential equation with a non-trivial first integral $I(u,v) = \ln u - u + 2\ln v - v$. A further more in-depth structural quality of this differential equation is the subject of this exercise and will hopefully shine a new light on the invariant.

**(2.2a)** What differential equation are satisfied by the functions $p = \ln u$ and $q = \ln v$, where $u$ and $v$ are solutions of (2.2)?

**(2.2b)** Show that the differential equation found in subproblem (2.2a) is Hamiltonian and give the corresponding Hamiltonian function $H(p,q)$.

HINT: Apply the transformation $p = \ln u$ and $q = \ln v$ to the invariant $I(u,v)$

### Problem 2.3 Cannon

Situated at the coordinate origin of the $(x_1, x_2)$-plane is a cannon which fires projectiles with initial velocity $v_0$ at an angle $\alpha$ with respect to the $x_1$-axis. The goal is to hit a target placed at the point $(250, 0)$. To model the trajectory of the projectile of mass $m$ we assume that it is affected by the gravitational force $mg$, acting in negative $x_2$-direction, as well as the air resistance

of magnitude $\rho v^2$, which is directed opposite to the instantaneous flight direction at all times. Newton's second law yields the autonomous ordinary differential equation of second order.

$$\ddot{\mathbf{x}} = g\begin{bmatrix} 0 \\ -1 \end{bmatrix} - \frac{\rho}{m}\|\dot{\mathbf{x}}\|\dot{\mathbf{x}} \tag{2.3.1}$$

describing the projectile's trajectory $\mathbf{x}(t)$. In addition, we obtain the initial values

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \dot{\mathbf{x}}(0) = v_0 \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}.$$

In the following, we will assume that $v_0 = 100$, $g = 10$, $m = 20$ and $\rho = 0.1$.

**(2.3a)** Convert (2.3.1) into an equivalent autonomous first order system $\dot{\mathbf{y}} = f(\mathbf{y})$. Determine the corresponding initial values for the first order system.

**(2.3b)** Implement a Python script

```
function cannon_trajectory
```

which calculates the trajectories of the projectile

- for a firing angle $\alpha = 20^\circ$ in the time interval $t \in [0, 5]$,
- for a firing angle $\alpha = 40^\circ$ in the time interval $t \in [0, 8.5]$,

using the standard integrator `integrate` in Python and which plots both solutions in a single figure.

**(2.3c)** Develop a Python function

```
function x1 = cannon_hit _fun(alpha)
```

which determines the $x_1$-coordinate of the point of impact of the projectile as a function of the firing angle $\alpha$, using the python functions `integrate`.

**(2.3d)** We wish to determine the firing angle $20^\circ < \alpha^* < 40^\circ$ for which the projectile will hit the aforementioned target. Formulate this problem in terms of a root-finding problem and solve it numerically by implementing a secant method in the python script `cannon_angle.py`.

**(2.3e)** Show that the energy of the projectile is a conserved quantity if the air resistance is negligible.

## Problem 2.4 Symplectic Transformation

**(2.4a)** Prove that a linear transformation $\boldsymbol{A} : \mathbb{R}^2 \to \mathbb{R}^2$ is symplectic, if and only if $\det \boldsymbol{A} = 1$.

We recall that $\boldsymbol{A}$ is symplectic iff

$$\boldsymbol{A}^\top \boldsymbol{J} \boldsymbol{A} = \boldsymbol{J},$$

where

$$\boldsymbol{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

**(2.4b)** Consider the transformation $(r, \phi) \mapsto (p, q)$, defined by

$$p = \psi(r) \cos \phi$$
$$q = \psi(r) \sin \phi$$

For which function $\psi(r)$ is a symplectic transformation?

We recall that a transformation $g$ is symplectic iff its Jacobian $g'$ satisfies

$$g'^{\top} \boldsymbol{J} g' = \boldsymbol{J}$$

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