H. Ammari
L. Baldassari
A. Scapin

Problem Sheet 10

Problem 10.1 Order and Stability of Multistep Method

Consider the problem

$$\begin{aligned} y'(t) &= f(t, y(t)), \quad t_0 < t < t_0 + T \\ y(t_0) &= y_0 \end{aligned}, \tag{10.1.1}$$

where $f \in C^3([t_0, t_0 + T], \mathbb{R}^m)$ satisfies the Lipschitz condition

$$\forall t, y, z : \ |f(t, y) - f(t, z)| \leq L|y - z|.$$

For $n \geq 1$ consider the following multistep scheme with constant time step $h_n = h$:

$$\begin{aligned} p_{n+1} &= y_{n-1} + 2hf(t_n, y_n) \\ y_{n+1} &= y_{n-1} + \frac{h}{3}[f(t_{n+1}, p_{n+1}) + 4f(t_n, y_n) + f(t_{n-1}, y_{n-1})] \end{aligned} \tag{10.1.2}$$

(10.1a) Study the weak stability of the scheme using the following definition. The 2-step method (10.1.2) is 'weakly' stable if there exists a constant C such that for any two sequences (y^2) and $(\tilde{y}^2)$ which have been generated by the same formulas but different initial data y^0, y^1 and $\tilde{y}^0, \tilde{y}^1$, respectively, we have $|y^2 - \tilde{y}^2| \leq C \max\{|y^0 - \tilde{y}^0|, |y^1 - \tilde{y}^1|\}$ as $\Delta t \to 0$. Note that this definition proves a weaker stability than Definition 4.22 in the lecture notes.

Solution:

Suppose we have two sets of initial values (y_0, y_1) and (y_0', y_1'), and y_2, y_2' are the results corresponding to these two initial values, then

$$y_2 - y_2' = y_0 - y_0' + \frac{h}{3}[f(t_2, p_2) - f(t_2, p_2') + 4(f(t_1, y_1) - f(t_1, y_1')) + f(t_0, y_0) - f(t_0, y_0')]$$

Notice that

$$p_2 - p_2' = (y_0 - y_0') + 2h(f(t_1, y_1) - f(t_1, y_1')),$$

so

$$\begin{aligned} |p_2 - p_2'| &\leq |y_0 - y_0'| + 2h|f(t_1, y_1) - f(t_1, y_1')| \\ &\leq |y_0 - y_0'| + 2hL|y_1 - y_1'| \end{aligned}$$

Back to main equation and taking abstract value on both sides, we have

$$
\begin{aligned}
|y_2 - y_2'| &\le |y_0 - y_0'| + \frac{h}{3}[|f(t_2, p_2) - f(t_2, p_2')| + 4|f(t_1, y_1) - f(t_1, y_1')| + |f(t_0, y_0) - f(t_0, y_0')|] \\
&\le |y_0 - y_0'| + \frac{h}{3}[L(|y_0 - y_0'| + 2hL|y_1 - y_1'|) + 4L|y_1 - y_1'| + L|y_0 - y_0'|] \\
&= (1 + \frac{2hL}{3})|y_0 - y_0'| + \frac{hL}{3}(2hL + 4)|y_1 - y_1'| \\
&\le C(h, L)\max\{|y_0 - y_0'|, |y_1 - y_1'|\},
\end{aligned}
$$

which proves the desired form of stability.

(10.1b) Prove that the order of truncation error defined by

$$
T_n(h) := \frac{1}{2h}\left[y(t_{n+1}) - y(t_{n-1}) - \frac{h}{3}[f(t_{n+1}, y(t_{n-1}) + 2hf(t_n, y(t_n))) + 4f(t_n, y(t_n)) + f(t_{n-1}, y(t_{n-1}))]\right]
$$

is at least 3.

Solution: Let

$$
\begin{aligned}
\mathrm{I} &:= y(t_{n+1}) - y(t_{n-1}), \\
\mathrm{II} &:= f(t_{n+1}, y(t_{n-1}) + 2hf(t_n, y(t_n))), \\
\mathrm{III} &:= f(t_n, y(t_n)), \\
\mathrm{IV} &:= f(t_{n-1}, y(t_{n-1})).
\end{aligned}
$$

By the same way as in Problem 7.2, Assignment 7, we have

$$
\begin{aligned}
\mathrm{I} &= \int_{t_{n-1}}^{t_{n+1}} f(t, y(t))\mathrm{d}t \\
&= \int_{t_{n-1}}^{t_{n+1}} f(t_{n-1}, y(t_{n-1})) + Df(t_{n-1}, y(t_{n-1}))(t - t_{n-1}) + D^2 f(t_{n-1}, y(t_{n-1}))\frac{(t - t_{n-1})^2}{2} + O(h^3)\mathrm{d}t \\
&= 2hf(t_{n-1}, y(t_{n-1})) + (f_t + f_y f)|_{(t_{n-1}, y(t_{n-1}))}2h^2 + \\
&\quad (f_{tt} + f_{ty}f + f_{yt}f + f_{yy}f^2 + f_y f_t + f_y^2 f)|_{(t_{n-1}, y(t_{n-1}))}\frac{4}{3}h^3 + O(h^4)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{II} &= f(t_{n-1}, y(t_{n-1})) + 2h(f_t + f(t_n, y(t_n))f_y)|_{(t_{n-1}, y(t_{n-1}))} \\
&\quad + 2h^2(f_t t + f(t_n, y(t_n))f_{yt} + f(t_n, y(t_n))f_{ty} + f^2(t_n, y(t_n))f_{yy})|_{(t_{n-1}, y(t_{n-1}))} + O(h^3) \\
&= f(t_{n-1}, y(t_{n-1})) + 2h(f_t + (f + h(f_t + ff_y))f_y|_{(t_{n-1}, y(t_{n-1}))} \\
&\quad + 2h^2(f_{tt} + ff_{yt} + ff_{ty} + f^2 f_{yy})|_{(t_{n-1}, y(t_{n-1}))} + O(h^3),
\end{aligned}
$$

the last step holds because $f(t_n, y(t_n)) = f(t_{n-1}, y(t_{n-1})) + O(h)$.

$$
\begin{aligned}
\mathrm{III} &= f(t_{n-1}, y(t_{n-1})) + (hf_t + (y(t_n) - y(t_{n-1}))f_y)|_{(t_{n-1}, y(t_{n-1}))} \\
&\quad + \frac{1}{2}(h^2 f_{tt} + h\Delta y f_{yt} + h\Delta y f_{ty} + (\Delta y)^2 f_{yy})|_{(t_{n-1}, y(t_{n-1}))} + O(h^3) \quad (\Delta y := y(t_n) - y(t_{n-1})) \\
&= f(t_{n-1}, y(t_{n-1})) + h(f_t + (f + \frac{h}{2}(f_t + ff_y)f_y|_{(t_{n-1}, y(t_{n-1}))} \\
&\quad + \frac{1}{2}(h^2 f_{tt} + h^2 ff_{yt} + h^2 ff_{ty} + h^2 f^2 f_{yy})|_{(t_{n-1}, y(t_{n-1}))} + O(h^3)))
\end{aligned}
\tag{10.1.3}
$$

the last step holds because $\Delta y = hf(t_{n-1}, y(t_{n-1})) + O(h)$.

Put everything together and compute

$$T_n(h) := \frac{1}{2h}(\mathrm{I} - \frac{h}{3}(\mathrm{II} + 4\mathrm{III} + \mathrm{IV})),$$

we find that, the $O(1)$ term, $O(h^1)$ terms and $O(h^2)$ terms get all cancelled out, so

$$T_n(h) = O(h^3),$$

which shows that the method is at least of order 3.

(10.1c) We introduce in the scheme an intermediate step:

$$\begin{aligned}
p_{n+1} &= y_{n-1} + 2hf(t_n, y_n) \\
c_{n+1} &= y_{n-1} + \frac{h}{3}[f(t_{n+1}, p_{n+1}) + 4f(t_n, y_n) + f(t_{n-1}, y_{n-1})] \\
y_{n+1} &= y_{n-1} + \frac{h}{3}[f(t_{n+1}, c_{n+1}) + 4f(t_n, y_n) + f(t_{n-1}, y_{n-1})]
\end{aligned}$$

Prove that, the order of truncation error of this method is 4.

HINT: Use the result in subproblem (b) and the fact that the error of Simpson's rule is of order 5, i.e.

$$\int_a^b f(x)\mathrm{d}x - \frac{b-a}{6}[f(a) + 4f((a+b)/2) + f(b)] = O((b-a)^5)$$

Solution: The truncation error of this method is

$$T_n'(h) := \frac{1}{2h}[y(t_{n+1}) - y(t_{n-1}) - \frac{h}{3}[f(t_{n+1}, \tilde{c}_{n+1}) + 4f(t_n, y(t_n)) + f(t_{n-1}, y(t_{n-1}))]],$$

where actually $\tilde{c}_{n+1} = y(t_{n+1}) - 2hT_n(h) = y(t_{n+1}) + O(h^4)$ by using the definition of $T_n(h)$ and result in (b). As a result, it holds that

$$\begin{aligned}
2hT_n'(h) &= y(t_{n+1}) - y(t_{n-1}) - \frac{h}{3}[f(t_{n+1}, y(t_{n+1})) + f_y(t_{n+1}, y(t_{n+1}))O(h^4) + O(h^5) \\
&\quad + 4f(t_n, y(t_n)) + f(t_{n-1}, y(t_{n-1}))] \\
&= y(t_{n+1}) - y(t_{n-1}) - \frac{h}{3}[f(t_{n+1}, y(t_{n+1})) + 4f(t_n, y(t_n)) + f(t_{n-1}, y(t_{n-1}))] + O(h^5) \\
&= O(h^5)
\end{aligned}$$

The last step is by result of Simpson's rule.

Problem 10.2 A Complex Hamiltonian Differential Equation

We will look at the complex differential equation

$$\mathrm{i}\dot{z} = \lambda z + |z|^2 z, \qquad \lambda \in \mathbb{R}. \tag{10.2.1}$$

(10.2a) Show that the function $I(z) := |z|^2$ is an invariant of the differential equation (10.2.1).

Solution: For the time-derivative of $I(z)$, it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(z) &= \frac{\mathrm{d}}{\mathrm{d}t}|z|^2 = \frac{\mathrm{d}}{\mathrm{d}t}z\overline{z} = \dot{z}\overline{z} + z\dot{\overline{z}} \\ &\overset{\text{diff. eq.}}{=} [-\mathrm{i}\lambda z - \mathrm{i}|z|^2 z]\overline{z} + z\overline{[-\mathrm{i}\lambda z - \mathrm{i}|z|^2 z]} \\ &= -\mathrm{i}\lambda|z|^2 - \mathrm{i}|z|^4 + \mathrm{i}\lambda|z|^2 + \mathrm{i}|z|^4 = 0.\end{aligned}$$

Thus $I(z)$ is an invariant.

(10.2b) Show that the differential equation (10.2.1) for $p = \mathrm{Re}(z), q = \mathrm{Im}(z)$ is equivalent to a Hamiltonian differential equation of the form

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p}.$$

Solution: We substitute $z = p + \mathrm{i}q$ in the differential equation and separate the received equation

$$\mathrm{i}(\dot{p} + \mathrm{i}\dot{q}) = -\dot{q} + \mathrm{i}\dot{p} = \lambda(p + \mathrm{i}q) + (p^2 + q^2)(p + \mathrm{i}q).$$

into its real and imaginary parts. In this way we receive

$$\begin{aligned}\dot{p} &= \lambda q + (p^2 + q^2)q, \\ -\dot{q} &= \lambda p + (p^2 + q^2)p.\end{aligned} \tag{10.2.2}$$

It remains to show that (10.2.2) is indeed a Hamiltonian differential equation. The ansatz $\frac{\partial H}{\partial q} = -\dot{p} = -\lambda q - (p^2 + q^2)q$ and integrating gives us

$$H(p, q) = -\frac{\lambda}{2}q^2 - \frac{1}{2}p^2q^2 - \frac{1}{4}q^4 + C(p),$$

where the constant $C(p)$ does not depend on q. Analogously, from $\frac{\partial H}{\partial p} = \dot{q} = -\lambda p - (p^2 + q^2)p$, we get

$$H(p, q) = -\frac{\lambda}{2}p^2 - \frac{1}{4}p^2 - \frac{1}{2}p^2q^2 + C(q)$$

With $C(q)$ independent of p. Equating and sorting of both expressions gives us

$$\frac{\lambda}{2}p^2 + \frac{1}{4}p^2 + C(p) = \frac{\lambda}{2}q^2 + \frac{1}{4}q^4 + C(q).$$

As the left side of above equation does not depend on q and the right hand side does not depend on p, both sides must be constant. We denote this constant with C and receive

$$C(p) = C - \frac{\lambda}{2}p^2 - \frac{1}{4}p^2.$$

We can substitute this above, and we get our Hamiltonian function

$$H(p, q) = -\frac{\lambda}{2}(p^2 + q^2) - \frac{1}{4}(p^2 + q^2)^2.$$

One can easily confirm that H is indeed a Hamiltonian of the system (10.2.2).

(10.2c) We will now look at the following generalisation of the above differential equation with a real, continuously differentiable function ψ:

$$\mathrm{i}\dot{z} = -\psi'(|z|^2)z.$$

Write this equation as a Hamiltonian differential equation and find an invariant.

Solution: Analogously to above, we receive, by setting $z = p + \mathrm{i}q$, substituting this in the equation and seperating real and imaginary parts,

$$\begin{aligned} \dot{p} &= -\psi'(p^2 + q^2)q, \\ -\dot{q} &= -\psi'(p^2 + q^2)p. \end{aligned}$$

One can easily convince oneself that above system is created by the Hamiltonian $H(p, q) = \frac{1}{2}\psi(p^2 + q^2)$. As we know, $H(z) = \frac{1}{2}\psi(|z|^2)$ is then an invariant of the system.

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