Problem 7.1 Error estimate for the trapezium rule method

We consider the trapezium rule method

\[ x_{n+1} = x_n + \frac{1}{2}h(f_{n+1} + f_n). \]

for the numerical solution of the initial value problem \( \frac{dx}{dt} = f(t, x) \), \( x(0) \) given, where \( f_n = f(t_n, x_n) \) and \( h = t_{n+1} - t_n \). Let us define the truncation error \( T_n \) as

\[ T_n := \frac{x(t_{n+1}) - x(t_n)}{h} - \frac{1}{2}\left( f(t_{n+1}, x(t_{n+1})) + f(t_n, x(t_n)) \right). \]

(7.1a) By integrating by parts the integral

\[ \int_{t_n}^{t_{n+1}} (t - t_{n+1})(t - t_n)x'''(t)dt, \]

show that

\[ T_n = -\frac{1}{12}h^2 x'''(\xi_n), \]

for some \( \xi_n \) in the interval \((t_n, t_{n+1})\), where \( x \) is the solution of the initial value problem.

(7.1b) Suppose that \( f \) satisfies the Lipschitz condition

\[ |f(t, x) - f(t, y)| \leq L|x - y| \]

for all real \( t, x, y \), where \( L \) is a positive constant independent of \( t \), and that \( |x'''(t)| \leq M \) for some positive constant \( M \) independent of \( t \). Show that the global error \( e_n = x(t_n) - x_n \) satisfies the inequality

\[ |e_{n+1}| \leq |e_n| + \frac{1}{2}hL(|e_{n+1}| + |e_n|) + \frac{1}{12}h^3 M. \]

(7.1c) For a uniform step \( h \) satisfying \( hL < 2 \) deduce that, if \( x_0 = x(t_0) \), then

\[ |e_n| \leq \frac{h^2 M}{12L} \left[ \left( \frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL} \right)^n - 1 \right]. \]
Problem 7.2  Truncation Error

Consider the following one-step method for the numerical solution of initial value problem \( x' = f(t, x), x(t_0) = x_0, f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \):

\[
x_{n+1} = x_n + \frac{1}{2} h (k_1 + k_2),
\]

where

\[
k_1 = f(t_n, x_n), \quad k_2 = f(t_n + h, x_n + hk_1).
\]

Show that the method is consistent and has truncation error

\[
T_n = \frac{1}{6} h^2 \left( f_x (f_t + f_x f) - \frac{1}{2} (f_{tt} + 2f_{tx} f + f_{xx} f^2) \right) + O(h^3)
\]

Problem 7.3  Roundoff Error Effects

In practical situations, computers always round off real numbers. In numerical methods rounding errors become important when the step size \( \Delta t \) is comparable with the precision of the computations. Thus, if taking rounding error into consideration, the Explicit Euler method will become the following perturbed scheme:

\[
x^{k+1} = x^k + \Delta t f(t_k, x^k) + (\Delta t) \mu^k + \rho^k,
\]

where \( \mu^k \) and \( \rho^k \) represent the errors in \( f \) and in the assembling, respectively. Assume that \( |\mu^k| \leq \mu \) and \( |\rho^k| \leq \rho \) for all \( k \) and \( f \in C^1 \). Let \( e^k := x(t_k) - x^k \), and try to prove that

\[
|e^{k+1}| \leq (1 + \Delta t C)|e^k| + \Delta t \mu + \rho + \sup_{\xi \in [t_k, t_{k+1}]} |Df(\xi)| \frac{1}{2} (\Delta t)^2,
\]

and hence

\[
|e^k| \leq e^{CT}|e^0| + \frac{\mu e^{CT}}{C} + \frac{\rho e^{CT}}{C\Delta t} + \frac{1}{2C} \sup_{\xi \in [0,T]} |Df(\xi)| e^{CT} \Delta t,
\]

where \( C_f \) is the Lipschitz constant for \( f \), and \( Df \) denotes the differentiation to \( f \) where \( f(t, x(t)) \) is regarded as a function with single parameter \( t \).

Introduce

\[
\phi(\Delta t) = \frac{\rho e^{CT}}{C\Delta t} + \frac{1}{2C} \sup_{\xi \in [0,T]} |Df(\xi)| e^{CT} \Delta t,
\]

when does \( \phi \) attain its minimum, and therefore what suggestion do you have for the minimal step size \( \Delta t \)?

Published on 03 April 2019.
To be submitted by 11 April 2019.