Problem 4.1 Discrete Gronwall Lemma

Prove the discrete Gronwall Lemma for constant $h$:

If the sequence $(\xi_k)_{k \in \mathbb{N}_0}$, $\xi_k \geq 0$ satisfies the inequality

$$
\xi_{k+1} \leq C h^{p+1} + (1 + L h) \xi_k, \quad k \in \mathbb{N}_0, \quad C, h \geq 0, \quad L > 0, \quad p \in \mathbb{N}^*
$$

then

$$
\xi_k \leq C h^p \frac{1}{L} (e^{k L h} - 1) + e^{k L h} \cdot \xi_0, \quad k \in \mathbb{N}_0.
$$

HINT: Show, by induction, that

$$
\xi_k \leq C h^p \frac{1}{L} [(1 + L h)^k - 1] + (1 + L h)^k \xi_0
$$

and use the convexity of the exponential function.

Problem 4.2 Exponential of matrices

Let $A, B$ be two $d \times d$ matrices ($d \geq 2$). Consider $x(t) \in \mathbb{R}^d$ solution to

$$
\begin{cases}
\frac{dx}{dt} = u(t)Ax(t) + (1 - u(t))Bx(t), \\
x(0) = x_0,
\end{cases}
$$

(4.2.1)

where $u : t \mapsto u(t) \in [0, 1]$ is a continuous function.

(4.2a) Prove using Cauchy-Lipschitz theorem that, for all $u$, there exists a unique solution $x$ of Eqn. (4.2.1).

(4.2b) Verify that the solution of

$$
\begin{cases}
\frac{dx}{dt} = Ax(t), \\
x(0) = x_0,
\end{cases}
$$

is given by $x(t) = e^{tA} x_0$ where $e^{tA} := \sum_{n \geq 0} \frac{(tA)^n}{n!}$.

(4.2c) Suppose that $u(t) = \chi_E(t) \in \{0, 1\}$ is the characteristic function of

$$
E = \bigcup_{n \geq 0} [t_{2n}, t_{2n+1}] \subset [0, T]
$$

where $(t_n)_{n \geq 0}$ is a strictly increasing sequence of real numbers in $[0, T]$ with $t_0 = 0$. Give an expression of $x(t)$ on each interval $[t_n, t_{n+1}]$, $n = 0, 1 \ldots$. $x(t)$ has to be continuous. Simplify this expression if $[A, B] = AB - BA = 0$ (i.e., $A$ and $B$ commute).
(4.2d) Let \( d = 2 \) and let
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

(i) Compute, for \( s, t > 0 \), \( e^{tA} \) and \( e^{sB} \).

(ii) Do \( A \) and \( B \) commute?

(iii) Verify that \( e^{tA} e^{sB} \neq e^{sB} e^{tA} \).

(iv) Verify that \( e^{tA} e^{sB} \neq e^{tA+sB} \).

Problem 4.3 Linear System

Consider
\[
\begin{align*}
\frac{dx}{dt} &= A(\delta, \mu)x(t) \quad \text{on } [0, T] \\
x(0) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{align*}
\]
where
\[
A(\delta, \mu) = \begin{pmatrix} -\delta & 1 \\ 0 & -\mu \end{pmatrix}
\]
and \( \delta, \mu \) are positive parameters.

(4.3a) Solve this problem explicitly when \( \mu = \delta \).

(4.3b) Solve it when \( \mu \neq \delta \).

(4.3c) Show explicitly that, for any fixed \( t > 0 \), if we take the limit as \( \mu \to \delta \), the two solutions become the same.

Problem 4.4 Second-order ODE

(4.4a) Consider the linear second-order ODE on \([1, 2]\) with parameter \( \beta \):
\[
\begin{align*}
t^2 \frac{d^2 x(t)}{dt^2} + t \frac{dx(t)}{dt} - \beta^2 x(t) &= 0 \\
x(t = 1) &= 1 \\
\frac{dx}{dt}(t = 1) &= 0
\end{align*}
\]
Verify that \( x(t) = \cosh(\beta \log t) \) is the solution to the IVP. Is it a continuous function of \( \beta \)? Can it be differentiated with respect to \( \beta \)?

Problem 4.5 Exponential of Matrix

This is a small problem on using MATLAB to calculate exponential and eigenvalues of matrices.
Let
\[
A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 \\ 1 & 4 \end{pmatrix}
\]
(4.5a) Use MATLAB to calculate $\exp(A)$, $\exp(B)$ and $\exp(A + B)$. Is $\exp(A) \exp(B) = \exp(A + B)$?

**HINT:** To calculate the exponential of matrix, use `expm` instead of `exp`.

(4.5b) Now let

$$C = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix},$$

use MATLAB to calculate $\exp(A) \exp(C)$ and $\exp(A + C)$. Is $\exp(A) \exp(C) = \exp(A + C)$? Can you briefly explain the reason?

(4.5c) use MATLAB to calculate the eigenvalues of $\exp(A) \exp(B)$ and $\exp(A + B)$. Do they have the same eigenvalues?

**HINT:** Use `eig` to calculate eigenvalues for matrices in MATLAB.

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