Problem 1.1 Simple Pendulum

(1.1a) A simple pendulum is one which can be considered to be a point mass suspended from a string or rod of negligible mass. Denote

- $l$ = length of the rod/string in meters
- $m$ = mass of the point mass in kilograms
- $g$ = acceleration due to gravity $= 9.81007 \text{ m/s}^2$ according to Wikipedia.

![Simple Pendulum Diagram](image)

Figure 1.1: Simple pendulum

Try to derive the differential equation that the angle $\theta$ follows as a function of time $t$.

**Solution:** Set coordinate system as shown in Figure 1.2. Let $(x(t), y(t))$ denote the position of mass point at time $t$. By Newton’s Law, the following equation system follows:

\[
\begin{align*}
F_x &= m \frac{d^2x}{dt^2} \\
F_y &= m \frac{d^2y}{dt^2}
\end{align*}
\]

and decomposition of force gives the following equation system

\[
\begin{align*}
F_x &= mg - T \cos \theta \\
F_y &= -T \sin \theta
\end{align*}
\]
Notice \((x, y)\) obeys the following geometric constraints

\[
\begin{align*}
x &= l \cos \theta \\
y &= l \sin \theta
\end{align*}
\]

Take derivative of \((x, y)\) with respect to \(t\), we have

\[
\begin{align*}
\dot{x} &= -l \sin \theta \dot{\theta} \\
\ddot{x} &= -l \cos \theta \dot{\theta}^2 - l \sin \theta \ddot{\theta} \\
\dot{y} &= l \cos \dot{\theta} \\
\ddot{y} &= -l \sin \theta \dot{\theta}^2 + l \cos \theta \ddot{\theta}
\end{align*}
\]

Combine everything together, we get

\[
-m(l \cos \theta \dot{\theta}^2 + l \sin \theta \ddot{\theta}) = mg - T \cos \theta \\
m(-l \sin \theta \dot{\theta}^2 + l \cos \theta \ddot{\theta}) = -T \sin \theta
\]

After eliminating \(T\), the equation system becomes

\[
ml \sin^2 \theta + \cos^2 \theta \ddot{\theta} = -mg \sin \theta,
\]

that is

\[
\ddot{\theta} = -\frac{g \sin \theta}{l}
\]

(1.1b) Let \(\theta = \pi/4, \dot{\theta} = 0\). Try to plot the graph of \(\theta\) using MATLAB function. MATLAB function \texttt{ode45} could be useful for you.

\textbf{Hint: } You may need to convert the second-order ODE into first-order ODE system to use \texttt{ODE45}.

\textbf{Solution:}

See \texttt{simplependulum.m}.
Listing 1.1: simplependulum.m

```matlab
function approx
l = 1;
g = 9.81007;
tspan = [0; 10];
x0 = [pi/4; 0];
fun=@(t,x)[x(2); -g*x(1)/l];
[t,y] = ode45(fun, tspan, x0);
plot(t,y(:,1))
end
```

The resulting plot is given in Figure 1.5.

Figure 1.3: Plot of \( \theta \)

(1.1c) Recall the Taylor expansion of \( \sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots \). We can substitute \( \sin \theta \) with \( \theta \) in our equation when \( \theta \) is small. Try to plot the graph for the approximated solution.

When \( \theta \approx \pi \), i.e. pendulum is almost inverted, try to apply the small angle approximation around \( \pi \). Try to plot the graph. What’s wrong?

**Solution:**

The graph for approximated solution is given as follows:
When the pendulum is almost inverted, its small angle approximation equation is:

\[ \ddot{\phi} - \frac{g \phi}{l} = 0 \]

where \( \phi = \theta - \pi \).

The graph for inverted pendulum is given as follows:

Obviously the outcome is wrong, because the direction of force \( T \) is different between \( \theta > \pi/2 \) and \( \theta < \pi/2 \).

**Problem 1.2  Mortgage payments**

(1.2a) Suppose an amount \( P \) of money, called principal, is borrowed at an interest \( I(100I\%) \) for a period of \( N \) years. Monthly payment of \( D/12 \) (\( D \) denotes the amount of annual payment) is made. How does \( D \) depend on \( P \), \( I \) and \( N \)? Try to derive an explicit formula for \( D = D(P, I, N) \).

**HINT:** Try to figure out how total debt changes in a small time period \( \Delta t \), and take \( \Delta t \) to 0 to get differential equation.

**Solution:**

Let \( y(t) \) be the amount owed at time \( t \) (measured in years), we want \( y(N) = 0 \). Within a small interval of time \( \Delta t \), the change \( \Delta y \) of total debt:

- is increased by compounding at interest \( I \); that is, \( \Delta y \) is increased by the amount of \( Iy(t)\Delta t \).

- is decreased by the amount paid back in the time interval \( \Delta t \). If \( D \) denotes this constant rate of payback, then \( D\Delta t \) is the amount paid back in the time interval \( \Delta t \).
Thus,

\[ \Delta y = I y \Delta t - D \Delta t, \]

i.e.

\[ \frac{\Delta y}{\Delta t} = I y - D. \]

Take \( \Delta t \) to 0 we get

\[ \frac{dy}{dt} = I y - D = I(y - \frac{D}{I}), \]

Let \( x = y - D/I \),

\[ \frac{dx}{dt} = I x, \]

i.e.

\[ \frac{dx}{x} = I dt \]

Integrate on both side, we get

\[ \ln x = It + C, \]

where \( C \) is a constant depending on initial value. From this equation we know that

\[ x = C \exp(It), \]

i.e.

\[ y = C \exp(It) + \frac{D}{I}. \]

We know that \( y(0) = P \), so

\[ C = P - \frac{D}{I}, \]
that is,
\[ y(t) = (P - \frac{D}{I}) \exp(It) + \frac{D}{I}. \]

At time \( N \), all debt should be paid, which means \( y(N) = 0 \), so
\[ 0 = (P - \frac{D}{I}) \exp(IN) + \frac{D}{I}, \]
so
\[ D = PI \frac{\exp IN}{\exp IN - 1}. \]

(1.2b) Suppose we are paying \( D_0 \) per year, then how long can we pay off the loan? Is there any constraint that \( D_0 \) has to satisfy?

**Solution:** This time,
\[ y(t) = (P - \frac{D_0}{I}) \exp(It) + \frac{D_0}{I}. \]
Let \( N \) to be the solution of \( y(N) = 0 \), i.e.
\[ (P - \frac{D_0}{I}) \exp(IN) + \frac{D_0}{I} = 0, \]
then
\[ N = \frac{1}{I} \ln \frac{D_0}{D_0 - IP}. \]
Notice that \( D_0 > IP \), otherwise the loan will never be paid off.

**Problem 1.3  Population dynamics**

In biological applications the population \( P \) of certain organisms at time \( t \) is sometimes assumed to obey the equation
\[ \frac{dP}{dt} = aP(1 - \frac{P}{E}) \tag{1.3.1} \]
where \( a \) and \( E \) are positive constants. This model is sometimes called the logistic growth model. \( P \) needs to be non-negative.

(1.3a) Find the equilibrium solutions, i.e. the solution that doesn’t change with \( t \).

**Solution:** Equilibrium solution \( P \) asks \( \frac{dP}{dt} = 0 \), i.e. \( P = 0 \) or \( P = E \).

(1.3b) From (1.3.1) determine the regions of \( P \) where \( P \) is increasing (decreasing) as a function of \( t \). Again using (1.3.1) find an expression for \( \frac{d^2P}{dt^2} \) in terms of \( P \) and the constants \( a \) and \( E \). From this expression find the regions of \( P \) where \( P \) is convex \( \frac{d^2P}{dt^2} > 0 \) and the regions where \( P \) is concave \( \frac{d^2P}{dt^2} < 0 \).

**Solution:** The monotone increasing region is \( 0 < P < E \), monotone decreasing region is \( P > E \).
The second order derivative of $P$ is
\[
\frac{d^2P}{dt^2} = \frac{d}{dt} \frac{dP}{dt} = \left( a - \frac{2aP}{E} \right) \frac{dP}{dt} = a^2 P \left( 1 - 2 \frac{P}{E} \right) \left( 1 - \frac{P}{E} \right)
\]

So $P$ is convex when $P > E$ or $0 < P < E/2$. $P$ is concave when $E/2 < P < E$.

(1.3c) Let $E = 100$, $a = 0.1$, $P_0 = 10$ and $P_1 = 150$. Plot the graph of $P$ as a function of $t$ on time interval $[0, 100]$ given $P_0$ and $P_1$ as initial values. What’s the behaviour of $P$ when $t$ getting bigger and bigger?

**Solution:**

See `population.m`.

Listing 1.2: `population.m`

```matlab
function population

e = 100;
a = 0.1;
x0 = 10;
x1 = 150;
tspan = [0 100];
fun = @(t,x) a*x*(1-x/e);
[t0,y0] = ode45(fun, tspan, x0);
[t1,y1] = ode45(fun, tspan, x1);
plot(t0,y0,t1,y1);
end
```

The resulting plot is given in Figure 1.6.

When $t$ gets bigger, both solution converges to 100, which is the equilibrium solution.

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Figure 1.6: Population trend of $P_0 = 10$ and $P_1 = 150$