Problem 10.1  Order and Stability of Multistep Method

Consider the problem
\[
y'(t) = f(t, y(t)), \quad t_0 < t < t_0 + T, \\
y(t_0) = y_0
\]
where \( f \in C^3([t_0, t_0 + T], \mathbb{R}^m) \) satisfies the Lipschitz condition
\[
\forall t, y, z : |f(t, y) - f(t, z)| \leq L|y - z|.
\]
For \( n \geq 1 \) consider the following multistep scheme with constant time step \( h_n = h \):
\[
p_{n+1} = y_{n-1} + 2hf(t_n, y_n) \\
y_{n+1} = y_{n-1} + \frac{h}{3}[f(t_{n+1}, p_{n+1}) + 4f(t_n, y_n) + f(t_{n-1}, y_{n-1})]
\]

\textbf{(10.1a)} Study the stability of the scheme.

\textbf{Solution:}

Suppose we have two sets of initial values \((y_0, y_1)\) and \((y'_0, y'_1)\), and \(y_2, y'_2\) are the results corresponding to these two initial values, then
\[
y_2 - y'_2 = y_0 - y'_0 + \frac{h}{3}[f(t_2, p_2) - f(t_2, p'_2) + 4(f(t_1, y_1) - f(t_1, y'_1)) + f(t_0, y_0) - f(t_0, y'_0)]
\]
Notice that
\[
p_2 - p'_2 = (y_0 - y'_0) + 2h(f(t_1, y_1) - f(t_1, y'_1)),
\]
so
\[
|p_2 - p'_2| \leq |y_0 - y'_0| + 2h|f(t_1, y_1) - f(t_1, y'_1)|
\]
Back to main equation and taking abstract value on both sides, we have
\[
|y_2 - y'_2| \leq |y_0 - y'_0| + \frac{h}{3}[|f(t_2, p_2) - f(t_2, p'_2)| + 4|f(t_1, y_1) - f(t_1, y'_1)| + |f(t_0, y_0) - f(t_0, y'_0)|]
\]
\[
\leq |y_0 - y'_0| + \frac{h}{3}L(|y_0 - y'_0| + 2hL|y_1 - y'_1|) + 4L|y_1 - y'_1| + L|y_0 - y'_0|
\]
\[
= (1 + \frac{2hL}{3})|y_0 - y'_0| + \frac{hL}{3}(2hL + 4)|y_1 - y'_1|
\]
\[
\leq C(h, L) \max\{|y_0 - y'_0|, |y_1 - y'_1|\},
\]
which proved its stability.
(10.1b) Prove that the order of truncation error defined by

\[ T_n(h) := \frac{1}{2h}[y(t_{n+1})-y(t_{n-1})-\frac{h}{3}[f(t_{n+1}, y(t_{n-1}))+2hf(t_n, y(t_n))]+4f(t_n, y(t_n))+f(t_{n-1}, y(t_{n-1}))]] \]

is at least 3.

**Solution:** Let

\[ I := y(t_{n+1}) - y(t_{n-1}), \]
\[ II := f(t_{n+1}, y(t_{n-1}))+2hf(t_n, y(t_n)), \]
\[ III := f(t_n, y(t_n)), \]
\[ IV := f(t_{n-1}, y(t_{n-1})). \]

By the same way as in Problem 7.2, Assignment 7, we have

\[
I = \int_{t_{n-1}}^{t_{n+1}} f(t, y(t))\,dt \\
= \int_{t_{n-1}}^{t_{n+1}} f(t_{n-1}, y(t_{n-1}))(t-t_{n-1}) + Df(t_{n-1}, y(t_{n-1}))(t-t_{n-1})^2 + O(h^3)\,dt \\
= 2hf(t_{n-1}, y(t_{n-1}))+\langle f_t + fyf \rangle|_{(t_{n-1}, y(t_{n-1}))}2h^2 + \\
(f_{tt} + f_{ty}f + fyf + fyf + f_{fy}f)|_{(t_{n-1}, y(t_{n-1}))} \frac{4}{3}h^3 + O(h^4)
\]

\[
II = f(t_{n-1}, y(t_{n-1}))+2hf(t_t + f_t, y(t_n))f(f_t + f_{yy})|_{(t_{n-1}, y(t_{n-1}))}2h^2 + \\
= f(t_{n-1}, y(t_{n-1}))+2hf(t_t + f_t, y(t_n))f(f_t + f_{yy})|_{(t_{n-1}, y(t_{n-1}))}2h^2 + \\
O(h^3),
\]

the last step holds because \( f(t_n, y(t_n)) = f(t_{n-1}, y(t_{n-1})) + O(h) \).

\[
III = f(t_{n-1}, y(t_{n-1}))+hf(t_t + f_t, y(t_n) - y(t_{n-1}))f|_{(t_{n-1}, y(t_{n-1}))} + \frac{1}{2}(h^2f_{tt} + h\Delta yf_{ty} + \Delta yf_{fy} + (\Delta y)^2f_{yy})|_{(t_{n-1}, y(t_{n-1}))} + O(h^3), \quad (\Delta y := y(t_n) - y(t_{n-1}))
\]

\[
= f(t_{n-1}, y(t_{n-1}))+hf(t_t + f_t, y(t_n) - y(t_{n-1}))f|_{(t_{n-1}, y(t_{n-1}))} + \frac{1}{2}(h^2f_{tt} + h^2f_{ty} + h^2f_{fy} + h^2f_{yy})|_{(t_{n-1}, y(t_{n-1}))} + O(h^3))
\]

(10.1.2)

the last step holds because \( \Delta y = hf(t_{n-1}, y(t_{n-1})) + O(h) \).

Put everything together and compute

\[ T_n(h) := \frac{1}{2h}(I - \frac{h}{3}(II + III + IV)), \]

we find that, the \( O(1) \) term, \( O(h^1) \) terms and \( O(h^2) \) terms get all cancelled out, so

\[ T_n(h) = O(h^3), \]

which shows that the method is at least of order 3.
(10.1c) We introduce in the scheme an intermediate step:

\[ p_{n+1} = y_{n-1} + 2hf(t_n, y_n) \]
\[ c_{n+1} = y_{n-1} + \frac{h}{3}[f(t_{n+1}, p_{n+1}) + 4f(t_n, y_n) + f(t_{n-1}, y_{n-1})] \]
\[ y_{n+1} = y_{n-1} + \frac{h}{3}[f(t_{n+1}, c_{n+1}) + 4f(t_n, y_n) + f(t_{n-1}, y_{n-1})] \]

Prove that, the order of truncation error of this method is 4.

**Hint:** Use the result in subproblem (b) and the fact that the error of Simpson’s rule is of order 5, i.e.

\[ \int_a^b f(x)dx - \frac{b-a}{6}[f(a) + 4f((a+b)/2) + f(b)] = O((b-a)^5) \]

**Solution:** The truncation error of this method is

\[ T_n'(h) := \frac{1}{2h}[y(t_{n+1}) - y(t_{n-1}) - \frac{h}{3}[f(t_{n+1}, \tilde{c}_{n+1}) + 4f(t_n, y(t_n)) + f(t_{n-1}, y(t_{n-1}))]], \]

where actually \( \tilde{c}_{n+1} = y(t_{n+1}) - 2hT_n(h) = y(t_{n+1}) + O(h^4) \) by using the definition of \( T_n(h) \) and result in (b). As a result, it holds that

\[ 2hT_n'(h) = y(t_{n+1}) - y(t_{n-1}) - \frac{h}{3}[f(t_{n+1}, y(t_{n+1})) + f_y(t_{n+1}, y(t_{n+1}))O(h^4) + O(h^5)] \]
\[ + 4f(t_n, y(t_n)) + f(t_{n-1}, y(t_{n-1}))] \]
\[ = y(t_{n+1}) - y(t_{n-1}) - \frac{h}{3}[f(t_{n+1}, y(t_{n+1})) + 4f(t_n, y(t_n)) + f(t_{n-1}, y(t_{n-1}))] + O(h^5) \]
\[ = O(h^5) \]

The last step is by result of Simpson’s rule.

**Problem 10.2 A Complex Hamiltonian Differential Equation**

We will look at the complex differential equation

\[ i\dot{z} = \lambda z + |z|^2z, \quad \lambda \in \mathbb{R}. \] (10.2.1)

(10.2a) Show that the function \( I(z) := |z|^2 \) is an invariant of the differential equation (10.2.1).

**Solution:** For the time-derivative of \( I(z) \), it holds that

\[ \frac{d}{dt} I(z) = \frac{d}{dt} |z|^2 = \frac{d}{dt} z \bar{z} = \dot{z}\bar{z} + z\dot{\bar{z}} \]
\[ \overset{\text{diff. eq.}}{=} [-i\lambda z - i|z|^2z\bar{z} + z[-i\lambda z - i|z|^2z] \]
\[ = -i\lambda |z|^2 - i|z|^4 + i\lambda |z|^2 + i|z|^4 = 0. \]

Thus \( I(z) \) is an invariant.
Show that the differential equation (10.2.1) for \( p = \text{Re}(z), q = \text{Im}(z) \) is equivalent to a Hamiltonian differential equation of the form

\[
\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}.
\]

**Solution:** We substitute \( z = p + iq \) in the differential equation and separate the received equation

\[
i(\dot{p} + i\dot{q}) = -\dot{q} + i\dot{p} = \lambda(p + iq) + (p^2 + q^2)(p + iq).
\]

into its real and imaginary parts. In this way we receive

\[
\dot{p} = \lambda q + (p^2 + q^2)q, \\
-\dot{q} = \lambda p + (p^2 + q^2)p.
\]

(10.2.2)

It remains to show that (10.2.2) is indeed a Hamiltonian differential equation. The ansatz \( \frac{\partial H}{\partial q} = -\dot{p} = -\lambda q - (p^2 + q^2)q \) and integrating gives us

\[
H(p, q) = -\frac{\lambda}{2}q^2 - \frac{1}{2}p^2q^2 - \frac{1}{4}q^4 + C(p),
\]

where the constant \( C(p) \) does not depend on \( q \). Analogously, from \( \frac{\partial H}{\partial p} = \dot{q} = -\lambda p - (p^2 + q^2)p \), we get

\[
H(p, q) = -\frac{\lambda}{2}p^2 - \frac{1}{4}p^2 - \frac{1}{2}p^2q^2 + C(q)
\]

With \( C(q) \) independent of \( p \). Equating and sorting of both expressions gives us

\[
\frac{\lambda}{2}p^2 + \frac{1}{4}p^2 + C(p) = \frac{\lambda}{2}q^2 + \frac{1}{4}q^4 + C(q).
\]

As the left side of above equation does not depend on \( q \) and the right hand side does not depend on \( p \), both sides must be constant. We denote this constant with \( C \) and receive

\[
C(p) = C - \frac{\lambda}{2}p^2 - \frac{1}{4}p^2.
\]

We can substitute this above, and we get our Hamiltonian function

\[
H(p, q) = -\frac{\lambda}{2}(p^2 + q^2) - \frac{1}{4}(p^2 + q^2)^2.
\]

One can easily confirm that \( H \) is indeed a Hamiltonian of the system (10.2.2).

(10.2c) We will now look at the following generalisation of the above differential equation with a real, continuously differentiable function \( \psi \):

\[
i\dot{z} = -\psi(|z|^2)z.
\]

Write this equation as a Hamiltonian differential equation and find an invariant.
Solution: Analogously to above, we receive, by setting $z = p + iq$, substituting this in the equation and separating real and imaginary parts,

$$\dot{p} = -\psi'(p^2 + q^2)q,$$
$$\dot{q} = -\psi'(p^2 + q^2)p.$$

One can easily convince oneself that above system is created by the Hamiltonian $H(p, q) = \frac{1}{2} \psi(p^2 + q^2)$. As we know, $H(z) = \frac{1}{2} \psi(|z|^2)$ is then an invariant of the system.