Problem 10.1 Order and Stability of Multistep Method

Consider the problem

\[ \begin{align*}
y'(t) &= f(t, y(t)), \quad t_0 < t < t_0 + T, \\
y(t_0) &= y_0
\end{align*} \]  

(10.1.1)

where \( f \in C^3([t_0, t_0 + T], \mathbb{R}^m) \) satisfies the Lipschitz condition

\[ \forall t, y, z : |f(t, y) - f(t, z)| \leq L|y - z|. \]

For \( n \geq 1 \) consider the following multistep scheme with constant time step \( h_n = h \):

\[ \begin{align*}
p_{n+1} &= y_{n-1} + 2hf(t_n, y_n) \\
y_{n+1} &= y_{n-1} + \frac{h}{3}[f(t_{n+1}, p_{n+1}) + 4f(t_n, y_n) + f(t_{n-1}, y_{n-1})]
\end{align*} \]  

(10.1a) Study the stability of the scheme.

(10.1b) Prove that the order of truncation error defined by

\[ T_n(h) := \frac{1}{2h}[y(t_{n+1}) - y(t_{n-1}) - \frac{h}{3}[f(t_{n+1}, y(t_{n-1})) + 2hf(t_n, y(t_n))) + 4f(t_n, y(t_n)) + f(t_{n-1}, y(t_{n-1}))]] \]

is at least 3.

(10.1c) We introduce in the scheme an intermediate step:

\[ \begin{align*}
p_{n+1} &= y_{n-1} + 2hf(t_n, y_n) \\
c_{n+1} &= y_{n-1} + \frac{h}{3}[f(t_{n+1}, p_{n+1}) + 4f(t_n, y_n) + f(t_{n-1}, y_{n-1})] \\
y_{n+1} &= y_{n-1} + \frac{h}{3}[f(t_{n+1}, c_{n+1}) + 4f(t_n, y_n) + f(t_{n-1}, y_{n-1})]
\end{align*} \]

Prove that, the order of truncation error of this method is 4.

Hint: Use the result in subproblem (b) and the fact that the error of Simpson’s rule is of order 5, i.e.

\[ \int_a^b f(x)dx = \frac{b-a}{6}[f(a) + 4f((a + b)/2) + f(b)] = O((b - a)^5) \]
Problem 10.2  A Complex Hamiltonian Differential Equation

We will look at the complex differential equation

\[ i \dot{z} = \lambda z + |z|^2 z, \quad \lambda \in \mathbb{R}. \]  \hspace{1cm} (10.2.1)

(10.2a) Show that the function \( I(z) := |z|^2 \) is an invariant of the differential equation (10.2.1).

(10.2b) Show that the differential equation (10.2.1) for \( p = \text{Re}(z), q = \text{Im}(z) \) is equivalent to a Hamiltonian differential equation of the form

\[ \dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}. \]

(10.2c) We will now look at the following generalisation of the above differential equation with a real, continuously differentiable function \( \psi \):

\[ i \dot{z} = -\psi'(|z|^2) z. \]

Write this equation as a Hamiltonian differential equation and find an invariant.

Published on 1 May 2019.
To be submitted by 9 May 2019.

Last modified on April 29, 2019