H. Ammari
L. Baldassari
A. Vanel

Spring Term 2019

Numerical Analysis II

ETH Zürich
D-MATH

Final Exam Spring 2019

Dice marks difficulty of corresponding problem. ⚀ stands for the easiest, and ⚅ stands for the hardest.

Problem 1 **[60 points]**

Assume that f is $\mathcal{C}^2$ in $[0,T] \times \mathbb{R}^2$. Consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x),\ t \in [0,T], \\ x(0) = x_0 \in \mathbb{R}^2. \end{cases} \tag{1.1}$$

For $\Delta t > 0$ small enough, consider the following schemes

$$x^{k+1} = \Phi^{(1)}_{\Delta t}(x^k) := x^k + \Delta t f(t_k, x^k), \tag{1.2}$$

and

$$x^{k+1} = \Phi^{(2)}_{\Delta t}(x^k) := x^k + \Delta t f(t_{k+1}, x^{k+1}). \tag{1.3}$$

(1a)

(i) ⚁ Is $\Phi^{(2)}_{\Delta t}$ the adjoint of $\Phi^{(1)}_{\Delta t}$? Justify.

(ii) ⚀ Prove that (1.2) and (1.3) are consistent with (1.1).

(iii) ⚂ Prove that (1.2) and (1.3) are of order one.

(1b) Let $\Psi_{\Delta t} := \Phi^{(2)}_{\frac{\Delta t}{2}} \circ \Phi^{(1)}_{\frac{\Delta t}{2}}$

(i) ⚀ Prove that

$$\Psi_{\Delta t}(x^k) = x^k + \frac{\Delta t}{2}\big(f(t_k, x^k) + f(t_{k+1}, x^{k+1})\big).$$

(ii) ⚀ Is $\Psi_{\Delta t}$ symmetric?

(iii) ⚀ Is $\Psi_{\Delta t}$ consistent with (1.1)?

(iv) ⚅ Prove that $\Psi_{\Delta t}$ is of order 2. Hint: compute $\int_{t_k}^{t_{k+1}} (t - t_{k+1})(t - t_k) x'''(t) dt$ in two different ways, using integration by part and the integral mean value theorem.

(1c) Suppose that $f(x) = J^{-1}Cx$, where C is a real symmetric 2×2 matrix and J is the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(i) ⚀ Prove that (1.1) is a Hamiltonian system associated with the Hamiltonian $H(x) = \frac{1}{2}x^\top Cx$, where $\top$ denotes the transpose.

(ii) ⚃ Compute the Jacobians $\left(\Phi_{\Delta t}^{(1)}\right)' := \dfrac{\partial x^{k+1}}{\partial x^k}$, $\left(\Phi_{\Delta t}^{(2)}\right)'$ and $(\Psi_{\Delta t})'$.

(iii) ⚄ Are any of the schemes defined by $\Phi_{\Delta t}^{(1)}$, $\Phi_{\Delta t}^{(2)}$, $\Psi_{\Delta t}$ symplectic?

(iv) ⚁ Are any of the schemes defined by $\Phi_{\Delta t}^{(1)}$, $\Phi_{\Delta t}^{(2)}$, $\Psi_{\Delta t}$ volume preserving?

(v) ⚁ Is H an invariant?

(vi) ⚁ Is H preserved by $\Psi_{\Delta t}$?

(1d) Let $C = I$, the identity matrix.

(i) ⚃ Fill in the templates `TrapezStep.m` and `TrapezSolve.m` to implement the scheme $\Psi_{\Delta t}$ with initial data $x_0 = (0, 1)^\top$ and end time $T = 2\pi$.

(ii) ⚀ Write down the exact solution.

(iii) ⚃ Plot the graphs of the exact and approximate solutions, Figure 1 is in phase space and Figure 2 shows each component over time using template `TrapezScript.m`.

(iv) ⚂ Plot the graph $H(t)$ in Figure 3 in template `TrapezScript.m`.

(v) ⚃ Check the order of $\Psi_{\Delta t}$ by filling in the template `TrapezOrder.m` for end time $T = \pi$..

Problem 2 **[40 points]**

Consider

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x),\ t \in [0,T], \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{2.1}$$

with $f \in C^\infty([0,T] \times \mathbb{R})$ satisfying the Lipschitz condition

$$|f(t,x) - f(t,y)| \le C_f |x-y|,\ \forall x,y \in \mathbb{R}, \forall t \in [0,T],$$

for some positive constant C_f.

(2a)

(i) ⚀ Does (2.1) have a unique solution $x(t) \in C^\infty([0,T])$?

(ii) ⚁ If we regard $x(t)$ also as a function of the initial value x_0, what is the equation satisfied by the derivative with respect to t of $\partial x(t)/\partial x_0$? Is it a linear equation?

(2b) Consider the numerical scheme

$$x^{k+1} = x^k + \frac{\Delta t}{4} f(t_k, x^k) + \frac{3\Delta t}{4} f\left(t_k + \frac{2\Delta t}{3}, x^k + \frac{2\Delta t}{3} f(t_k, x^k)\right), \tag{2.2}$$

where $\Delta t > 0$ is small enough and $t_k = k\Delta t$ for $k \in \mathbb{N}$.

(i) ⚀ Prove that (2.2) is consistent with (2.1).

(ii) ⚅ Prove that the truncation error can be expressed as

$$T_k(\Delta t) = \frac{(\Delta t)^2}{6} \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t_k) \frac{\partial f}{\partial x}(t_k, x(t_k)) + \mathcal{O}((\Delta t)^3).$$

(2c) Consider the second order differential equation

$$\begin{cases} \dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \cos x, \\ x(0) = \dfrac{\pi}{2}, \\ \dfrac{\mathrm{d}x}{\mathrm{d}t}(0) = 1. \end{cases} \tag{2.3}$$

(i) ⚀ Rewrite the problem into a first-order ODE system.

(ii) ⚄ Solve the problem (2.4) using the scheme (2.2) by filling in the template `methodPrb2c.m`. Plot the solution $x(t)$ for $t \in [0, 10]$ with the template `RunPrb2c.m`.

(2d) Consider the scheme

$$
\begin{aligned}
x^{k+1} &= x^k + \frac{\Delta t}{4}(k_1 + 3k_3), \\
k_1 &= f(t_k, x^k), \\
k_2 &= f(t_k + \frac{\Delta t}{3}, x^k + \frac{\Delta t}{3} k_1), \\
k_3 &= f(t_k + \frac{2\Delta t}{3}, x^k + \frac{2\Delta t}{3} k_2).
\end{aligned} \tag{2.4}
$$

(i) ⚃ Is (2.5) a Runge-Kutta method?

(ii) ⚁ Is (2.5) consistent with (2.1)? What is its order?

(iii) ⚂ Is (2.5) stable? Is it convergent?

(iv) ⚃ Implement (2.5) by filling in the template `rk3.m` for the following ODE:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin t + x(t), \quad t \in [0, 1], \quad x(0) = 0,$$

and check the order of the method by filling in the template `RKmethodscript.m`.