Final Exam Spring 2019

Dice marks difficulty of corresponding problem. □ stands for the easiest, and ▼ stands for the hardest.

**Problem 1** [60 points]

Assume that \( f \) is \( C^2 \) in \([0, T] \times \mathbb{R}^2\). Consider the initial value problem

\[
\begin{aligned}
\frac{dx}{dt} &= f(t, x), \quad t \in [0, T], \\
\quad x(0) &= x_0 \in \mathbb{R}^2.
\end{aligned}
\]  

(1.1)

For \( \Delta t > 0 \) small enough, consider the following schemes

\[
x^{k+1} = \Phi_{\Delta t}^{(1)}(x^k) := x^k + \Delta t f(t_k, x^k),
\]

(1.2)

and

\[
x^{k+1} = \Phi_{\Delta t}^{(2)}(x^k) := x^k + \Delta t f(t_{k+1}, x^{k+1}).
\]

(1.3)

(1a)

(i) □ Is \( \Phi_{\Delta t}^{(2)} \) the adjoint of \( \Phi_{\Delta t}^{(1)} \)? Justify.

(ii) □ Prove that (1.2) and (1.3) are consistent with (1.1).

(iii) □ Prove that (1.2) and (1.3) are of order one.

(1b) Let \( \Psi_{\Delta t} := \Phi_{\Delta t}^{(2)} \circ \Phi_{\Delta t}^{(1)} \)

(i) □ Prove that

\[
\Psi_{\Delta t}(x^k) = x^k + \frac{\Delta t}{2} \left( f(t_k, x^k) + f(t_{k+1}, x^{k+1}) \right).
\]

(ii) □ Is \( \Psi_{\Delta t} \) symmetric?

(iii) □ Is \( \Psi_{\Delta t} \) consistent with (1.1)?

(iv) ▼ Prove that \( \Psi_{\Delta t} \) is of order 2. Hint: compute \( \int_{t_k}^{t_{k+1}} (t-t_{k+1})(t-t_k)x'''(t)dt \) in two different ways, using integration by part and the integral mean value theorem.
(1c) Suppose that $f(x) = J^{-1}C x$, where $C$ is a real symmetric $2 \times 2$ matrix and $J$ is the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(i) √ Prove that (1.1) is a Hamiltonian system associated with the Hamiltonian $H(x) = \frac{1}{2} x^\top C x$, where $\top$ denotes the transpose.

(ii) ⊗ Compute the Jacobians $(\Phi^{(1)}_{\Delta t})' := \frac{\partial x^{k+1}}{\partial x^k}, (\Phi^{(2)}_{\Delta t})'$ and $(\Psi_{\Delta t})'$.

(iii) ⊗ Are any of the schemes defined by $\Phi^{(1)}_{\Delta t}, \Phi^{(2)}_{\Delta t}, \Psi_{\Delta t}$ symplectic?

(iv) ⊗ Are any of the schemes defined by $\Phi^{(1)}_{\Delta t}, \Phi^{(2)}_{\Delta t}, \Psi_{\Delta t}$ volume preserving?

(v) ⊗ Is $H$ an invariant?

(vi) ⊗ Is $H$ preserved by $\Psi_{\Delta t}$?

(1d) Let $C = I$, the identity matrix.

(i) ⊗ Fill in the templates TrapezStep.m and TrapezSolve.m to implement the scheme $\Psi_{\Delta t}$ with initial data $x_0 = (0, 1)^\top$ and end time $T = 2\pi$.

(ii) √ Write down the exact solution.

(iii) ⊗ Plot the graphs of the exact and approximate solutions, Figure 1 is in phase space and Figure 2 shows each component over time using template TrapezScript.m.

(iv) ⊗ Plot the graph $H(t)$ in Figure 3 in template TrapezScript.m.

(v) ⊗ Check the order of $\Psi_{\Delta t}$ by filling in the template TrapezOrder.m for end time $T = \pi$..
Consider
\[
\begin{align*}
\frac{dx}{dt} &= f(t, x), \ t \in [0, T], \\
x(0) &= x_0 \in \mathbb{R},
\end{align*}
\]
with \( f \in C^\infty([0, T] \times \mathbb{R}) \) satisfying the Lipschitz condition
\[
|f(t, x) - f(t, y)| \leq C_f |x - y|, \ \forall x, y \in \mathbb{R}, \ \forall t \in [0, T],
\]
for some positive constant \( C_f \).

(2a)

(i) Does (2.1) have a unique solution \( x(t) \in C^\infty([0, T]) \)?

(ii) If we regard \( x(t) \) also as a function of the initial value \( x_0 \), what is the equation satisfied by the derivative with respect to \( t \) of \( \frac{\partial x(t)}{\partial x_0} \)? Is it a linear equation?

(2b) Consider the numerical scheme
\[
x^{k+1} = x^k + \frac{\Delta t}{4} f(t_k, x^k) + \frac{3\Delta t}{4} f(t_k + \frac{2\Delta t}{3}, x^k + \frac{2\Delta t}{3} f(t_k, x^k)),
\]
where \( \Delta t > 0 \) is small enough and \( t_k = k\Delta t \) for \( k \in \mathbb{N} \).

(i) Prove that (2.2) is consistent with (2.1).

(ii) Prove that the truncation error can be expressed as
\[
T_k(\Delta t) = \frac{(\Delta t)^2}{6} \frac{d^2 x}{dt^2}(t_k) \frac{\partial f}{\partial x}(t_k, x(t_k)) + O((\Delta t)^3).
\]

(2c) Consider the second order differential equation
\[
\begin{align*}
\frac{d^2 x}{dt^2} &= \cos x, \\
x(0) &= \frac{\pi}{2}, \\
\frac{dx}{dt}(0) &= 1.
\end{align*}
\]

(i) Rewrite the problem into a first-order ODE system.

(ii) Solve the problem (2.4) using the scheme (2.2) by filling in the template `methodPrb2c.m`. Plot the solution \( x(t) \) for \( t \in [0, 10] \) with the template `RunPrb2c.m`. 
(2d) Consider the scheme

\[
x^{k+1} = x^k + \frac{\Delta t}{4} (k_1 + 3k_3),
\]

(2.4)

\[
k_1 = f(t_k, x^k),
\]

\[
k_2 = f(t_k + \frac{\Delta t}{3}, x^k + \frac{\Delta t}{3} k_1),
\]

\[
k_3 = f(t_k + \frac{2\Delta t}{3}, x^k + \frac{2\Delta t}{3} k_2).
\]

(i) ☐ Is (2.5) a Runge-Kutta method?

(ii) ☐ Is (2.5) consistent with (2.1)? What is its order?

(iii) ☐ Is (2.5) stable? Is it convergent?

(iv) ☐ Implement (2.5) by filling in the template \texttt{rk3.m} for the following ODE:

\[
\frac{dx}{dt} = \sin t + x(t), \quad t \in [0, 1], \quad x(0) = 0,
\]

and check the order of the method by filling in the template \texttt{RKmethodscript.m}. 