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# Numerical Analysis II

ETH Zürich
D-MATH

# Final Exam Spring 2018

Dice marks difficulty of corresponding problem. ⚀ stands for the easiest, and ⚅ stands for the hardest.

## Problem 1 **[50 points]**

Consider

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x),\ t \in [0, T], \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{1.1}$$

with $f \in C^\infty([0, T] \times \mathbb{R})$ satisfying the Lipschitz condition

$$|f(t, x) - f(t, y)| \leq C_f|x - y|,\ \forall x, y \in \mathbb{R}, \forall t \in [0, T],$$

for some positive constant $C_f$.

**(1a)**

(i) ⚀ Does (1.1) have a unique solution $x(t) \in C^\infty([0, T])$?

(ii) ⚁ If we regard $x(t)$ also as a function of the initial value $x_0$, what is the equation satisfied by the derivative respect to $t$ of $\partial x(t)/\partial x_0$? Is it a linear equation?

**(1b)** ⚁ Consider the numerical scheme

$$x^{k+1} = x^k + \Delta t f(t_k + \theta\Delta t, (1 - \theta)x^k + \theta x^{k+1}) \tag{1.2}$$

where $0 \leq \theta \leq 1$, $\Delta t > 0$ is small enough and $t_k = k\Delta t$ for $k \in \mathbb{N}$.

For which $0 \leq \theta \leq 1$ is (1.2) explicit? For which $0 \leq \theta \leq 1$ is (1.2) implicit? Is (1.2) a one-step method? Is (1.2) a two-step method? Prove your conclusions.

**(1c)** Let $\Phi(t, x, \Delta t)$ be defined by

$$\Phi(t, x, \Delta t) = f(t + \theta\Delta t, x + \theta\Delta t\Phi(t, x, \Delta t))$$

so that (1.2) can be written in the form

$$x^{k+1} = x^k + \Delta t\Phi(t_k, x^k, \Delta t).$$

(i) ⚀ Prove that the scheme (1.2) is consistent with (1.1).

(ii) ⚃ Define the truncation error by

$$T_k(\Delta t) = \frac{x(t+\Delta t) - x(t)}{\Delta t} - \Phi(t, x(t), \Delta t)$$

where $x(t)$ is the solution to (1.1).

Compute the order of the scheme (1.2) in terms of $0 \leq \theta \leq 1$. For which value of $\theta$, the scheme is of order 2? Prove your result.

**(1d)**

(i) ⚁ For $x, y \in \mathbb{R}$, $t \in [0, T]$ and $\Delta t > 0$, prove that

$$|\Phi(t, x, \Delta t) - \Phi(t, y, \Delta t)| \leq C_f(|x - y| + \theta\Delta t|\Phi(t, x, \Delta t) - \Phi(t, y, \Delta t)|).$$

(ii) ⚁ For $\Delta t > 0$ such that $C_f\theta\Delta t < 1$, prove that the scheme (1.2) is stable.

(iii) ⚀ Is (1.2) for solving (1.1) convergent when $C_f\theta\Delta t < 1$? Prove that.

**(1e)** Now consider the following specific problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = tx + t^3,\ t \in [0, T], \\ x(0) = 0. \end{cases} \tag{1.3}$$

The explicit solution to (1.3) is given by

$$x(t) = 2e^{t^2/2} - t^2 - 2.$$

(i) ⚁ Use template `ExplicitMethod.m` to implement numerical scheme (1.2) when $\theta = 0$ for (1.3).

(ii) ⚂ Use template `ImplicitMethod.m` to implement numerical scheme (1.2) when $\theta = 1$ for (1.3). Use *Newton's method* to solve implicit equation for *this* problem.

(iii) ⚂ Use template `MidpointMethod.m` to implement numerical scheme (1.2) when $\theta = 1/2$ for (1.3). Use *fixed point iteration* to solve implicit equation for *this* problem.

(iv) ⚁ Now take step sizes $dt = 1/2, (1/2)^2, \cdots, (1/2)^{10}$ and end time $T_0 = 1$. Call the previous three functions in template `ConvergenceRate.m` to calculate the error between approximated solutions and exact results of each step size for each method at time $T_0 = 1$. The program will automatically print convergence order of the three methods.

## Problem 2 **[50 points]**

Let $A$ be a real symmetric $2 \times 2$ matrix. Let $J$ denote the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Consider the system of linear equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = J^{-1}Ax, \\ x(0) = x_0 \in \mathbb{R}^2. \end{cases} \tag{2.1}$$

**(2a)**

(i) ⚁ Prove that (2.1) is a Hamilton system associated with the Hamiltonian $H(x) = \frac{1}{2}x^\top Ax$, where $x^\top$ denotes the transpose of $x$.

(ii) ⚁ Define the flow $\Phi_t$ associated with (2.1) by $\Phi_t(x_0) = x(t)$. Prove that $\Phi_t(x_0) = e^{tJ^{-1}A}x_0$.

(iii) ⚂ Is $\Phi_t$ symmetric? Does $\Phi_t$ preserve the energy $H$? Is $\Phi_t$ volume preserving? Prove your results.

**(2b)** Consider the system of equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}q = p, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}p = -\omega^2 q, \\ p(0) = p_0 \in \mathbb{R},\ q(0) = q_0 \in \mathbb{R}, \end{cases} \tag{2.2}$$

where $\omega > 0$.

(i) ⚁ Find matrix $A$, such that (2.2) can be reformulated in the form of (2.1) for $x = (p, q)^\top$.

(ii) ⚂ Let

$$S = \begin{pmatrix} \sqrt{\omega} & 0 \\ 0 & \frac{1}{\sqrt{\omega}} \end{pmatrix}$$

Verify that $J^{-1}A$ gets transformed to

$$\hat{A} = S^{-1}J^{-1}AS = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

and (2.2) to

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{q} = \omega\hat{p}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{p} = -\omega\hat{q}. \end{cases} \tag{2.3}$$

(iii) ⚂ Prove that the flow $\hat{\Phi}_t$ associated with (2.3) is given by

$$\hat{\Phi}_t(x_0) = e^{tS^{-1}J^{-1}AS}x_0 = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} x_0$$

and deduce that the flow $\Phi_t$ associated with (2.2) is given by

$$\Phi_t(x_0) = \begin{pmatrix} \cos\omega t & -\omega\sin\omega t \\ \omega^{-1}\cos\omega t & \sin\omega t \end{pmatrix} x_0.$$

**(2c)**

(i) ⚃ Consider the scheme

$$\begin{cases} \hat{p}^{k+1} = \hat{p}^k - \omega \Delta t \hat{q}^k \\ \hat{q}^{k+1} = \hat{q}^k + \omega \Delta t \hat{p}^k \end{cases} \tag{2.4}$$

for solving (2.3). Is (2.4) explicit or implicit? Is (2.4) symplectic? Is (2.4) symmetric? Prove your results.

(ii) ⚂ Write (2.4) in the form

$$\begin{pmatrix} \hat{p}^{k+1} \\ \hat{q}^{k+1} \end{pmatrix} = \hat{R}(\Delta t) \begin{pmatrix} \hat{p}^k \\ \hat{q}^k \end{pmatrix} \tag{2.5}$$

and compute the eigenvalues and eigenvectors of $\hat{R}(\Delta t)$ in terms of those of $\hat{A}$. Is (2.4) stable? Prove your results.

(iii) ⚄ Consider the scheme

$$\hat{x}^{k+1} = \hat{x}^k + \Delta t \hat{A}\left(\frac{\hat{x}^k + \hat{x}^{k+1}}{2}\right). \tag{2.6}$$

Write (2.6) in the form of (2.5) and compute the eigenvalues and eigenvectors of $\hat{R}(\Delta t)$ in this case. Does (2.6) preserve the energy? Is (2.6) symplectic? Is (2.6) stable? Prove your results.

(iv) ⚂ Prove whether (2.6) is symmetric or not.

**(2d)** Consider the Hamiltonian systems

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{q} = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{p} = -\omega \hat{q}, \\ p(0) = p_0 \in \mathbb{R},\ q(0) = q_0 \in \mathbb{R}. \end{cases} \tag{2.7}$$

and

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{q} = \omega \hat{p}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{p} = 0, \\ p(0) = p_0 \in \mathbb{R},\ q(0) = q_0 \in \mathbb{R}. \end{cases} \tag{2.8}$$

(i) ⚃ Let $\Psi^{(1)}_{\Delta t}$ and $\Psi^{(2)}_{\Delta t}$ be the associated flows of system (2.7) and (2.8). Write down the scheme $\Psi^{(1)}_{\Delta t} \circ \Psi^{(2)}_{\Delta t}$ for solving (2.3). Is this scheme symplectic? Is this scheme symmetric? Prove your results and write down its adjoint.

(ii) ⚄ Write down the scheme $\Psi^{(1)}_{\Delta t/2} \circ \Psi^{(2)}_{\Delta t} \circ \Psi^{(1)}_{\Delta t/2}$. Prove that this scheme is symplectic and it is of second order.

(iii) ⚁ Use template `SplittingMethod.m` to implement numerical scheme $\Psi^{(1)}_{\Delta t/2} \circ \Psi^{(2)}_{\Delta t} \circ \Psi^{(1)}_{\Delta t/2}$ onto equation system (2.3). Set simulation time interval to be $[0, 10]$, initial value at $T = 0$ to be $(p(0), q(0)) = (1, 2)$, number of total step to be $N = 500$, and frequency $\omega = \pi$. Template will plot the graph of exact solution $(p(t), q(t))$ and approximated solution $(p_k, q_k)$ in the end.