

End-term Exam 2019

Problem 1

[28 Marks]

Consider

$$\begin{cases} \frac{dx}{dt} = f(t, x), & t \in [0, T] \\ x(0) = x_0 \in \mathbb{R} \end{cases} \quad (1.1)$$

with $f \in C^\infty$ subject to the Lipschitz condition $|f(t, x) - f(t, y)| \leq C_f|x - y|$ for all $x, y \in \mathbb{R}$, for all $t \in [0, T]$.

Consider the scheme

$$x^{k+1} = x^k + \frac{\Delta t}{2}(\kappa_1 + \kappa_2) \quad (1.2)$$

where

$$\begin{aligned} \kappa_1 &= f(t_k, x^k), \\ \kappa_2 &= f(t_{k+1}, x^k + \Delta t \kappa_1). \end{aligned}$$

(1a) Is (1.2) explicit or implicit? Is (1.2) a one-step or a two-step method?

Explicit Implicit

One-step method Two-step Method

(1b) Find the coefficients $(a_{i,j})$, (b_i) , (c_i) such that the scheme (1.2) can be rewritten in the form

$$\begin{aligned} x_{i,k} &= x^k + \Delta t \sum_{j=1}^m a_{i,j} f(t_{j,k}, x_{j,k}) \\ x^{k+1} &= x^k + \Delta t \sum_{i=1}^m b_i f(t_{i,k}, x_{i,k}) \\ t_{i,k} &= t_k + c_i \Delta t \end{aligned} \quad (1.3)$$

(1c) Is (1.2) consistent? Calculate the order of this Runge-Kutta scheme.

(1d) Recall that the Runge-Kutta method defined by (1.3) is stable if

$$\Delta t C_f \rho(|A|) < 1 \tag{1.4}$$

where $|A| = (|a_{ij}|)_{i,j}$ and ρ is the spectral radius of a matrix. Is scheme (1.2) stable? Is (1.2) convergent?

Now, consider another numerical scheme for (1.1), defined by

$$x^{k+1} = x^k + \frac{\Delta t}{2} [3f(t_k, x^k) - f(t_{k-1}, x^{k-1})]. \tag{1.5}$$

(1e) What kind of method is (1.5)?

- Explicit One-step Explicit Two-step
Implicit One-step Implicit Two-step

(1f) Define the truncation error by

$$T_k(\Delta t) = \frac{x(t_{k+1}) - x(t_k) - \frac{\Delta t}{2}[3f(t_k, x(t_k)) - f(t_{k-1}, x(t_{k-1}))]}{\Delta t} \quad (1.6)$$

where $t_{k\pm 1} = t_k \pm \Delta t$.

Is scheme (1.5) consistent with (1.1)?, i.e. , does $T_k(\Delta t) = O(\Delta t)$ hold? Prove it.

Problem 2**[32 Marks]**

Let T and V be two smooth real valued functions. Consider the system of equations

$$\begin{cases} \frac{dp}{dt} = -V'(q), \\ \frac{dq}{dt} = T'(p), \end{cases} \quad (2.1)$$

where T' and V' are the derivatives of T and V , and $p, q \in \mathbb{R}$.

Consider the numerical scheme (for $\Delta t > 0$)

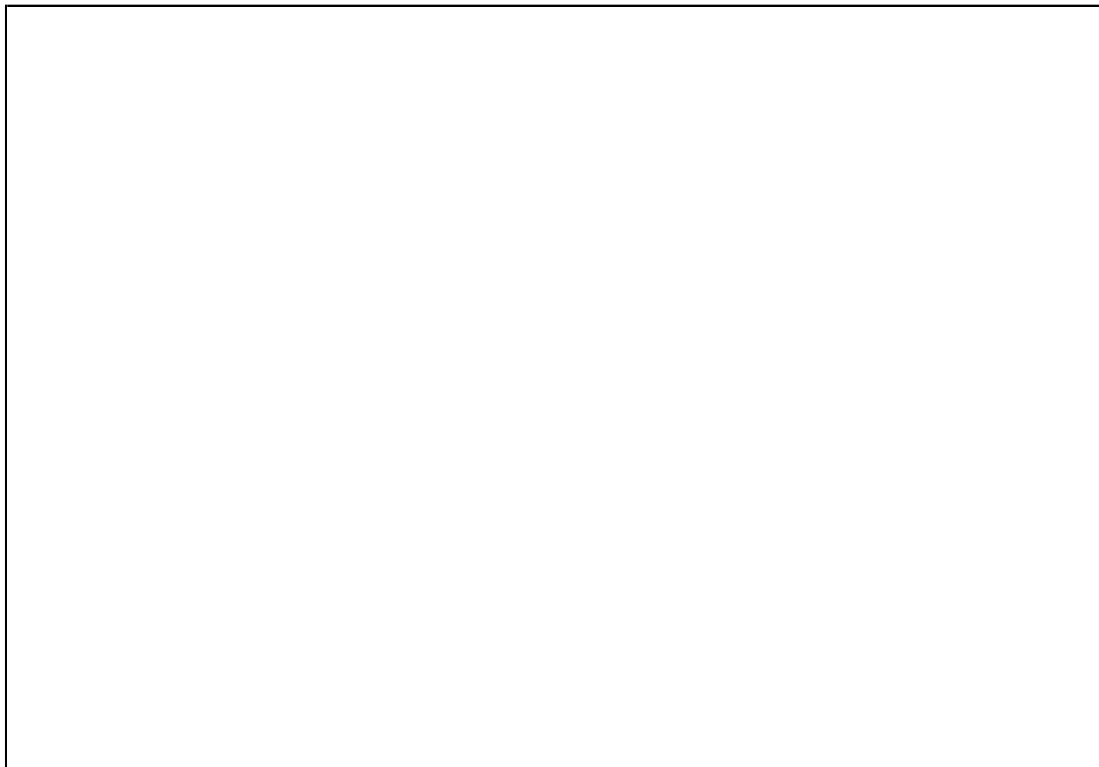
$$\begin{cases} p^{k+1} = p^k - \Delta t V'(q^k), \\ q^{k+1} = q^k + \Delta t T'(p^{k+1}) = q^k + \Delta t T'(p^k - \Delta t V'(q^k)), \end{cases} \quad (2.2)$$

and define the numerical flow

$$\Phi_{\Delta t} : (p^k, q^k) \mapsto (p^{k+1}, q^{k+1}). \quad (2.3)$$

(2a) Compute the Jacobian $\Phi'_{\Delta t}$ of the numerical flow $\Phi_{\Delta t}$ defined by

$$\Phi'_{\Delta t}(p^k, q^k) = \frac{\partial(p^{k+1}, q^{k+1})}{\partial(p^k, q^k)} := \begin{pmatrix} \frac{\partial p^{k+1}}{\partial p^k} & \frac{\partial p^{k+1}}{\partial q^k} \\ \frac{\partial q^{k+1}}{\partial p^k} & \frac{\partial q^{k+1}}{\partial q^k} \end{pmatrix}.$$



(2b) Is (2.2) symplectic? i.e. does the numerical flow satisfy

$$(\Phi'_{\Delta t})^\top J \Phi'_{\Delta t} = J,$$

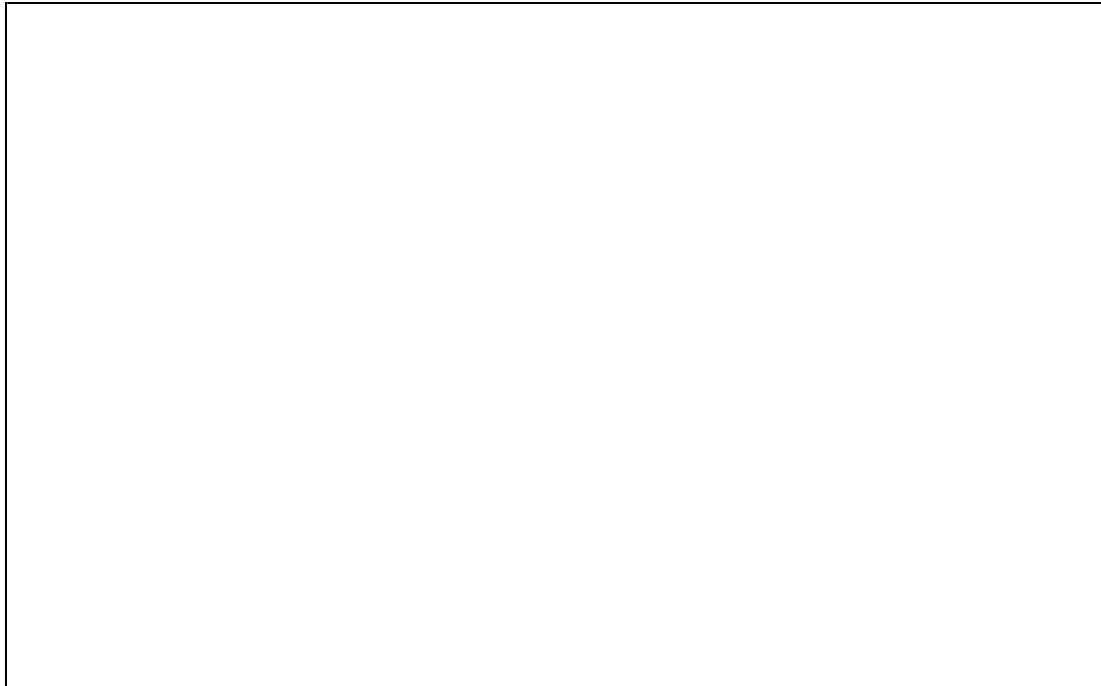
where $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $^\top$ denotes the transpose of a matrix? Prove it.



(2c) Is (2.2) symmetric, i.e. does

$$\Phi_{\Delta t}^* = \Phi_{\Delta t}?$$

Justify your answer. Here $\Phi_{\Delta t}^*$ is defined by $\Phi_{\Delta t}^* = (\Phi_{-\Delta t})^{-1}$, i.e. by replacing Δt by $-\Delta t$ and exchanging k and $k + 1$.



(2d) Is $(\Phi_{\Delta t/2}^*)^* = \Phi_{\Delta t/2}$?

Yes No

(2e) The composition $\Phi_{\Delta t/2}^* \circ \Phi_{\Delta t/2}$ is

symmetric. not symmetric.

(2f) The composition $\Phi_{\Delta t/2}^* \circ \Phi_{\Delta t/2}$ is

symplectic. not symplectic.

(2g) The composition $\Phi_{\Delta t/2}^* \circ \Phi_{\Delta t/2}$ is

a Symplectic Euler method. a Leapfrog method.

(2h) Let $x = (p, q)^\top$. Rewrite explicitly (2.1) in the form

$$\frac{dx}{dt} = J^{-1} \nabla H(x).$$

Is $f(x) = J^{-1} \nabla H(x)$ divergence-free? Is $\Phi_{\Delta t}$ defined by (2.3) volume preserving? Explain why.

