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Numerical Analysis II

ETH Zürich
D-MATH

End-term Exam 2019

Problem 1 **[28 Marks]**

Consider

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), \quad t \in [0,T] \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{1.1}$$

with $f \in C^\infty$ subject to the Lipschitz condition $|f(t,x) - f(t,y)| \leq C_f |x-y|$ for all $x, y \in \mathbb{R}$, for all $t \in [0,T]$.

Consider the scheme

$$x^{k+1} = x^k + \frac{\Delta t}{2}(\kappa_1 + \kappa_2) \tag{1.2}$$

where

$$\kappa_1 = f(t_k, x^k),$$
$$\kappa_2 = f(t_{k+1}, x^k + \Delta t \kappa_1).$$

(1a) Is (1.2) explicit or implicit? Is (1.2) a one-step or a two-step method?

Explicit □ Implicit □

One-step method □ Two-step Method □

(1b) Find the coefficients $(a_{i,j})$, (b_i), (c_i) such that the scheme (1.2) can be rewritten in the form

$$\begin{aligned} x_{i,k} &= x^k + \Delta t \sum_{j=1}^{m} a_{i,j} f(t_{j,k}, x_{j,k}) \\ x^{k+1} &= x^k + \Delta t \sum_{i=1}^{m} b_i f(t_{i,k}, x_{i,k}) \\ t_{i,k} &= t_k + c_i \Delta t \end{aligned} \tag{1.3}$$

(1c) Is (1.2) consistent? Calculate the order of this Runge-Kutta scheme.

(1d) Recall that the Runge-Kutta method defined by (1.3) is stable if

$$\Delta t C_f \rho(|A|) < 1 \tag{1.4}$$

where $|A| = (|a_{ij}|)_{i,j}$ and ρ is the spectral radius of a matrix. Is scheme (1.2) stable? Is (1.2) convergent?

Now, consider another numerical scheme for (1.1), defined by

$$x^{k+1} = x^k + \frac{\Delta t}{2}\big[3f(t_k, x^k) - f(t_{k-1}, x^{k-1})\big]. \tag{1.5}$$

(1e) What kind of method is (1.5)?

Explicit One-step ☐ Explicit Two-step ☐
Implicit One-step ☐ Implicit Two-step ☐

(1f) Define the truncation error by

$$T_k(\Delta t) = \frac{x(t_{k+1}) - x(t_k) - \frac{\Delta t}{2}[3f(t_k, x(t_k)) - f(t_{k-1}, x(t_{k-1}))]}{\Delta t} \tag{1.6}$$

where $t_{k\pm1} = t_k \pm \Delta t$.

Is scheme (1.5) consistent with (1.1)?, i.e. , does $T_k(\Delta t) = O(\Delta t)$ hold? Prove it.

Problem 2 **[32 Marks]**

Let T and V be two smooth real valued functions. Consider the system of equations

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = -V'(q), \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = T'(p), \end{cases} \tag{2.1}$$

where T' and V' are the derivatives of T and V, and $p, q \in \mathbb{R}$.

Consider the numerical scheme (for $\Delta t > 0$)

$$\begin{cases} p^{k+1} = p^k - \Delta t\, V'(q^k), \\ q^{k+1} = q^k + \Delta t\, T'(p^{k+1}) \quad = \quad q^k + \Delta t\, T'(p^k - \Delta t\, V'(q^k)), \end{cases} \tag{2.2}$$

and define the numerical flow

$$\Phi_{\Delta t} : (p^k, q^k) \mapsto (p^{k+1}, q^{k+1}). \tag{2.3}$$

(2a) Compute the Jacobian $\Phi'_{\Delta t}$ of the numerical flow $\Phi_{\Delta t}$ defined by

$$\Phi'_{\Delta t}(p^k, q^k) = \frac{\partial(p^{k+1}, q^{k+1})}{\partial(p^k, q^k)} := \begin{pmatrix} \dfrac{\partial p^{k+1}}{\partial p^k} & \dfrac{\partial p^{k+1}}{\partial q^k} \\ \dfrac{\partial q^{k+1}}{\partial p^k} & \dfrac{\partial q^{k+1}}{\partial q^k} \end{pmatrix}.$$

(2b) Is (2.2) symplectic? i.e. does the numerical flow satisfy

$$(\Phi'_{\Delta t})^\top J \Phi'_{\Delta t} = J,$$

where $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $^\top$ denotes the transpose of a matrix? Prove it.

(2c) Is (2.2) symmetric, i.e. does

$$\Phi^*_{\Delta t} = \Phi_{\Delta t}?$$

Justify your answer. Here $\Phi^*_{\Delta t}$ is defined by $\Phi^*_{\Delta t} = (\Phi_{-\Delta t})^{-1}$, i.e. by replacing Δt by $-\Delta t$ and exchanging k and $k+1$.

(2d) Is $(\Phi^*_{\Delta t/2})^* = \Phi_{\Delta t/2}$?

Yes ☐ No ☐

(2e) The composition $\Phi^*_{\Delta t/2} \circ \Phi_{\Delta t/2}$ is

symmetric. ☐ not symmetric. ☐

(2f) The composition $\Phi^*_{\Delta t/2} \circ \Phi_{\Delta t/2}$ is

symplectic. ☐ not symplectic. ☐

(2g) The composition $\Phi^*_{\Delta t/2} \circ \Phi_{\Delta t/2}$ is

a Symplectic Euler method. ☐ a Leapfrog method. ☐

(2h) Let $x = (p, q)^\top$. Rewrite explicitly (2.1) in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = J^{-1}\nabla H(x).$$

Is $f(x) = J^{-1}\nabla H(x)$ divergence-free? Is $\Phi_{\Delta t}$ defined by (2.3) volume preserving? Explain why.