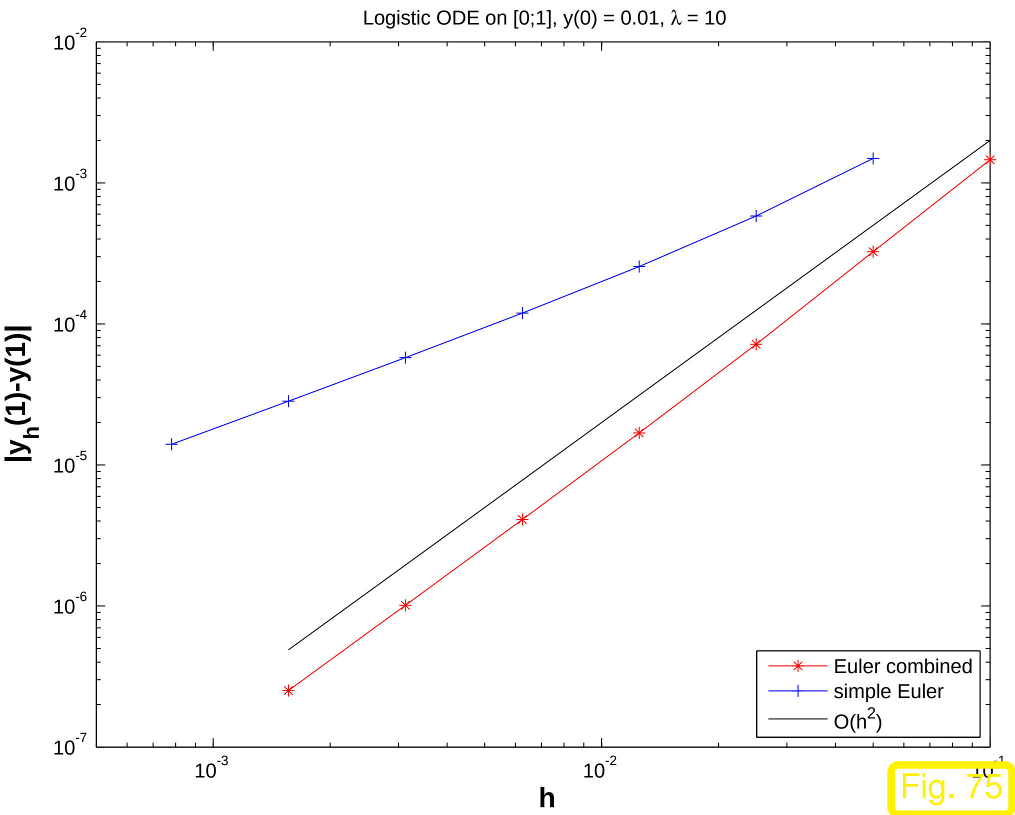
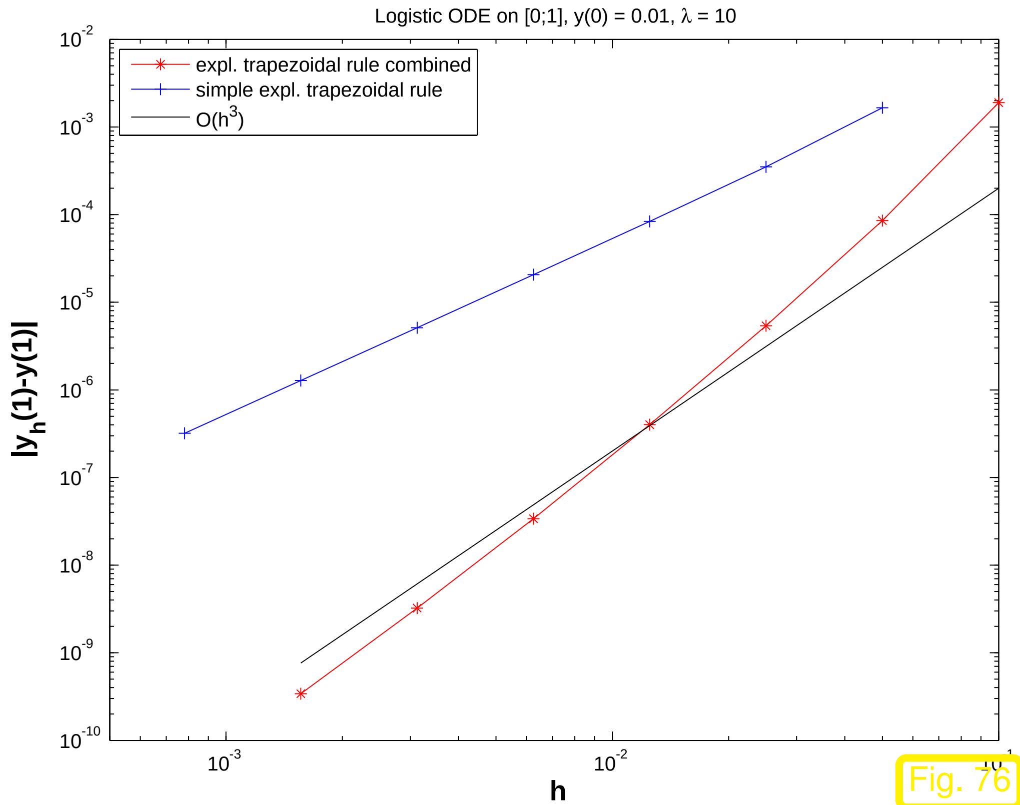


Euler method



Explicit trapezoidal rule

◇

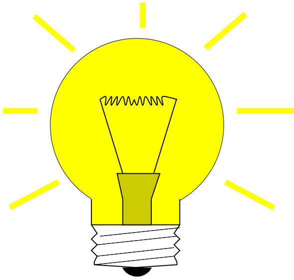
2.4.2 Extrapolation idea

Abstract frame:

Problem: $\mathbf{\Pi} : X \mapsto \mathbb{R}^d$, we look for $\mathbf{\Pi}(x_0)$ for fixed $x_0 \in X$, $X \hat{=}$ data space

Family of numerical methods $\left\{\mathbf{\Pi}_h : X \mapsto \mathbb{R}^d\right\}_h$ ➤ approximations $\mathbf{\Pi}_h(x_o) \approx \mathbf{\Pi}(x_0)$

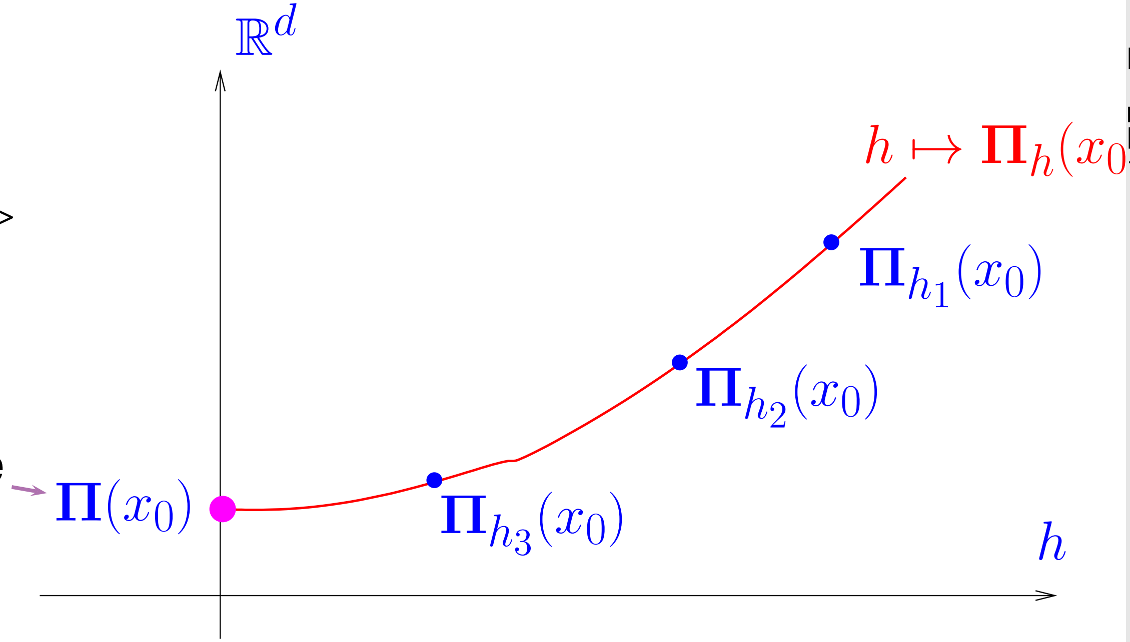
$\mathbf{\Pi}_h$ depends on scalar discretization parameter $h > 0$ (e.g., time step)



- Compute $\mathbf{\Pi}_h(x_0)$ for $h \in \{h_1, \ldots, h_k\}$ ("series of step sizes", $h_i > h_{i+1}$)
- Compute interpolation polynomial $\mathbf{p} \in (\mathcal{P}_{k-1})^d$ with
$$\mathbf{p}(h_i) = \mathbf{\Pi}_{h_i}(x_0), \quad i = 1, \ldots, k \quad .$$
- Better (?) approximation $\boxed{\mathbf{\Pi}(x_0) \approx \mathbf{p}(0)}$

Visualization: ▷

Idea of extrapolation method



Algorithm: Aitken-Neville scheme [13, Sect. 9.4] $\rightarrow$ course "Numerical methods"

Recursive computation of values of interpolation polynomials for $h=0$, $p=1$:

$$T_{i1} := \mathbf{\Pi}_{h_i}(x_0)\ , \quad i=1,\dots,k\ , \tag{2.4.5}$$

$$T_{il} := T_{i,l-1} + \frac{T_{i,l-1}-T_{i-1,l-1}}{\dfrac{h_{i-l+1}}{h_i}-1}\ , \quad 2\le l\le k\ . \tag{2.4.6}$$

Extrapolation tableau ▷

$$\begin{array}{ccccccc}
T_{11} & & & & & & \\
 & \searrow & & & & & \\
T_{21} & \rightarrow & T_{22} & & & & \\
\vdots & & & \ddots & & & \\
T_{k-1,1} & \rightarrow & \cdots & \rightarrow & T_{k-1,k-1} & & \\
 & \searrow & & \searrow & & \searrow & \\
T_{k1} & \rightarrow & \cdots & \rightarrow & T_{k,k-1} & \rightarrow & T_{kk}
\end{array}$$

MATLAB-CODE : Aitken-Neville extrapolation

```
function T = anexpol(y,h)
k = length(h);
T(1) = y(1);
for i=2:k
  T(i) = y(i);
  for l=i-1:-1:1
    T(l)=T(l+1)+(T(l+1)-T(l))/...
          (h(l)/h(i)-1);
  end
end
```

$\eta_l : \eta_i$

$$\begin{array}{ccccccc}
\mathtt{T(1)} & & \mathtt{T(2)} & & \mathtt{T(3)} & \cdots & \mathtt{T(k)} \\
\hline
T_{11}=y_1 & & & & & & \\
\downarrow & & & & & & \\
T_{22} & \leftarrow & T_{21}=y_2 & & & & \\
\downarrow & & \downarrow & & & & \\
T_{33} & \leftarrow & T_{32} & \leftarrow & T_{31}=y_3 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
T_{kk} & \leftarrow & T_{k,k-1} & \leftarrow & \cdots & \leftarrow & T_{k,1}
\end{array}$$

Output: lowest row of tableau decreasing

☞ Extrapolation "works", if

- $\lim_{h\to 0} \mathbf{\Pi}_h(x_0) = \mathbf{\Pi}(x_0)$ $\hat{=}$ convergence,
- $h \mapsto \mathbf{\Pi}_h(x_0)$ "behaves like a polynomial for small h."

Definition 2.4.7 ((Truncated) asymptotic expansion)**.**
$h \mapsto \mathbf{\Pi}_h(x_0)$ ($x_0 \in X$ fixed) has a (truncated) asymptotic expansion in h up to the order k, if there exist constants(*) $\alpha_0, \alpha_1, \dots, \alpha_k \in \mathbb{R}^d$ and a function $h \mapsto R_k(h)$ that is uniformly bounded for sufficiently small h such that

$$\mathbf{\Pi}_h(x_0) = \alpha_0 + \alpha_1 h + \alpha_2 h^2 + \dots + \alpha_k h^k + R_k(h) h^{k+1} \quad \textit{for small } h > 0 \; .$$

$(*)$ α_i constants $\hat{=}$ α_i independent of h !

Theorem 2.4.8 (Convergence of extrapolated values)**.**
Let $\mathbf{\Pi}_h(x_0)$ *have an asymptotic expansion in* h *up to the order* k *according to Def. 2.4.7. Then, the values from the extrapolation tableau, compare* (2.4.5), (2.4.6), *satisfy for sufficiently small* $h_j > 0$

$$\left\|T_{i,l} - \alpha_0\right\| \leq \|\alpha_l\| \, h_{i-l+1} \cdot \cdots \cdot h_i + C \cdot \sum_{j=i-l+1}^{i} \|R_k(h_j)\| \, h_j^{l+1} , \quad 1 \leq i, l \leq k ,$$

where $C > 0$ *only depends on the ratios* $h_i : h_j$.

P. Grohs

rev 63606, February 17, 2014