

Lecture 6: Finite difference methods

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Finite difference methods

- **Finite difference methods**: basic numerical solution methods for **partial differential equations**.
- Obtained by replacing the **derivatives** in the equation by the appropriate **numerical differentiation formulas**.
- Numerical scheme: accurately approximate the true solution.
- Basic finite difference schemes for the **heat** and the **wave equations**.

Finite difference methods

- Numerical algorithms for the heat equation
Finite difference approximations
- Consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \gamma \frac{\partial^2 u}{\partial x^2} = 0, & x \in [0, 1], t \geq 0, \\ u(t, 0) = u(t, 1) = 0, & t \geq 0, \\ u(0, x) = u_0(x), & x \in [0, 1], \end{cases}$$

$\gamma > 0$: thermal conductivity.

Finite difference methods

- Numerical approximation to the solution u :
 - **Rectangular mesh** consisting of points (t_k, x_j) with
$$0 = t_0 < t_1 < t_2 < \dots \quad \text{and} \quad 0 = x_0 < x_1 < \dots < x_{N+1} = 1.$$
 - Uniform time step and spatial mesh sizes:

$$\Delta t = t_{k+1} - t_k, \quad \Delta x = x_{j+1} - x_j = \frac{1}{N}.$$

- Numerical approximation of u at the mesh point (t_k, x_j) :

$$u_j^k \approx u(t_k, x_j) \quad \text{where} \quad t_k = k\Delta t, \quad x_j = j\Delta x.$$

- **Dirichlet boundary conditions** $u(t, 0) = u(t, 1) = 0, t \geq 0 \Rightarrow$

$$u_0^k = u_{N+1}^k = 0 \quad \text{for all } k > 0.$$

Finite difference methods

- Finite difference approximations for the derivatives.
- Approximation of the **second order space derivative**:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(t_k, x_j) &\approx \frac{u(t_k, x_{j-1}) - 2u(t_k, x_j) + u(t_k, x_{j+1}))}{(\Delta x)^2} + O((\Delta x)^2) \\ &\approx \frac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{(\Delta x)^2} + O((\Delta x)^2).\end{aligned}$$

- Approximation of the **time derivative**:

$$\frac{\partial u}{\partial t}(t_k, x_j) \approx \frac{u(t_{k+1}, x_j) - u(t_k, x_j)}{\Delta t} + O(\Delta t) \approx \frac{u_j^{k+1} - u_j^k}{\Delta t} + O(\Delta t).$$

Finite difference methods

- **Explicit scheme:**

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} + \gamma \frac{-u_{j-1}^k + 2u_j^k - u_{j+1}^k}{(\Delta x)^2} = 0$$

for $k \geq 0$ and $j \in \{1, \dots, N\}$.

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$$\mu := \frac{\gamma \Delta t}{(\Delta x)^2};$$

- Vector whose entries are the numerical approximations to the solution values at time t_k at the interior nodes:

$$\mathbf{u}^{(k)} := (u_1^k, u_2^k, \dots, u_N^k)^\top \approx (u(t_k, x_1), u(t_k, x_2), \dots, u(t_k, x_N))^\top.$$

Finite difference methods

- Matrix form:

$$u^{(k+1)} = Au^{(k)};$$

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$$A := \begin{pmatrix} 1-2\mu & \mu & & & \\ \mu & 1-2\mu & \mu & & \\ & \mu & 1-2\mu & \mu & \\ & & \ddots & \ddots & \ddots \\ & & & \mu & 1-2\mu & \mu \\ & & & & \mu & 1-2\mu \end{pmatrix}.$$

- A: symmetric and tridiagonal.

Finite difference methods

- Consistency, stability, and convergence
- General finite difference method:

$$F_{\Delta t, \Delta x}(\{u_{j+n}^{k+m}\}_{m^- \leq m \leq m^+, n^- \leq n \leq n^+}) = 0,$$

m^\pm, n^\pm : width of the stencil of the scheme.

Finite difference methods

- **DEFINITION: Consistency and order**
 - Finite difference scheme: **consistent with $F(u) = 0$** if, for any smooth solution $u(x, t)$, **truncation error:**

$$F_{\Delta t, \Delta x}(\{u(t_{k+m}, x_{j+n})\}_{m^- \leq m \leq m^+, n^- \leq n \leq n^+}) \rightarrow 0$$

- as Δt and $\Delta x \rightarrow 0$ **independently**.
- Scheme: **of order p in time and order q in space** if **truncation error:** of the order of $O((\Delta t)^p + (\Delta x)^q)$ as Δt and $\Delta x \rightarrow 0$.

Finite difference methods

- **THEOREM:**

- Explicit scheme: **consistent** with the heat equation, **of order one in time and two in space.**
- Moreover, if

$$(*) \quad \frac{\gamma \Delta t}{(\Delta x)^2} = \frac{1}{6},$$

then, explicit scheme: **of order two in time and four in space.**

Finite difference methods

- **PROOF:**
 - **Taylor expansion** of $v(t, x) \in C^6$ evaluated at (t, x) ,

$$\begin{aligned} & \frac{v(t + \Delta t, x) - v(t, x)}{\Delta t} \\ & + \gamma \frac{-v(t, x - \Delta x) + 2v(t, x) - v(t, x + \Delta x)}{(\Delta x)^2} \\ & = \left(\frac{\partial v}{\partial t} - \gamma \frac{\partial^2 v}{\partial x^2} \right)(t, x) + \frac{\Delta t}{2} \frac{\partial^2 v}{\partial t^2}(t, x) - \frac{\gamma(\Delta x)^2}{12} \frac{\partial^4 v}{\partial x^4}(t, x) \\ & + O((\Delta t)^2 + (\Delta x)^4). \end{aligned}$$

Finite difference methods

- v : solution of the heat equation \Rightarrow truncation error goes to zero as $\Delta t, \Delta x \rightarrow 0 \Rightarrow$ explicit scheme: consistent.
- Scheme: of order 1 in time and 2 in space.
- Suppose that (*) holds \Rightarrow terms in Δt and $(\Delta x)^2$ cancel out since

$$\frac{\partial^2 v}{\partial t^2} = \gamma \frac{\partial^3 v}{\partial t \partial x^2} = \gamma^2 \frac{\partial^4 v}{\partial x^4}.$$

- Explicit scheme: of order 2 in time and 4 in space.

Finite difference methods

- **DEFINITION: Stability**
 - Finite difference scheme: **stable with respect to the norm $\|\cdot\|_r$** defined by

$$\|u^{(k)}\|_r := \left(\sum_{j=1}^N \Delta x |u_j^k|^r \right)^{\frac{1}{r}}, \quad 1 \leq r \leq +\infty,$$

$u^{(k)}$, if there exists a positive constant C independent of Δt and Δx s.t.

$$\|u^{(k)}\|_r \leq C \|u^{(0)}\|_r \quad \text{for all } k \geq 0.$$

Finite difference methods

- **DEFINITION: Linear scheme**
 - Finite difference scheme: **linear** if scheme: linear with respect to its arguments u_{j+n}^{k+m} .
- Linear finite difference scheme:

$$u^{(k+1)} = Au^{(k)},$$

A : **iteration matrix**.

Finite difference methods

- \Rightarrow

$$u^{(k+1)} = A^{k+1} u^{(0)}.$$

- **Stability** \Leftrightarrow

$$\|A^k u^{(0)}\|_r \leq C \|u^{(0)}\|_r, \quad \text{for all } k \geq 0 \text{ and } u^{(0)} \in \mathbb{R}^N.$$

- Matrix norm

$$\|M\|_r = \sup_{u \in \mathbb{R}^N, u \neq 0} \frac{\|Mu\|_r}{\|u\|_r}.$$

- **Stability** \Leftrightarrow

$$\|A^k\|_r \leq C, \quad \text{for all } k \geq 0.$$

Finite difference methods

- **Stability in the L^∞ norm**
- L^∞ -norm:

$$\|u^{(k)}\|_\infty := \sup_{1 \leq j \leq N} |u_j^k|.$$

- **Implicit scheme:**

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} + \gamma \frac{-u_{j-1}^{k+1} + 2u_j^{k+1} - u_{j+1}^{k+1}}{(\Delta x)^2} = 0$$

for $k \geq 0$ and $j \in \{1, \dots, N\}$.

- Implicit scheme **well defined**: $u^{(k+1)}$ can be obtained from $u^{(k)}$ by inverting the definite positive matrix

$$\begin{pmatrix} 1 + 2\mu & -\mu & & & \\ -\mu & 1 + 2\mu & -\mu & & \\ & -\mu & 1 + 2\mu & -\mu & \\ & \ddots & \ddots & \ddots & \\ & & -\mu & 1 + 2\mu & -\mu \\ & & & -\mu & 1 + 2\mu \end{pmatrix}.$$

Finite difference methods

- **THEOREM:**

- (i) **Explicit scheme:** **stable** with respect to the L^∞ **norm** iff the **Courant-Friedrichs-Lewy** (CFL) condition holds:

$$2\gamma\Delta t \leq (\Delta x)^2.$$

- (ii) **Implicit scheme:** **unconditionally stable** with respect to the L^∞ **norm**.

Finite difference methods

- Stability in the L^2 norm
- Consider the heat equation with the **periodic boundary conditions**

$$u(t, x + 1) = u(t, x) \quad \text{for all } x \in [0, 1], \quad t \geq 0.$$

- For any $u^{(k)} = (u_j^k)_{j=0, \dots, N}$, we associate a piecewise constant function $u^{(k)}(x)$, periodic with period 1, defined on $[0, 1]$ by

$$u^{(k)}(x) := u_j^k \quad \text{for } x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}},$$

$$x_{j+\frac{1}{2}} = (j + \frac{1}{2})\Delta x, \quad j = 0, \dots, N, \quad x_{-\frac{1}{2}} = 0, x_{N+1+\frac{1}{2}} = 1.$$

Finite difference methods

- **Fourier series** of $u^{(k)}$:

$$u^{(k)}(x) = \sum_{n \in \mathbb{Z}} \hat{u}_n^{(k)} e^{2\pi i n x},$$

$$\hat{u}_n^{(k)} := \int_0^1 u^{(k)}(x) e^{-2\pi i n x} dx.$$

- **Plancherel's** formula \Rightarrow

$$\int_0^1 |u^{(k)}(x)|^2 dx = \sum_{n \in \mathbb{Z}} |\hat{u}_n^{(k)}|^2.$$

- **Property of Fourier series of periodic functions:**

$$v^{(k)}(x) = u^{(k)}(x + \Delta x) \Rightarrow \hat{v}_n^{(k)} = \hat{u}_n^{(k)} e^{2\pi i n \Delta x}.$$

Finite difference methods

- Explicit scheme:

$$\frac{u^{k+1}(x) - u^k(x)}{\Delta t} + \gamma \frac{-u^k(x - \Delta x) + 2u^k(x) - u^k(x + \Delta x)}{(\Delta x)^2} = 0.$$

- Fourier transform \Rightarrow

$$\hat{u}_n^{(k+1)} = \left(1 - \frac{\gamma \Delta t}{(\Delta x)^2} (e^{-2\pi i n \Delta x} + 2 - e^{2\pi i n \Delta x}) \right) \hat{u}_n^{(k)}.$$

Finite difference methods

- Equivalently,

$$\hat{u}_n^{(k+1)} = \alpha(n)\hat{u}_n^{(k)} = \alpha(n)^{k+1}\hat{u}^{(0)}(n)$$

with

$$\alpha(n) := 1 - \frac{4\gamma\Delta t}{(\Delta x)^2}(\sin(\pi n\Delta x))^2.$$

- $\Rightarrow \hat{u}_n^{(k)}$: bounded as $k \rightarrow +\infty$ iff the **amplification factor** $\alpha(n)$ satisfies

$$|\alpha(n)| \leq 1 \quad \text{for all } n \in \mathbb{Z}.$$

Finite difference methods

- **Implicit scheme:**

$$\frac{u^{k+1}(x) - u^k(x)}{\Delta t} + \gamma \frac{-u^{k+1}(x - \Delta x) + 2u^{k+1}(x) - u^{k+1}(x + \Delta x)}{(\Delta x)^2} = 0.$$

- **Fourier transform** \Rightarrow

$$\hat{u}_n^{(k+1)} = \beta(n) \hat{u}_n^{(k)} = \beta(n)^{k+1} \hat{u}^{(0)}(n),$$

$$\beta(n) := \left(1 + \frac{4\gamma\Delta t}{(\Delta x)^2} (\sin(\pi n \Delta x))^2 \right)^{-1}.$$

Finite difference methods

- **THEOREM:**

- (i) **Explicit** scheme: **stable** with respect to the L^2 norm iff the **CFL condition**

$$2\gamma\Delta t \leq (\Delta x)^2$$

holds.

- (ii) **Implicit** scheme: **unconditionally stable** with respect to the L^2 norm.

Finite difference methods

- **Convergence**
- **THEOREM:** Lax theorem
 - u : smooth solution of the heat equation.
 - Suppose that the finite difference scheme for computing the numerical solution u_j^k : **linear, consistent, and stable with respect to the norm $\| \cdot \|_r$.**
 - Let $e_j^k := u_j^k - u(t_k, x_j)$ and $e^{(k)} = (e_1^k, e_2^k, \dots, e_N^k)^\top$.
 - Assume that $u_j^0 = u_0(x_j)$.

Finite difference methods

- Then,

$$\lim_{\Delta t, \Delta x \rightarrow 0} \left(\sup_{t_k \leq T} \|e^{(k)}\|_r \right) = 0 \quad \text{for all } T > 0.$$

- Moreover, if the scheme: **of order p in time and q in space**, then there exists a constant $C_T > 0$ s.t.

$$\sup_{t_k \leq T} \|e^{(k)}\|_r \leq C_T ((\Delta t)^p + (\Delta x)^q).$$

Finite difference methods

- **PROOF:**

- $u^{(k+1)} = Au^{(k)}$; A : iteration matrix; $\tilde{u}_j^k = u(t_k, x_j)$.
- **Consistency** \Rightarrow there exists $\epsilon^{(k)}$ s.t.

$$\tilde{u}^{(k+1)} = A\tilde{u}^{(k)} + (\Delta t)\epsilon^{(k)} \quad \text{and} \quad \lim_{\Delta t, \Delta x \rightarrow 0} \|\epsilon^{(k)}\|_r = 0,$$

uniformly in k .

- Scheme: **of order p in time and q in space** \Rightarrow

$$\|\epsilon^{(k)}\|_r \leq C((\Delta t)^p + (\Delta x)^q).$$

Finite difference methods

- \Rightarrow

$$e^{(k+1)} = Ae^{(k)} - \Delta t \epsilon^k.$$

- By induction,

$$e^{(k)} = A^k e^{(0)} - \Delta t \sum_{l=1}^k A^{k-l} \epsilon^{l-1}.$$

- **Stability** \Rightarrow

$$\|A^k\|_r \leq C'$$

for some positive constant C' .

- \Rightarrow

$$\|e^{(k)}\|_r \leq (\Delta t)kCC'((\Delta t)^p + (\Delta x)^q) \leq TCC'((\Delta t)^p + (\Delta x)^q).$$

Finite difference methods

- Numerical algorithms for the wave equation
- Consider the wave equation

$$\left\{ \begin{array}{l} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t \geq 0, \\ u(t, x+1) = u(t, x), \quad 0 < x < 1, \quad t \geq 0, \\ u(0, x) = u_0(x), \quad 0 < x < 1, \\ \frac{\partial u}{\partial t}(0, x) = u_1(x), \quad 0 < x < 1, \end{array} \right.$$

$c > 0$: wave speed.

- Suppose

$$(**) \quad \int_0^1 u_1(x) dx = 0.$$

Finite difference methods

- Standard finite difference scheme for solving the wave equation:
 θ -centered scheme

$$\left\{ \begin{array}{l} \frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{(\Delta t)^2} + \theta c^2 \frac{-u_{j-1}^{k+1} + 2u_j^{k+1} - u_{j+1}^{k+1}}{(\Delta x)^2} \\ + (1 - 2\theta)c^2 \frac{-u_{j-1}^k + 2u_j^k - u_{j+1}^k}{(\Delta x)^2} + \theta c^2 \frac{-u_{j-1}^{k-1} + 2u_j^{k-1} - u_{j+1}^{k-1}}{(\Delta x)^2} = 0, \end{array} \right.$$

$$0 \leq \theta \leq 1/2.$$

Finite difference methods

- If $\theta = 0$, then the scheme: **explicit**;
- Scheme: **implicit** if $\theta \neq 0$.
- **Initial conditions** expressed by

$$u_j^0 = u_0(x_j) \quad \text{and} \quad \frac{u_j^1 - u_j^0}{\Delta t} = \int_{x_{j-1/2}}^{x_{j+1/2}} u_1(x) dx;$$

- $\Rightarrow (**)$: **satisfied by the numerical solution.**

Finite difference methods

- **THEOREM:**

- If $1/4 \leq \theta \leq 1/2$, then the θ -centered scheme: **unconditionally stable** with respect to the L^2 norm.
- If $0 \leq \theta < 1/4$, scheme: **stable provided** that the **CFL condition**

$$\frac{c\Delta t}{\Delta x} < \sqrt{\frac{1}{1-4\theta}}$$

holds and **unstable** if $c\Delta t/\Delta x > 1/\sqrt{1-4\theta}$.