Lecture 3: Linear systems

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Linear systems

- Linear systems:
  - Exponential of a matrix;
  - Linear systems with constant coefficients;
  - Linear system with non-constant real coefficients;
  - Second order linear equations;
  - Linearization and stability for autonomous systems.

Linear systems

- Exponential of a matrix:
  - $\mathbb{M}_d(\mathbb{C})$: vector space of $d \times d$ matrices with entries in $\mathbb{C}$.
  - $GL_d(\mathbb{C}) \subset \mathbb{M}_d(\mathbb{C})$: group of invertible matrices.
  - DEFINITION: Matrix norm

$$\|A\| = \max_{|y|=1} |Ay|.$$

Linear systems

- LEMMA: Properties of the norm
  - $|Ay| \leq \|A\| \, |y|$ for all $y \in \mathbb{C}^d$;
  - $\|A + B\| \leq \|A\| + \|B\|$ for all $A, B \in \mathbb{M}_d(\mathbb{C})$;
  - $\|AB\| \leq \|A\| \, \|B\|$ for all $A, B \in \mathbb{M}_d(\mathbb{C})$.

Linear systems

- LEMMA: Jordan-Chevalley decomposition
  - $A \in \mathbb{M}_d(\mathbb{C})$.
  - There exists $C \in GL_d(\mathbb{C})$ s.t. A has a unique decomposition

$$C^{-1}AC = D + N;$$

  - D: Diagonal; N: Nilpotent (i.e., $N^d = 0$);

$$ND = DN.$$

Linear systems

- Exponential of a matrix.
- DEFINITION:
    - For $A \in \mathbb{M}_d(\mathbb{C})$,

$$e^A = \sum_{n \geq 0} \frac{A^n}{n!}.$$

Linear systems

- Properties:
  - Exponential of the sum: $A, B \in \mathbb{M}_d(\mathbb{C})$,

$$\text{If } AB = BA \Rightarrow e^{A+B} = e^A e^B.$$

  - Conjugation and exponentiation:
    - $A, B \in \mathbb{M}_d(\mathbb{C})$ and $C \in GL_d(\mathbb{C})$ s.t. $A = C^{-1}BC$.
    - $$e^A = C^{-1}e^B C.$$

    - PROOF:

$$e^A = \sum_{n\geq 0} \frac{A^n}{n!} = \sum_{n\geq 0} \frac{(C^{-1}BC)^n}{n!} = \sum_{n\geq 0} \frac{C^{-1}B^nC}{n!} = C^{-1}e^BC;$$

Linear systems

- Exponential of a diagonalizable matrix:
  - A: diagonalizable

$$A = C^{-1} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix} C;$$

  - $\lambda_1, \dots, \lambda_d \in \mathbb{C}$ and $C \in GL_d(\mathbb{C})$.
  - $\Rightarrow$

$$e^A = C^{-1} \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_d} \end{pmatrix} C.$$

Linear systems

- Exponential of a block matrix:
  - $A_j \in \mathbb{M}_{h_j}(\mathbb{C})$ for $j = 1, ..., p$; A: block matrix of the form

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix}.$$

  -

$$e^A = \begin{pmatrix} e^{A_1} & & 0 \\ & \ddots & \\ 0 & & e^{A_p} \end{pmatrix}.$$

Linear systems

- Derivative: $A \in \mathbb{M}_d(\mathbb{C})$,

$$\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A.$$

Linear systems

- Linear systems with constant coefficients
  - $A \in \mathbb{M}_d(\mathbb{C}))$: **independent** of t.
  - $f \in \mathcal{C}^0([0, T])$.
  - Linear ODE with constant coefficients:

$$(*) \quad \begin{cases} \dfrac{dx}{dt} = Ax(t) + f(t), \quad t \in [0, T], \\ x(0) = x_0 \in \mathbb{R}^d. \end{cases}$$

  - $$|A(x-y)| \leq \|A\||x-y| \quad \text{for all } x, y \in \mathbb{C}^d,$$
  - Cauchy-Lipschitz theorem $\Rightarrow$ there exists a unique solution x to $(*)$.
  - $(*)$ **autonomous** system of equations.

Linear systems

- If $d = 1$ (i.e., $A = a \in \mathbb{C}$), then by the method of integrating factors,

$$x(t) = e^{at} x_0 + \int_0^t e^{a(t-s)} f(s) ds.$$

- General case ($d \geq 1$), if $f = 0$,

$$x(t) = e^{tA} x_0.$$

Linear systems

- For an arbitrary f,

$$\frac{d}{dt}(e^{-tA}x) = e^{-tA}f(t),$$

- $\Rightarrow$

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}f(s)ds.$$

Linear systems

- Linear system with non-constant real coefficients
  - Homogeneous case;
  - Inhomogeneous case.
- Homogeneous case:
  - $\mathbb{M}_d(\mathbb{R})$: vector space of $d \times d$ matrices with entries in $\mathbb{R}$.
  - PROPOSITION:
    - $A : [0, T] \to \mathbb{M}_d(\mathbb{R})$: continuous.
    - S: linear subspace of $\mathcal{C}^1([0, T]; \mathbb{R}^d)$ of dimension d:

$$S = \left\{ x \in \mathcal{C}^1([0, T]; \mathbb{R}^d) : x \text{ satisfies } \frac{dx}{dt} = A(t)x \right\}$$

Linear systems

- PROOF:
  - $x, y \in S \Rightarrow$ for any $\alpha, \beta \in \mathbb{R}$, $\alpha x + \beta y \in \mathcal{C}^1([0, T]; \mathbb{R}^d)$: also a solution.
  - $\Rightarrow S$: linear subspace of $\mathcal{C}^1([0, T]; \mathbb{R}^d)$.

Linear systems

- Dimension of $S = d$:
  - Define $F : S \to \mathbb{R}^d$ by $F[x] = x(t_0)$ for some $t_0 \in [0, T]$.
  - F: linear:

$$F[\alpha x + \beta y] = \alpha x(t_0) + \beta y(t_0) = \alpha F[x] + \beta F[y].$$

  - F: injective,

$$F[x] = 0 \quad \Rightarrow \quad x = 0;$$

    - x solves

$$\frac{dx}{dt} = A(t)x(t)$$

      with the initial condition $x(t_0) = 0$.
    - Cauchy-Lipschitz theorem $\Rightarrow x = 0$.

Linear systems

- F: surjective: for any $x_0 \in \mathbb{R}^d$,

$$\begin{cases} \frac{dx}{dt} = A(t)x(t), \quad t \in [0, T], \\ x(t_0) = x_0, \end{cases}$$

has a solution $x \in \mathcal{C}^1([0, T]; \mathbb{R}^d)$.

Linear systems

- PROPOSITION:
  - $x_1, \ldots, x_d \in S$;
  - $[x_1, \ldots, x_d]$: $d \times d$ matrix with columns $x_1, \ldots, x_d \in \mathbb{R}^d$;
  - det: determinant of a matrix;
  - Equivalent statements:

    (i) $\{x_1, ..., x_d\}$: basis of S;

    (ii) $\det[x_1(t), ..., x_d(t)] \neq 0$ for all $t \in [0, T]$.

    (iii) $\det[x_1(t_0), ..., x_d(t_0)] \neq 0$ for some $t_0 \in [0, T]$.

Linear systems

- PROOF:
  - (i) $\Leftrightarrow$ (ii).
  - (i) $\Rightarrow$ (iii): $\{x_1, ..., x_d\}$: basis of $S \Rightarrow \{F[x_1], ..., F[x_d]\}$: basis of $\mathbb{R}^d$.
  - (iii) $\Rightarrow$ (i): t_0 s.t. (iii) holds; $F : S \to \mathbb{R}^d$: isomorphism relative to t_0.
  - $F^{-1} : \mathbb{R}^d \to S$: isomorphism $\Rightarrow$ $x_1 = F^{-1}[x_1(t_0)], \ldots, x_d = F^{-1}[x_d(t_0)]$: basis of S.

Linear systems

- DEFINITION: Fundamental matrix
    - If (i), (ii) or (iii): holds $\Rightarrow x_1, \ldots, x_d$: fundamental system of solutions of the differential equation $\frac{dx}{dt} = A(t)x$.
    - $X = [x_1, \ldots, x_d]$: fundamental matrix of the equation.

Linear systems

- DEFINITION: Wronskian determinant
  - $x_1, ..., x_d \in S$.
  - Wronskian determinant $w \in \mathcal{C}^1([0, T]; \mathbb{R})$ of $x_1, \ldots, x_d$:

$$w(t) = \det[x_1(t), ..., x_d(t)].$$

Linear systems

- THEOREM:
  - $x_1, ..., x_d \in S$; $w \in \mathcal{C}^1([0,T];\mathbb{R}^d)$: Wronskian determinant of $x_1, \ldots, x_d$.
  - w solves the differential equation

$$(**) \quad \frac{dw}{dt} = (\mathrm{tr} A(t))w \quad \text{for } t \in [0,T].$$

  - $\mathrm{t}r$: trace of a matrix.

Linear systems

- PROOF:
  - If $x_1, \ldots, x_d$: linearly dependent $\rightarrow w = 0$ and $(**)$ trivially holds.
  - Suppose that $x_1, \ldots, x_d$: linearly independent, i.e., $w(t) \neq 0$ for all $t \in [0, T]$.
  - $X : [0, T] \rightarrow \mathbb{M}_d(\mathbb{R})$: fundamental matrix having as columns the solutions $x_1, \ldots, x_d$, i.e.,

$$X(t) = (x_{ij}(t))_{i,j=1,\ldots,d}, \quad t \in [0, T],$$

$x_j = (x_{1j}, \ldots, x_{dj})^\top$ for $j = 1, \ldots, d$.

Linear systems

- z_j: solution of

$$\begin{cases} \frac{dz_j}{dt} = A(t)z_j(t), \\ z_j(t_0) = e_j, \end{cases}$$

$\{e_j\}_{j=1,\ldots,d}$: standard unit orthonormal basis in $\mathbb{R}^d$.

Linear systems

- $\Rightarrow \{z_1, \ldots, z_d\}$: basis of the space of solutions to $dz/dt = Az$.
- There exists $C \in GL_d(\mathbb{R}^d)$ s.t.

$$X(t) = CZ(t), \quad t \in [0, T],$$

  $Z = [z_1, \ldots, z_d]$.
- $v(t) := \det Z(t)$ solves

$$\frac{dv}{dt}(t_0) = \mathrm{tr} A(t_0).$$

- $Z(t_0) = I \Rightarrow v(t_0) = 1$.

Linear systems

- Definition of the determinant of a matrix $\Rightarrow$

$$\frac{dv}{dt}(t) = \frac{d}{dt}\sum_{\sigma \in S_d} (-1)^{\operatorname{sgn}\sigma} \prod_{i=1}^{d} z_{i\sigma(i)}(t)$$

$$= \sum_{\sigma \in S_d} (-1)^{\operatorname{sgn}\sigma} \sum_{j=1}^{d} \frac{d}{dt} z_{j\sigma(j)}(t) \prod_{i \neq j} z_{i\sigma(i)}(t);$$

S_d: set of all permutations of the d elements $\{1, 2, \ldots, d\}$; $\operatorname{sgn}\sigma$: signature of the permutation σ.

Linear systems

- $$\prod_{i \neq j} z_{i\sigma(i)}(t_0) = 0 \quad \text{unless } \sigma = \text{identity};$$

- $$\begin{aligned} \frac{dz_{jj}}{dt}(t_0) &= (A(t_0) z_j(t_0))_j \\ &= \sum_{h=1}^{d} a_{jh}(t_0) z_{hj}(t_0) = \sum_{h=1}^{d} a_{jh}(t_0) \delta_{hj}(t_0) \\ &= a_{jj}(t_0). \end{aligned}$$

- $\Rightarrow$
$$\frac{dv}{dt}(t_0) = \sum_{j=1}^{d} a_{jj}(t_0) = \mathrm{tr} A(t_0).$$

Linear systems

- Differentiation of

$$w = \det X = \det(CZ) = (\det C) \det Z = (\det C) v;$$

- $\Rightarrow$

$$\frac{dw}{dt}(t_0) = (\det C) \frac{dv}{dt}(t_0) = (\det C) \mathrm{tr} A(t_0).$$

- $v(t_0) = 1 \Rightarrow$

$$\frac{dw}{dt}(t_0) = \mathrm{tr} A(t_0) w(t_0).$$

Linear systems

- REMARK:
  - $t_0 \in [0, T]$.
  - Abel's identity or Liouville's formula:

$$w(t) = w(t_0) e^{\int_{t_0}^{t} \mathrm{tr} A(s)\, ds} \quad \text{for } t \in [0, T].$$

  - It suffices to check that the determinant of the fundamental matrix: nonzero for one $t_0 \in [0, T]$.

Linear systems

- Inhomogeneous case
- Inhomogeneous linear differential equation:

$$(***) \quad \left\{ \frac{dx}{dt} = A(t)x + f(t); \right.$$

- $A(t) \in \mathcal{C}^0([0, T]; \mathbb{M}_d(\mathbb{R}))$ and $f \in \mathcal{C}^0([0, T]; \mathbb{R}^d)$.
- X: fundamental matrix for the homogeneous equation $dx(t)/dt = A(t)x(t)$,

$$\frac{dX}{dt} = AX \quad \text{and} \quad \det X \neq 0 \quad \text{for all } t \in [0, T].$$

- Any solution x to the homogeneous equation:

$$x(t) = X(t)c, \quad t \in [0, T],$$

for some (column) vector $c \in \mathbb{R}^d$.

Linear systems

- Method of integrating factors: $c \in \mathcal{C}^1([0, T]; \mathbb{R}^d)$.
- $$\frac{dx}{dt} = \frac{dX}{dt} c + X \frac{dc}{dt} = AXc + X \frac{dc}{dt} = Ax + X \frac{dc}{dt}.$$
- $\Rightarrow X \frac{dc}{dt} = f(t)$.
- X: invertible $\Rightarrow$
$$\frac{dc}{dt} = X^{-1} f(t).$$
- $\Rightarrow$
$$c(t) = c_0 + \int_0^t X(s)^{-1} f(s) ds,$$
for some $c_0 \in \mathbb{R}^d$.

Linear systems

- THEOREM:
  - X: fundamental matrix for the homogeneous equation $dx/dt = Ax$.
  - For all $c_0 \in \mathbb{R}^d$,

$$(****) \quad x(t) = X(t)\big(c_0 + \int_0^t X(s)^{-1} f(s) ds\big)$$

    solution to $(***)$.
  - Any solution to $(***)$: of the form $(****)$ for some $c_0 \in \mathbb{R}^d$.
- Formula $(****)$: Duhamel's formula.

Linear systems

- PROOF:
  - First statement: already proved.
  - Second statement:
    - x_2: solution to $(***)$.
    - $$\frac{d}{dt}(x_2 - x(t)) = A(x_2 - x),$$
    - $\Rightarrow x_2 - x = Xc_1$ for some $c_1 \in \mathbb{R}^d$.

Linear systems

- Second order linear equations
- $d = 1$:

$$\frac{d^2x}{dt^2} = f(t, x, \frac{dx}{dt}),$$

for a given scalar function f.

- Linear ODE if f: linear in x and dx/dt,

$$f(t, x, \frac{dx}{dt}) = g(t) - p(t)\frac{dx}{dt} - q(t)x,$$

g, p, q: functions of t but not of x.

- $$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x = g(t).$$

- Initial conditions:

$$x(t_0) = x_0, \quad \frac{dx}{dt}(t_0) = x_0', \quad x_0, x_0' \in \mathbb{R}^d.$$

Linear systems

- **Homogeneous** if $g = 0$ and **inhomogeneous** otherwise.
- ODE with constant coefficients: $p(t)$ and $q(t)$: constant.
- Suppose $p, q \in \mathcal{C}^0([0, T])$.
  - If NOT: points at which either p or q fail to be continuous: **singular points**.
  - EXAMPLES:

Bessel's equation: $p(t) = \dfrac{1}{t}, q(t) = 1 - \dfrac{\nu}{t^2}, \quad (\text{at } t = 0);$

Legendre's equation: $p(t) = \dfrac{2t}{1-t^2}, q(t) = \dfrac{n(n+1)}{1-t^2}, n \in \mathbb{N}$

(at $t = \pm 1$).

Linear systems

- THEOREM:
    - Suppose that $p, q, g \in \mathcal{C}^0([0, T], \mathbb{R}^d)$.
    - There exists a unique solution $x(t)$ on $[0, T]$.

Linear systems

- Structure of the general solution.
- DEFINITION:
  - Two functions x_1 and x_2 on $[0, T]$: **linearly independent** if neither of them is a multiple of the other.
  - Otherwise, x_1 and x_2 on $[0, T]$: **linearly dependent**.
- PROPOSITION:
  - w: Wronskian determinant

$$w(t) := x_1(t)\frac{dx_2}{dt}(t) - x_2(t)\frac{dx_1}{dt}(t) = \det \begin{pmatrix} x_1 & x_2 \\ \frac{dx_1}{dt} & \frac{dx_2}{dt} \end{pmatrix}.$$

  - $w(t)$: not zero at some $t_0 \in [0, T] \Rightarrow x_1$ and x_2: linearly independent.

Linear systems

- PROOF:
  - Prove: x_1 and x_2: linearly dependent $\Rightarrow w(t) = 0$ for all $t \in [0, T]$.
  - Suppose x_1 and x_2: linearly dependent.
  - Nontrivial solution (α_1, α_2):

$$\begin{cases} \alpha_1 x_1 + \alpha_2 x_2 = 0, \\ \alpha_1 \dfrac{dx_1}{dt} + \alpha_2 \dfrac{dx_2}{dt} = 0, \end{cases} \qquad \text{for all } t \in [0, T],$$

  - $\Rightarrow$

$$w(t) = \det \begin{pmatrix} x_1 & x_2 \\ \dfrac{dx_1}{dt} & \dfrac{dx_2}{dt} \end{pmatrix} = 0, \qquad \text{for all } t \in [0, T].$$

Linear systems

- PROPOSITION:
  - If x_1 and x_2: solutions on $[0, T]$.
  - $w(t)$: either identically zero or not equal to zero at any point of $[0, T]$.
- PROOF:
  - $$w'(t) = x_1 \frac{d^2 x_2}{dt^2} - x_2 \frac{d^2 x_1}{dt^2}.$$
  - x_1, x_2: solutions $\Rightarrow$
    $$\frac{d^2 x_i}{dt^2} = -p(t)\frac{dx_i}{dt} - q(t)x_i, \quad i = 1, 2.$$
  - $\Rightarrow$
    $$\frac{dw}{dt} = -p(t)(x_1 \frac{dx_2}{dt} - \frac{dx_1}{dt} x_2) = -p(t)w(t).$$
  - $w(t) = w(t_0)e^{-\int_{t_0}^{t} p(s)ds}$: either identically zero or never vanishes depending on $w(t_0)$.

Linear systems

- Structure of the general solution to the homogeneous system.
- THEOREM:
    - Suppose that x_1 and x_2: solutions for $g = 0$.
    - Suppose that x_1 and x_2: linearly independent.
    - General solution: of the form $c_1 x_1 + c_2 x_2$; c_1 and c_2: constant coefficients.

Linear systems

- PROOF:
  - $\tilde{x}$: arbitrary solution with the initial condition $\tilde{x}(t_0) = \tilde{x}_0, d\tilde{x}/dt(t_0) = \tilde{x}_0'$.
  - Consider the system of equations for (c_1, c_2)

$$\begin{cases} c_1 x_1(t_0) + c_2 x_2(t_0) = \tilde{x}_0, \\ c_1 \frac{dx_1}{dt}(t_0) + c_2 \frac{dx_2}{dt}(t_0) = \tilde{x}_0'. \end{cases}$$

  - $x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} \neq 0$ at $t = t_0 \Rightarrow$ there exists a unique nontrivial solution $(c_1, c_2) = (\tilde{c}_1, \tilde{c}_2)$.
  - Existence and uniqueness theorem for the initial value problem of the second order ODE $\Rightarrow \tilde{c}_1 x_1 + \tilde{c}_2 x_2 = \tilde{x}$.

Linear systems

- Linear n-th order ODE with constant coefficients
- Solve a linear n-th order ODE with constant coefficients.
- Consider
$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 x = 0,$$
$a_i \in \mathbb{R}$ for $i = 0, \ldots, n-1$.
- General solution:
$$x(t) = c_1 x_1 + \ldots + c_n x_n;$$
- $\{x_i\}_{i=1}^n$: set of linearly independent solutions (a fundamental set of solutions) and c_i: constant coefficients.

Linear systems

- $W(t)$: Wronskian determinant of the set $\{x_1, ..., x_n\}$,

$$W(t) = \det \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ \frac{dx_1}{dt} & & & \\ \vdots & & & \vdots \\ \frac{d^{n-1}}{dt^{n-1}}x_1 & \frac{d^{n-1}}{dt^{n-1}}x_2 & \dots & \frac{d^{n-1}}{dt^{n-1}}x_n \end{bmatrix}.$$

- If $w(t_0) \neq 0$ for some $t_0 \Rightarrow (x_1, ..., x_n)$ forms a fundamental set of solution.
- Solve the equation through the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0.$$

- Equation: derived by guessing a solution $x(t) = e^{\lambda t}$ with $\lambda \in \mathbb{C}$.

Linear systems

- Characteristic equation: n complex roots $\hat{\lambda}_j$ counted with their multiplicities l_j.
- Rewritten in the form
$$\prod_{j=1}^{m}(\lambda - \hat{\lambda}_j)^{l_j} = 0$$
with $\sum_{j=1}^{m} l_j = n$.
- General solution $x(t)$: linear combination of $t^k e^{\hat{\lambda}_j t}$ for $0 \leq k < l_j$ and $j = 1, \ldots, m$.
- In particular, if $m = n$, then $x(t)$: linear combination of $e^{\hat{\lambda}_j t}$.

Linear systems

- THEOREM:
  - $\hat{\lambda}_j, 1 \leq j \leq m$,: zeros of the characteristic polynomial;
  - l_j: corresponding multiplicities.
  - n linearly independent solutions:

$$x_{j,k}(t) = t^k e^{\hat{\lambda}_j t}, \quad 0 \leq k < l_j, \quad 1 \leq j \leq m.$$

  - Any other solution can be written as a linear combination of these solutions.

Linear systems

- Reduction of order
- Method for finding a second solution to the homogeneous second order ODE when a first solution: known by reducing the order.
- x_1: a solution.
- $$x(t) = v(t)x_1(t).$$
- $$\frac{dx}{dt}(t) = \frac{dv}{dt}x_1 + v\frac{dx_1}{dt}$$
- $$\frac{d^2x}{dt^2}(t) = \frac{d^2v}{dt^2}x_1 + 2\frac{dv}{dt}\frac{dx_1}{dt} + v\frac{d^2x_1}{dt^2}.$$

Linear systems

- $\Rightarrow$

$$\frac{d^2 v}{dt^2} + (p + 2\frac{(dx_1/dt)}{x_1})\frac{dv}{dt} = 0.$$

- $u = dv/dt \Rightarrow$ first order ODE

$$\frac{du}{dt} + (p + 2\frac{(dx_1/dt)}{x_1})u = 0.$$

- $\Rightarrow$

$$u(t) = ce^{-\int^t (p+2\frac{(dx_1/dt)}{x_1})ds} = \frac{c}{(x_1(t))^2} e^{-\int^t p(s)ds}.$$

- $v = \int^t u(s)ds \Rightarrow$

$$x(t) = x_1(t)\int^t u(s)ds.$$

Linear systems

- Example:
  - $$\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} - 2x = 0.$$
  - $x_1(t) = e^{t^2}$: solution.
  - $x(t) = e^{t^2} v(t)$:
    $$\frac{d^2v}{dt^2} + 2t\frac{dv}{dt} = 0.$$
  - Solution:
    $$\frac{dv}{dt} = e^{-t^2},$$
  - $\Rightarrow$
    $$v(t) = \int_0^t e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(t),$$
  - erf: Gauss error function.
  - $\Rightarrow$
    $$x_2(t) = e^{t^2}\,\mathrm{erf}(t).$$

Linear systems

- Linearization and stability for autonomous systems:
    - Linear systems;
    - Nonlinear systems.

Linear systems

- Linear systems
- $A \in \mathbb{M}_d(\mathbb{R})$: independent of t.
- Linear system of ODEs:

$$\begin{cases} \frac{dx}{dt} = Ax(t), \quad t \in [0, +\infty[, \\ x(0) = x_0 \in \mathbb{R}^d. \end{cases}$$

- There exists $C \in GL_d(\mathbb{C})$ s.t.

$$C^{-1}AC = D + N,$$

  where D is diagonal, N is nilpotent, and $ND = DN$.
- $\lambda_j, j = 1, \ldots, J$: (distinct) eigenvalues of A.
- m_j: multiplicity of λ_j; $E_j = \ker(A - \lambda_j I)^{m_j}$: characteristic subspace associated with λ_j.
- $\oplus E_j = \mathbb{C}^d$.

Linear systems

- DEFINITION:
- Linear system: **stable** if there exists a positive constant C_0 s.t.

$$|x(t)| \leq C_0 |x_0| \quad \text{for all } t \in [0, +\infty[.$$

- LEMMA:
- Linear system: **stable** iff $\Re\lambda_j < 0$ or $\Re\lambda_j = 0$ and $N|_{E_j} = 0$ for $j = 1, \ldots, J$.

Linear systems

- PROOF:
  - $\widetilde{x}(t) = Cx(t)$ and $\widetilde{x}_0 = Cx_0$.
  - $\widetilde{x}(t) = e^{tD+tN}\widetilde{x}_0, \quad t \in [0, +\infty[$.
  - $DN = ND \Rightarrow$

$$\widetilde{x}(t) = \Big(\sum_{i=0}^{d-1} \frac{(tN)^i}{i!}\Big) e^{tD}\widetilde{x}_0, \quad t \in [0, +\infty[.$$

  - $\widetilde{x}_0$ belongs to the vector eigenspace associated with the eigenvalue $\lambda_j \Rightarrow$

$$\widetilde{x}(t) = e^{t\lambda_j}\Big(\sum_{i=0}^{d-1} \frac{(tN)^i}{i!}\Big)\widetilde{x}_0, \quad t \in [0, +\infty[.$$

  - $\Rightarrow x(t)$ satisfies the stability estimate for some positive constant C_0 iff $\Re\lambda_j < 0$ or $\Re\lambda_j = 0$ and $N|_{E_j} = 0$.

Linear systems

- Nonlinear systems
- Autonomous system: $f \in \mathcal{C}^1$,

$$\begin{cases} \frac{dx}{dt} = f(x), \\ x(0) = x_0 \in \mathbb{R}^d, \end{cases}$$

- $f(x^*) = 0$: x^*: an equilibrium point.

Linear systems

- THEOREM: Local stability
  - Suppose that all the eigenvalues λ of the Jacobian $f'(x^*)$ of f at x^*: negative real parts.
  - There exists $\delta > 0$ s.t. if $|x_0 - x^*| \leq \delta$, then

$$|x(t) - x^*| \to 0 \quad \text{as } t \to +\infty.$$

Linear systems

- PROOF:
  - Linearized system: $A = f'(x^*)$

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t), \quad t \geq 0, \\ y(0) = x_0 - x^*. \end{cases}$$

  - Explicit solution $y(t) = e^{tA}(x_0 - x^*)$ for $t \geq 0$.
  - $\Re\lambda < 0$ for any eigenvalue λ of $f'(x^*)$: negative real parts.
  - There exists $r > 0$ s.t.

$$|e^{tA}z| \leq C_0 e^{-rt}|z| \quad \text{for all } z \in \mathbb{R}^d,$$

C_0: depends only on f.

Linear systems

- Small perturbation of the linearized system:

$$\begin{cases} \dfrac{dx}{dt} = A(x - x^*) + g(x), \\ x(0) = x^*, \end{cases}$$

- $$g(x) = |x - x^*|\epsilon(x), \quad \text{with } \epsilon \in \mathcal{C}^0 \quad \text{and } \epsilon(x^*) = 0.$$

Linear systems

- There exists $\delta_0 > 0$ s.t. for all $\delta \in]0, \delta_0[$,

$$\sup\{|g(x)| : |x - x^*| \leq \delta\} < \frac{r\delta}{C_0}.$$

- It suffices to prove that if $|x_0 - x^*| < \min(\delta, \delta/C_0)$, then

$$|x(t) - x^*| \leq \delta \quad \text{for all } t \geq 0.$$

- $$x(t) - x^* = e^{tA}(x_0 - x^*) + \int_0^t e^{(t-s)A} g(x(s))\, ds,$$

- $\Rightarrow$

$$\begin{aligned} |x(t) - x^*| &\leq e^{-rt}\left(C_0|x_0 - x^*| + \int_0^t e^{-r(t-s)} C_0 |g(x(s))|\, ds\right) \\ &\leq e^{-rt}\left(C_0|x_0 - x^*| + (1 - e^{-rt})\frac{C_0}{r} \sup\{|g(x(s)| : 0 \leq s \leq t\}\right). \end{aligned}$$

Linear systems

- For all $t \geq 0$,

$$|x(t) - x^*| \leq \max\left(C_0|x_0 - x^*|, \frac{C_0}{r}\sup\{|g(x(s))| : 0 \leq s \leq t\}\right).$$

- Introduce

$$T := \inf\{t > 0 : |x(t) - x^*| \geq \delta\}.$$

- Assume that T: finite $\Rightarrow$

$$|x(t) - x^*| \leq \delta \quad \text{for all } t \in [0, T], \quad |x(T) - x^*| = \delta.$$

- $\Rightarrow$ Contradiction.

Linear systems

- DEFINITION:
  - A function $V \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$: **Lyapunov function** for the ODE if
    - $$V(x^*) < V(x) \quad \text{for any } x \neq x^*;$$
    - $$f(x) \cdot V'(x) \leq 0 \quad \text{for any } x \in \mathbb{R}^d.$$

Linear systems

- EXAMPLE:

  (i) Consider

$$\begin{cases} \dfrac{dx_1}{dt} = x_2, \\ \dfrac{dx_2}{dt} = -2x_1 - x_2. \end{cases}$$

  - $x* = (0,0)$: equilibrium point and

$$V(x) = x_1^2 + \frac{1}{2}x_2^2 : \quad \text{Lyapunov function.}$$

Linear systems

(ii) Suppose that $f(x) = -\nabla \Phi(x)$. Suppose that the potential Φ: smooth and there exists x^* s.t. $\Phi(x^*) < \Phi(x)$ for any $x \neq x^*$. Then

$$V = \Phi : \quad \text{Lyapunov function}.$$

Linear systems

- THEOREM:
  - Suppose that there exists a Lyapunov function V.
  - $\Rightarrow$ For any $\epsilon > 0$, there exists $\delta > 0$, s.t.

$$\sup_{t \geq 0} |x(t) - x^*| \leq \epsilon$$

provided that $|x_0 - x^*| \leq \delta$.

Linear systems

- PROOF:
  - Condition on V implies that for fixed $\epsilon > 0$, there exists $\gamma > 0$ (sufficiently small) s.t.

$$\left\{x : |x - x^*| \leq 2\epsilon, V(x) \leq V(x^*) + \gamma\right\} \subset \left\{x : |x - x^*| \leq \epsilon\right\}.$$

  - Choose δ $(0 < \delta < \epsilon)$ s.t.

$$\left\{x : |x - x^*| \leq \delta\right\} \subset \left\{x : |x - x^*| \leq 2\epsilon, V(x) \leq V(x^*) + \gamma\right\}.$$

Linear systems

- Fundamental property of a Lyapunov function V:

$$\frac{d}{dt}V(x(t)) = f(x(t)) \cdot V'(x(t)) \leq 0, \quad t \geq 0;$$

- $\Rightarrow$

$$V(x(t)) \leq V(x_0) \leq V(x^*) + \gamma \quad \text{if } |x_0 - x^*| \leq \delta.$$

- $$|x(s) - x^*| \leq 2\epsilon \quad \text{for all } s \geq 0,$$

  since otherwise, there would exist $t > 0$ s.t. $|x(t) - x^*| = 2\epsilon$.
- $V(x(t)) \leq V(x^*) + \gamma \Rightarrow$ contradiction.

Linear systems

- THEOREM: Global stability
  - Suppose that there exists $V \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ satisfying
  $$V(x^*) < V(x) \quad \text{for any } x \neq x^*$$
  s.t.
  $$f(x) \cdot V'(x) < 0 \qquad \text{for any } x \neq x^*.$$
  - Suppose that the set $\{x : V(x) \leq V(x^*)\}$: bounded.
  - $\Rightarrow$ The solution $x(t)$ converges to x^* as $t \to +\infty$.

Linear systems

- PROOF:
  - $V(x(t)) \leq V(x_0) \Rightarrow \{x(t) : t \geq 0\}$: bounded.
  - $$\int_0^{+\infty} \left| f(x(t)) \cdot V'(x(t)) \right| dt$$
    $$= \int_0^{+\infty} -f(x(t)) \cdot V'(x(t))\, dt \leq V(x_0) - V^*;$$
    $V^* := \lim_{t \to +\infty} V(x(t))$.
  - $(x(t))_{t \geq 0}$: bounded $\Rightarrow V^* > -\infty$.

Linear systems

- $(t_n)_{n\in\mathbb{N}}$ s.t. $x(t_n) \to \widetilde{x}$ and $f(x(t_n)) \cdot V'(x(t_n)) \to 0$ as $n \to +\infty$.
- $\Rightarrow$
$$f(\widetilde{x}) \cdot V'(\widetilde{x}) = 0;$$
- $\Rightarrow \widetilde{x} = x^*$.