

# Lecture 2: Existence, uniqueness, and regularity in the Lipschitz case

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# Existence, uniqueness, and regularity

- **Banach fixed point theorem**
- **DEFINITION: Contraction** Let
  - $(X, d)$ : metric space.
  - $F : X \rightarrow X$ : **contraction** if there exists  $0 < \lambda < 1$  s.t. for all  $x, y \in X$

$$d(F(x), F(y)) \leq \lambda d(x, y).$$

- **THEOREM: Banach fixed point theorem**
  - $(X, d)$ : **complete** metric space (i.e., every Cauchy sequence of elements of  $X$ : convergent);
  - $F : X \rightarrow X$ : **contraction**.
  - There exists a **unique**  $x \in X$  s.t.

$$F(x) = x.$$

# Existence, uniqueness, and regularity

- Gronwall's lemma

LEMMA: Gronwall's lemma

- $I = [0, T]; \phi \in C^0(I)$ .
- There exist two constants  $\alpha, \beta \in \mathbb{R}, \beta \geq 0$ , s.t.

$$(*) \quad \phi(t) \leq \alpha + \beta \int_0^t \phi(s) ds \quad \text{for all } t \in I.$$

- $\Rightarrow$

$$\phi(t) \leq \alpha e^{\beta t} \quad \text{for all } t \in I.$$

# Existence, uniqueness, and regularity

- **PROOF:**

- $\varphi : I \rightarrow \mathbb{R}$

$$\varphi(t) := \alpha + \beta \int_0^t \phi(s) ds.$$

- $\phi \in \mathcal{C}^0 \Rightarrow \varphi \in \mathcal{C}^1,$

$$\frac{d\varphi}{dt} = \beta\phi(t) \quad \text{for all } t \in I.$$

- $(*) \Rightarrow$

$$\frac{d\varphi}{dt} \leq \beta\varphi.$$

# Existence, uniqueness, and regularity

- $\psi(t) := \exp(-\beta t)\varphi(t)$  for  $t \in I$ ,

$$\begin{aligned}\frac{d\psi}{dt} &= -\beta e^{-\beta t}\varphi(t) + e^{-\beta t}\frac{d\varphi}{dt} \\ &= e^{-\beta t}\left(-\beta\varphi(t) + \frac{d\varphi}{dt}\right) \leq 0.\end{aligned}$$

- $\psi(0) = \varphi(0) = \alpha \Rightarrow \psi(t) \leq \alpha$  for  $t \in I$ ,

$$\varphi(t) \leq \alpha e^{\beta t};$$

- $\Rightarrow \phi(t) \leq \varphi(t) \leq \alpha e^{\beta t}$  for all  $t \in I$ .

# Existence, uniqueness, and regularity

- **Cauchy-Lipschitz theorem**
- $I = [0, T]$ ;  $d$ : positive integer;  $f : I \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ .
- Suppose that  $f \in C^0(I \times \mathbb{R}^d)$ .
- **DEFINITION: Lipschitz condition**
  - There exists a constant  $C_f \geq 0$  s.t., for any  $x_1, x_2 \in \mathbb{R}^d$  and any  $t \in I$ ,

$$(**) \quad |f(t, x_1) - f(t, x_2)| \leq C_f |x_1 - x_2|.$$

- $f$  satisfies a **Lipschitz condition** on  $I$ .
- $C_f$ : **Lipschitz constant** for  $f$ .

# Existence, uniqueness, and regularity

- **THEOREM: Cauchy-Lipschitz theorem**

- Consider

$$\begin{cases} \frac{dx}{dt} = f(t, x), & t \in [0, T], \\ x(0) = x_0, & x_0 \in \mathbb{R}^d. \end{cases}$$

- If  $f \in \mathcal{C}^0(I \times \mathbb{R}^d)$  satisfies the **Lipschitz condition (\*\*)** on  $[0, T]$ , then there **exists a unique solution**  $x \in \mathcal{C}^1(I)$  on  $[0, T]$ .

# Existence, uniqueness, and regularity

- **PROOF:**

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$$x(t) = x_0 + \int_0^t f(s, x(s)) ds, \quad \forall t \in [0, T].$$

- Define  $F : C^0([0, T]; \mathbb{R}^d) \rightarrow C^0([0, T]; \mathbb{R}^d)$  by

$$F(y) := x_0 + \int_0^t f(s, y(s)) ds.$$

- For  $y \in C^0([0, T]; \mathbb{R}^d)$ , **norm** of  $y$ :

$$\|y\| := \sup_{t \in [0, T]} \{|y(t)| e^{-C_f t}\};$$

- $C_f$ : Lipschitz constant for  $f$ .
- Equivalent to the usual norm  $\sup_{t \in [0, T]} |y(t)| \Rightarrow C^0([0, T]; \mathbb{R}^d)$   
equipped with the new norm: **complete**.



# Existence, uniqueness, and regularity

- Compute

$$\begin{aligned}\|F[y_1] - F[y_2]\| &= \sup_{t \in [0, T]} |F[y_1](t) - F[y_2](t)| e^{-C_f t} \\ &\leq \sup_{t \in [0, T]} e^{-C_f t} \int_0^t |f(s, y_1(s)) - f(s, y_2(s))| ds \\ &\leq \sup_{t \in [0, T]} e^{-C_f t} C_f \int_0^t |y_1(s) - y_2(s)| ds \\ &\leq \sup_{t \in [0, T]} e^{-C_f t} C_f \int_0^t e^{C_f s} e^{-C_f s} |y_1(s) - y_2(s)| ds \\ &\leq \sup_{t \in [0, T]} \{e^{-C_f t} C_f \int_0^t e^{C_f s} ds\} \|y_1 - y_2\| \\ &\leq (1 - e^{-C_f T}) \|y_1 - y_2\|.\end{aligned}$$

# Existence, uniqueness, and regularity

- **Banach fixed point theorem** in a complete metric space  $\Rightarrow$  there exists a unique  $y \in C^0([0, T]; \mathbb{R}^d)$  s.t.  $F(y) = y$ .
- $\Rightarrow$  Existence and uniqueness of a solution.
- **Picard iteration**  $y^{(n+1)} = F[y^{(n)}]$  : **Cauchy sequence** and **converges to the unique fixed point**  $y$ .

# Existence, uniqueness, and regularity

- **REMARK:**
  - **Existence and uniqueness theorem:** holds true if  $\mathbb{R}^d$ : replaced with a **Banach space** (a complete normed vector space).
  - Same proof.

# Existence, uniqueness, and regularity

- **REMARK:**

- If  $f$ : continuous, there is no guarantee that the initial value problem possesses a unique solution.

- **EXAMPLE:**

- Consider

$$\frac{dx}{dt} = x^{\frac{2}{3}}, \quad x(0) = 0.$$

- There are **two solutions** given by  $x_1(t) = \frac{t^3}{27}$  and  $x_2(t) = 0$ .

# Existence, uniqueness, and regularity

- **THEOREM:** Cauchy-Peano existence theorem
  - $f$ : **continuous**.
  - There **exists a solution**  $x(t)$ : at least defined for small  $t$ .
- **PROOF:** Use **Arzela-Ascoli theorem**.

# Existence, uniqueness, and regularity

- **DEFINITION: Equicontinuity**

- A family of functions  $\mathcal{F}$ : **equicontinuous** on  $[a, b]$  if for any given  $\epsilon > 0$ , there exists  $\delta > 0$  s.t.

$$|f(t) - f(s)| < \epsilon$$

whenever  $|t - s| < \delta$  for every function  $f \in \mathcal{F}$  and  $t, s \in [a, b]$ .

- **DEFINITION: Uniform boundedness**

- A family of continuous functions  $\mathcal{F}$  on  $[a, b]$ : **uniformly bounded** if there exists a positive number  $M$  s.t.  $|f(t)| \leq M$  for every function  $f \in \mathcal{F}$  and  $t \in [a, b]$ .

# Existence, uniqueness, and regularity

- **THEOREM:** Arzela-Ascoli
  - Suppose that the sequence of functions  $\{f_n(t)\}_{n \in \mathbb{N}}$  on  $[a, b]$ : **uniformly bounded** and **equicontinuous**.
  - There exists a subsequence  $\{f_{n_k}(t)\}_{k \in \mathbb{N}}$ : **uniformly convergent** on  $[a, b]$ .

# Existence, uniqueness, and regularity

- **EXAMPLE:**

- Consider

$$\frac{dx}{dt} = x^2, \quad x(0) = x_0 \neq 0.$$

- **Separation of variables**  $\Rightarrow$

$$\frac{dx}{x^2} = dt.$$

- $\Rightarrow$

$$-\frac{1}{x} = \int \frac{dx}{x^2} = t + C,$$

- $\Rightarrow$

$$x = -\frac{1}{t + C}.$$

- $x(0) = x_0 \Rightarrow$

$$x(t) = \frac{x_0}{1 - x_0 t}.$$



# Existence, uniqueness, and regularity

- If  $x_0 > 0$ ,  $x(t)$  blows up when  $t \rightarrow \frac{1}{x_0}$  from below.
- If  $x_0 < 0$ , the singularity: in the past ( $t < 0$ ).
- **Only solution defined for all positive and negative  $t$ : constant solution  $x(t) = 0$ , corresponding to  $x_0 = 0$ .**

# Existence, uniqueness, and regularity

- Continuity of the solution.
- THEOREM:
  - $f$  satisfies the Lipschitz condition.
  - $x_1(t)$  and  $x_2(t)$ : two solutions of corresponding to the initial data  $x_1(0)$  and  $x_2(0)$ , respectively.
  - Continuity with respect to the initial data:

$$|x_1(t) - x_2(t)| \leq e^{C_f t} |x_1(0) - x_2(0)| \quad \text{for all } t \in [0, T].$$

# Existence, uniqueness, and regularity

- **PROOF:**

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$$\begin{aligned}\frac{d}{dt}|x_1(t) - x_2(t)|^2 &= 2(f(t, x_1(t)) - f(t, x_2(t)))(x_1(t) - x_2(t)) \\ &\leq 2C_f|x_1(t) - x_2(t)|^2, \quad t \in [0, T],\end{aligned}$$

- $\Rightarrow$

$$\frac{d}{dt} \left( |x_1(t) - x_2(t)|^2 e^{-2C_f t} \right) \leq 0.$$

- Integration from 0 to  $t$ :

$$|x_1(t) - x_2(t)|^2 e^{-2C_f t} \leq |x_1(0) - x_2(0)|^2.$$

- $\Rightarrow$

$$|x_1(t) - x_2(t)| \leq |x_1(0) - x_2(0)| e^{C_f t}.$$

# Existence, uniqueness, and regularity

- Differentiability with respect to the initial data.
- Formal: differentiate the solution  $x$  with respect to the initial data  $\Rightarrow$

$$(***) \quad \begin{cases} \frac{d}{dt} \frac{\partial x(t)}{\partial x_0} = \frac{\partial f}{\partial x}(t, x(t)) \frac{\partial x(t)}{\partial x_0}, \\ \frac{\partial x(t)}{\partial x_0} = 1. \end{cases}$$

- THEOREM:
  - $f \in \mathcal{C}^1$ .
  - $x_0 \mapsto x(t)$ : differentiable and  $\partial x(t)/\partial x_0$ : unique solution of the linear equation (\*\*\*) .

# Existence, uniqueness, and regularity

- **PROOF:**

- $\Delta x(t, x_0, h) := x(t, x_0 + h) - x(t, x_0)$ : **difference quotient**.
- **Mean-value theorem**  $\Rightarrow$

$$\begin{aligned}\Delta x(t, x_0, h) &= h + \int_0^t (f(s, x(s, x_0 + h)) - f(s, x(s, x_0))) ds \\ &= h + \int_0^t (f(s, x(s, x_0) + \Delta x(s, x_0, h)) - f(s, x(s, x_0))) ds \\ &= h + \int_0^t \frac{\partial f}{\partial x}(s, x(s, x_0) + \tau \Delta x) \Delta x ds.\end{aligned}$$

- $\tau = \tau(s, x_0, h) \in [0, 1]$ .

# Existence, uniqueness, and regularity

- There exists a positive constant  $M$  s.t.  $|\frac{\partial f}{\partial x}| < M \Rightarrow$

$$|\Delta x| \leq |h| + M \int_0^t |\Delta x(s, x_0, h)| ds.$$

- Gronwall's lemma  $\Rightarrow$

$$|\Delta x(t, x_0, h)| \leq |h|e^{Mt}.$$

# Existence, uniqueness, and regularity

- $v(t)$ : unique solution of (\*\*\*) .
- Compute

$$\begin{aligned} & \frac{\Delta x(t, x_0, h)}{h} - v(t) \\ &= \int_0^t \left( \frac{f(s, x(s, x_0 + h)) - f(s, x(s, x_0))}{h} - \frac{\partial f}{\partial x}(s, x(s, x_0))v(s) \right) ds \\ &= \int_0^t \frac{\Delta x(s, x_0, h)}{h} \left[ \frac{\partial f}{\partial x}(s, x(s, x_0) + \tau \Delta x(s, x_0, h)) - \frac{\partial f}{\partial x}(s, x(s, x_0)) \right] ds \\ & \quad + \int_0^t \frac{\partial f}{\partial x}(s, x(s, x_0)) \left( \frac{\Delta x(s, x_0, h)}{h} - v(s) \right) ds. \end{aligned}$$

# Existence, uniqueness, and regularity

- **Uniform continuity** of  $\frac{\partial f}{\partial x}$   $\Rightarrow$  For any  $\epsilon > 0$  there exists  $h_0 > 0$  s.t., for any  $|h| \leq h_0$ , the first term on the right-hand side: of order  $O(\epsilon)$ .
- **Gronwall's lemma**  $\Rightarrow$  for  $|h|$  small enough,

$$\left| \frac{\Delta x(t, x_0, h)}{h} - v \right| \leq \epsilon M T e^{MT}.$$

- $\Rightarrow x_0 \mapsto x(t)$ : **differentiable** and its derivative given by

$$\frac{\partial x}{\partial x_0} = v.$$



# Existence, uniqueness, and regularity

- **Stability**

**THEOREM:**

- Two ODEs on  $[0, T]$ :

$$\frac{dx}{dt} = f(t, x) \quad \text{and} \quad \frac{dy}{dt} = g(t, y).$$

- $f$  satisfies the **Lipschitz condition on  $[0, T]$**  and there exists  $\epsilon > 0$  s.t., for any  $x \in \mathbb{R}^d$ ,  $t \in [0, T]$ ,

$$|f(t, x) - g(t, x)| \leq \epsilon.$$

- **Strong continuity:**

$$|x(t) - y(t)| \leq |x(0) - y(0)|e^{C_f t} + \frac{\epsilon}{C_f}(e^{C_f t} - 1), \quad t \in [0, T].$$

# Existence, uniqueness, and regularity

- **REMARK:**  $g$  may not satisfy a Lipschitz condition.
- **PROOF:**
  - Compute:

$$\begin{aligned}\frac{d}{dt}|x(t) - y(t)|^2 &= 2(f(t, x(t)) - g(t, y(t)))(x(t) - y(t)) \\ &= 2(f(t, x(t)) - f(t, y(t)))(x(t) - y(t)) \\ &\quad + 2(f(t, y(t)) - g(t, y(t)))(x(t) - y(t)).\end{aligned}$$

# Existence, uniqueness, and regularity

- $\Rightarrow$

$$\begin{aligned} \frac{d}{dt}|x(t) - y(t)|^2 &\leq \left| \frac{d}{dt}|x(t) - y(t)|^2 \right| \\ &\leq 2|f(t, x(t)) - f(t, y(t))| |x(t) - y(t)| \\ &\quad + 2|f(t, y(t)) - g(t, y(t))| |x(t) - y(t)| \\ &\leq 2C_f |x(t) - y(t)|^2 + 2\epsilon |x(t) - y(t)| \\ &\leq 2C_f |x(t) - y(t)|^2 + 2\epsilon \sqrt{|x(t) - y(t)|^2}. \end{aligned}$$

# Existence, uniqueness, and regularity

- $h(t) := |x(t) - y(t)|^2$ :

$$\frac{dh}{dt} \leq 2C_f h + 2\epsilon\sqrt{h}.$$

- Consider

$$\begin{cases} \frac{du}{dt} = 2C_f u + 2\epsilon\sqrt{u}, \\ u(0) = |x(0) - y(0)|^2. \end{cases}$$

- $C_f > 0, u(0) \geq 0 \Rightarrow \frac{du}{dt}$ : always non-negative when  $t \geq 0$ ;
- $\Rightarrow u$ : **increasing**.

# Existence, uniqueness, and regularity

- Let  $z(t) := \sqrt{u(t)}$  and suppose that  $h(0) > 0$ .

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$$\begin{cases} \frac{dz}{dt} - C_f z = \epsilon, & t \in [0, T], \\ z(0) = \sqrt{u(0)}. \end{cases}$$

- $\Rightarrow$

$$\sqrt{u(t)} = z(t) = \sqrt{u(0)}e^{C_f t} + \frac{\epsilon}{C_f}(e^{C_f t} - 1).$$

# Existence, uniqueness, and regularity

- **Contradiction:** There exists  $t_1 \in [0, T]$  s.t.  $h(t_1) > u(t_1)$ .
- $t_0 := \sup\{t : h(t) \leq u(t)\}$ .
- Continuity:  $h(t_0) = u(t_0)$ .

# Existence, uniqueness, and regularity

- Prove:

$$\frac{d}{dt} \left( (h(t) - u(t)) e^{-(2C_f + \frac{2\epsilon}{\sqrt{u(0)}})t} \right) \leq 0.$$

- $\Rightarrow$  Integration from  $t_0$  to  $t \Rightarrow$

$$(h(t) - u(t)) e^{-(2C_f + \frac{2\epsilon}{\sqrt{u(0)}})t} \leq h(t_0) - u(t_0).$$

- $u(t_0) = h(t_0) \Rightarrow$

$$h(t) \leq u(t) \quad \text{for } t \in [t_0, t_1].$$

- Contradiction.

# Existence, uniqueness, and regularity

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$$\begin{aligned}\frac{d}{dt}(h(t) - u(t)) &\leq 2C_f(h(t) - u(t)) + 2\epsilon(\sqrt{h(t)} - \sqrt{u(t)}) \\ &= 2C_f(h(t) - u(t)) + 2\epsilon \frac{h(t) - u(t)}{\sqrt{h(t)} + \sqrt{u(t)}}.\end{aligned}$$

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$$\begin{aligned}\frac{d}{dt} \left( (h(t) - u(t)) e^{-(2C_f + \frac{2\epsilon}{\sqrt{u(0)}})t} \right) \\ \leq 2(h(t) - u(t)) \left( C_f + \frac{\epsilon}{\sqrt{u(t_0)}} \right).\end{aligned}$$



# Existence, uniqueness, and regularity

- $h(t) \leq u(t)$  for  $t \in [0, T] \Rightarrow$

$$\begin{aligned} |x(t) - y(t)| &\leq \sqrt{u(t)} \\ &= \sqrt{u(0)}e^{C_f t} + \frac{\epsilon}{C_f}(e^{C_f t} - 1) \\ &= \sqrt{h(0)}e^{C_f t} + \frac{\epsilon}{C_f}(e^{C_f t} - 1). \end{aligned}$$

- $\Rightarrow$

$$|x(t) - y(t)| \leq |x(0) - y(0)|e^{C_f t} + \frac{\epsilon}{C_f}(e^{C_f t} - 1).$$

# Existence, uniqueness, and regularity

- If  $h(0) = 0$ :

$$\begin{cases} \frac{du_n}{dt} = 2C_f u_n + 2\epsilon\sqrt{u_n}, & t \in [0, T], \\ u_n(0) = \frac{1}{n}, \end{cases}$$

- **Explicit solution:**

$$u_n(t) = \left[ \frac{1}{\sqrt{n}} e^{C_f t} + \frac{\epsilon}{C_f} (e^{C_f t} - 1) \right]^2.$$

# Existence, uniqueness, and regularity

- Only need to prove that for each  $n \in \mathbb{N}$ ,

$$h(t) \leq u_n(t) \quad \text{for all } t \in [0, T].$$

- Letting  $n \rightarrow +\infty$ ,  $u_n \rightarrow u \Rightarrow h(t) \leq u(t)$ .

# Existence, uniqueness, and regularity

- Proof by **contradiction**:
  - Suppose that there exists  $t_1 > 0$  s.t.  $h(t_1) > u_n(t_1)$ .
  - $t_0$ : the largest  $t$  in  $0 < t \leq t_1$  s.t.  $h(t) \leq u_n(t)$ .
  - **Continuity** of  $h(t)$  and  $u_n(t) \Rightarrow$

$$h(t_0) = u_n(t_0) > 0,$$

and  $h(t) > u_n(t)$  on  $(t_0, t_0 + \epsilon)$ , a small right-neighborhood of  $t_0$ .

- **Impossible** according to the discussion in the **case  $h(0) > 0$** .

# Existence, uniqueness, and regularity

- **Regularity**

- **THEOREM:**

- $f \in \mathcal{C}^n$  for  $n \geq 0$ .
- $\Rightarrow x \in \mathcal{C}^{n+1}$ .

- **PROOF:**

- Proof by **induction**.
- Case  $n = 0$ : clear.
- If  $f \in \mathcal{C}^n$  then  $x$ : at least of class  $\mathcal{C}^n$ , by the inductive assumption.
- The function  $t \mapsto f(t, x(t)) = dx(t)/dt \in \mathcal{C}^n$ .
- $\Rightarrow x(t) \in \mathcal{C}^{n+1}$ .

- **REMARK:**

- $f$ : **real analytic** function  $\Rightarrow x$ : **real analytic**.