Lecture 1: Some basics

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Some basics

- What is a differential equation ?
- Some methods of resolution:
  - Separation of variables;
  - Change of variables;
  - Method of integrating factors.
- Important examples of ODEs:
  - Autonomous ODEs;
  - Exact equations;
  - Hamiltonian systems.

Some basics

- Ordinary differential equation (ODE): equation that contains one or more derivatives of an unknown function $x(t)$.
- Equation may also contain x itself and constants.
- ODE of order n if the n-th derivative of the unknown function is the highest order derivative in the equation.

Some basics

- Examples of ODEs:
  - Membrane equation as a neuron model:

$$C\frac{dx(t)}{dt} + gx(t) = f(t),$$

    $x(t)$: membrane potential, i.e., the voltage difference between the inside and the outside of the neuron; $f(t)$: current flow due to excitation; C: capacitance; g: conductance (the inverse of the resistance) of the membrane.
  - Linear ODE of order 1.

Some basics

- Theta model: one-dimensional model for the spiking of a neuron.

$$\frac{d\theta(t)}{dt} = 1 - \cos\theta(t) + (1 + \cos\theta(t))f(t);$$

  $f(t)$: inputs to the model.

- $\theta \in [0, 2\pi]$; $\theta = \pi$ the neuron spikes $\rightarrow$ produces an action potential.
- Change of variables, $x(t) = \tan(\theta(t)/2)$, $\rightarrow$ quadratic model

$$(*) \quad \frac{dx(t)}{dt} = x^2(t) + f(t).$$

- Population growth under competition for resources:

$$(**) \quad \frac{dx(t)}{dt} = rx(t) - \frac{r}{k}x^2(t);$$

  r and k: positive parameters; $x(t)$: number of cells at time instant t, $rx(t)$: growth rate and $-(r/k)x^2(t)$: death rate.

- $(*)$ and $(**)$: Nonlinear ODEs of order 1.

Some basics

- FitzHugh-Nagumo model:

$$\begin{cases} \dfrac{dV}{dt} = f(V) - W + I, \\ \dfrac{dW}{dt} = a(V - bW); \end{cases}$$

- V: membrane potential, W: recovery variable, and I: magnitude of stimulus current.
- $f(V)$: polynomial of third degree, and a and b: constant parameters.
- FitzHugh-Nagumo model: two-dimensional simplification of the Hodgkin-Huxley model of spike generation in squid giant axons.
- Mathematical properties of excitation and propagation from the electrochemical properties of sodium and potassium ion flow.
- System of nonlinear ODEs of order 1.

Some basics

- Langevin equation of motion for a single particle:

$$\frac{dx(t)}{dt} = -ax(t) + \eta(t);$$

- $x(t)$: position of the particle at time instant t, $a > 0$: coefficient of friction, and η: random variable that represents some uncertainties or stochastic effects perturbing the particle.
- Diffusion-like motion from the probabilistic perspective of a single microscopic particle moving in a fluid medium.
- Linear stochastic ODE of order 1.

Some basics

- Vander der Pol equation:

$$\frac{d^2x(t)}{dt^2} - a(1 - x^2(t))\frac{dx(t)}{dt} + x(t) = 0;$$

- a: positive parameter, which controls the nonlinearity and the strength of the damping.
- Generate waveforms corresponding to electrocardiogram patterns.
- Nonlinear ODE of order 2.

Some basics

- Higher order ODEs: $\Omega \subset \mathbb{R}^{n+2}$ and $n \in \mathbb{N}$.
- ODE of order n:

$$F(t, x(t), \frac{dx}{dt}(t), \ldots, \frac{d^n x}{dt^n}(t)) = 0;$$

- x: real-valued unknown function and $dx(t)/dt, \ldots, d^n x(t)/dt^n$: its derivatives.
- $\varphi \in \mathcal{C}^n(I)$: solution of the differential equation if I: open interval, for all $t \in I$,

$$(t, \varphi(t), \frac{\partial \varphi}{\partial t}(t), \ldots, \frac{\partial^n \varphi}{\partial t^n}(t)) \in \Omega$$

  and

$$F(t, \varphi(t), \frac{\partial \varphi}{\partial t}(t), \ldots, \frac{\partial^n \varphi}{\partial t^n}(t)) = 0.$$

- x: vector valued function, $x(t) \in \mathbb{R}^d$, $\rightarrow$ $\Omega \subset \mathbb{R} \times \mathbb{R}^{(n+1)d}$.

Some basics

- n-th order ODE:

$$(***) \quad x^{(n)}(t) = f(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}), \quad t \in I.$$

- $x(t) \in \mathbb{R}^d$ and $f : I \times \mathbb{R}^{nd} \to \mathbb{R}^d$.
- Initial condition:

$$(x(t_0), x'(t_0), x''(t_0), \ldots, x^{(n-1)}(t_0))^\top.$$

- Reduce the high order ODE $(***)$ into a first order ODE:

$$y(t) := (x(t), dx(t)/dt, \ldots, d^{n-1}x(t)/dt^{n-1})^\top \in \mathbb{R}^{nd}$$

  and

$$F(t, y) := (y_2, \ldots, y_n, f(t, y_1, \ldots, y_n))^\top$$

  for $y = (y_1, \ldots, y_n)^\top \in \mathbb{R}^{nd}$ and $y_i \in \mathbb{R}^d$ for $i = 1, 2, \ldots, n$.
- $(***)$ equivalent to the following first order ODE:

$$\frac{dy}{dt} = F(t, y(t)).$$

Some basics

- EXAMPLE:
  - Consider the second order ODE:

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x(t) = g(t).$$

  - $\Rightarrow$

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} x \\ \frac{dx}{dt} \end{bmatrix} &= \begin{bmatrix} \frac{dx}{dt} \\ -p(t)\frac{dx}{dt} - q(t)x(t) + g(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}\begin{bmatrix} x \\ \frac{dx}{dt} \end{bmatrix} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}.\end{aligned}$$

Some basics

- ODEs:
  - Existence of solutions;
  - Uniqueness of solutions with suitable initial conditions;
  - Regularity and stability of solutions (e.g. dependence on the initial conditions, large time stability, higher regularity);
  - Computation of solutions.
- Existence of solutions: fixed point theorems; implicit function theorem in Banach spaces.
- Uniqueness: more difficult.
- Explicit solutions: only in a very few special cases.
- Numerical solutions.

Some basics

- Some methods of resolution:
  - Separation of variables;
  - Change of variables;
  - Method of integrating factors.

Some basics

- Separation of variables:
  - I and J: open intervals;
  - $f \in \mathcal{C}^0(I)$ and $g \in \mathcal{C}^0(J)$: continuous functions.
  - Solutions to the first order equation

$$(****) \quad \frac{dx}{dt} = f(t)g(x).$$

  - $t_0 \in I$ and $x_0 \in J$.
  - $g(x_0) = 0$ for some $x_0 \in J \to x(t) = x_0$ for $t \in I$: solution to $(****)$.
  - Suppose $g(x_0) \neq 0 \to g \neq 0$ in a neighborhood of $x_0 \Rightarrow$

$$\frac{dx}{g(x)} = f(t)dt.$$

  - Integration $\Rightarrow$

$$\int \frac{dx}{g(x)} = \int f(t)dt + c;$$

    c: constant uniquely determined by the initial condition.

Some basics

- F and G: primitives of f and $1/g$.
- $G'(x) \neq 0 \Rightarrow G$: strictly monotonic $\rightarrow$ invertible.
- Solution:

$$x(t) = G^{-1}(F(t) + c).$$

- Method of separation of variables.
- $(****)$: separable equation.

Some basics

- EXAMPLE:
  - Consider the following ODE:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1+2t}{\cos x(t)}, \\ x(0) = \pi. \end{cases}$$

  - $g(x) = 1/\cos x$ and $f(t) = 1 + 2t$.
  - g: defined for $x \neq \pi/2 + k\pi, k \in \mathbb{Z}$.
  - Separation of variables,

$$\cos x dx = 1 + 2t dt.$$

  - Integration,

$$\sin x(t) = t^2 + t + C,$$

    for some constant $C \in \mathbb{R}$.
  - Initial condition $x(0) = \pi \Rightarrow C = 0$.

Some basics

- Taking the arcsin $\Rightarrow x(t) = \arcsin(t^2 + t)$: not the solution because $x(0) = \arcsin(0) = 0$.
- arcsin: inverse of sin on $[-\pi/2, \pi/2]$; $x(t)$: takes the values in a neighborhood of π.
- $w(t) = x(t) - \pi \to w(0) = x(0) - \pi = 0 \Rightarrow w(t) = -\arcsin(t^2 + t)$.
- Correct solution:
$$x(t) = \pi - \arcsin(t^2 + t).$$

Some basics

- Change of variables:
  - Consider the following ODE:

$$\frac{dx}{dt} = f\Big(\frac{x(t)}{t}\Big);$$

    $f : I \subset \mathbb{R} \to \mathbb{R}$: continuous function on some open interval $I \subset \mathbb{R}$.
  - change of variable $x(t) = ty(t)$; $y(t)$: new unknown function,

$$\frac{dx}{dt} = y(t) + t\frac{dy}{dt} = f(y(t)),$$

  - Separable equation for y:

$$\frac{dy}{f(y) - y} = \frac{dt}{t}.$$

  - Solution by the method of separation of variables.

Some basics

- EXAMPLE:
  - Consider
$$\frac{dx}{dt} = \frac{t^2 + x^2}{xt}.$$
  - $f(s) = s + 1/s$ with $s = x/t$.
  - Change of variable: $y(t) = x(t)/t \Rightarrow ydy = dt/t$
  - $\Rightarrow$
$$(1/2)y^2 = \ln t + C.$$
  - $\Rightarrow$
$$x(t) = \pm t\sqrt{2(\ln t + C)}.$$

Some basics

- Method of integrating factors
  - Consider
$$\frac{dx(t)}{dt} = f(t).$$
  - Integration
$$x(t) = x(0) + \int_0^t f(s)\, ds.$$
  - Consider
$$\frac{dx}{dt} + p(t)x(t) = g(t);$$
    p and g: functions of t.
  - Left-hand side: expressed as the derivative of the unknown quantity $\leftarrow$ Multiply by $\mu(t)$.

Some basics

- $\mu(t)$ s.t.

$$\mu(t)\frac{dx}{dt} + \mu(t)p(t)x(t) = \frac{d}{dt}(\mu(t)x(t)).$$

- Taking derivatives $\Rightarrow$

$$(1/\mu)d\mu/dt = p(t) \quad \text{or} \quad \frac{d}{dt}\ln\mu(t) = p(t).$$

- Integration $\Rightarrow$

$$\mu(t) = \exp(\int_0^t p(s)ds),$$

  up to a multiplicative constant.

- Transformed equation:

$$\frac{d}{dt}(\mu(t)x(t)) = \mu(t)g(t).$$

- $\Rightarrow$

$$x(t) = \frac{1}{\mu(t)}\Big(\int_0^t \mu(s)g(s)ds\Big) + \frac{C}{\mu(t)};$$

  C: determined from the initial condition $x(0) = x_0$.

- $\mu(t)$: integrating factor.

Some basics

- EXAMPLE:
  - Consider

$$\begin{cases} \dfrac{dx}{dt} + \dfrac{1}{t+1} x(t) = (1+t)^2, \qquad t \geq 0, \\ x(0) = 1. \end{cases}$$

  - $p(t) = 1/(t+1)$ and $g(t) = (1+t)^2$.
  - Integrating factor:

$$\mu(t) = \exp(\int_0^t p(s)ds) = e^{\ln(t+1)} = t+1.$$

  - $\Rightarrow$

$$x(t) = \frac{1}{t+1} \int_0^t (s+1)^3 ds + \frac{C}{t+1} = \frac{(t+1)^3}{4} + \frac{C - \frac{1}{4}}{t+1}.$$

  - Initial condition $x(0) = 1 \Rightarrow C = 1$.

Some basics

- EXAMPLE: (Bernoulli's equation)
  - Consider
  $$\frac{dx}{dt} + p(t)x(t) = g(t)x^{\alpha}(t).$$
  - $\alpha \notin \{0, 1\}$.
  - Change of variable: $x = z^{\frac{1}{1-\alpha}}$,
  $$\frac{dx}{dt} = \frac{1}{1-\alpha} z^{\frac{\alpha}{1-\alpha}} \frac{dz}{dt}.$$
  - Linear equation:
  $$\frac{dz}{dt} + (1-\alpha)p(t)z(t) = (1-\alpha)g(t).$$
  - Solved by the method of integrating factors.

Some basics

- Important examples of ODEs:
  - Autonomous ODEs;
  - Exact equations;
  - Hamiltonian systems.

Some basics

- Autonomous ODEs:
  - DEFINITION: $\dfrac{dx(t)}{dt} = f(t, x(t))$: autonomous if f: independent of t.
  - Any ODE can be rewritten as an autonomous ODE on a higher-dimensional space.
  - $y = (t, x(t)) \rightarrow$ autonomous ODE

$$\frac{dy(t)}{dt} = F(y(t));$$

  -

$$F(y) = \begin{pmatrix} 1 \\ f(t, x(t)) \end{pmatrix}.$$

Some basics

- Exact equations:
  - $\Omega = I \times \mathbb{R} \subset \mathbb{R}^2$ with $I \subset \mathbb{R}$: open interval.
  - $f, g \in \mathcal{C}^0(\Omega)$.
  - Solution $x \in \mathcal{C}^1(I)$ of the ODE:

$$f(t, x(t)) + g(t, x(t))\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

    satisfying the initial condition $x(t_0) = x_0$ for some $(t_0, x_0) \in \Omega$.
  - Differential form:

$$\omega = f(t, x)\mathrm{d}t + g(t, x)\mathrm{d}x.$$

  - DEFINITION: Differential form: exact if there exists $F \in \mathcal{C}^1(\Omega)$ s.t.

$$\omega = \mathrm{d}F = \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial x}\mathrm{d}x.$$

  - F: potential of ω.
  - Differential equation: exact equation.

Some basics

- THEOREM: Implicit function theorem
    - Suppose that $F(t, x)$: continuously differentiable in a neighborhood of $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$ and $F(t_0, x_0) = 0$.
    - Suppose that $\partial F / \partial x(t_0, x_0) \neq 0$.
    - Then there exists a $\delta > 0$ and $\epsilon > 0$ s.t. for each t satisfying $|t - t_0| < \delta$, there exists a unique x s.t. $|x - x_0| < \epsilon$ for which $F(t, x) = 0$.
    - This correspondence defines a function $x(t)$ continuously differentiable on $\{|t - t_0| < \delta\}$ s.t.

$$F(t, x) = 0 \Leftrightarrow x = x(t).$$

Some basics

- THEOREM:
  - Suppose that ω: exact form with potential F s.t.

$$\frac{\partial F}{\partial x}(t_0, x_0) \neq 0.$$

  - $F(t,x) = F(t_0, x_0)$ implicitly defines a function $x \in \mathcal{C}^1(I)$ for some open interval I containing t_0, which solves

$$f(t, x(t)) + g(t, x(t)) \frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

    with the initial condition $x(t_0) = x_0$.
  - Solution: unique on I.

Some basics

- PROOF:
  - Suppose without loss of generality that $F(t_0, x_0) = 0$.
  - **Implicit function theorem** $\Rightarrow$ there exists $\delta, \eta > 0$ and $x \in \mathcal{C}^1(t_0 - \delta, t_0 + \delta)$ s.t. $\{(t,x) \in \Omega : |t - t_0| < \delta, |x - x_0| < \eta,$
$$F(t,x) = 0\} = \{(t, x(t)) \in \Omega : |t - t_0| < \delta\}.$$
  - By differentiating the identity $F(t, x(t)) = 0$,
$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t} F(t, x(t)) &= \frac{\partial F}{\partial t}(t, x(t)) + \frac{\partial F}{\partial x}(t, x(t)) \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= f(t, x(t)) + g(t, x(t)) \frac{\mathrm{d}x}{\mathrm{d}t}. \end{aligned}$$
  - $\Rightarrow x(t)$: solution of the differential equation.
  - $x(t_0) = x_0$.
  - If $z \in \mathcal{C}^1(I)$: solution s.t. $z(t_0) = x_0$, then
$$\frac{\mathrm{d}}{\mathrm{d}t} F(t, z(t)) = 0 \Longrightarrow F(t, z(t)) = F(t_0, z(t_0)) = 0 \Longrightarrow z(t) = x(t).$$

Some basics

- DEFINITION:
  - $f, g \in \mathcal{C}^1(\Omega)$.
  - Differential form $\omega = f\mathrm{d}t + g\mathrm{d}x$: **closed** in Ω if
  $$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial t}$$
  for all $(t, x) \in \Omega$.
- PROPOSITION:
  - Exact differential form $\omega = f\mathrm{d}t + g\mathrm{d}x$ with a potential $F \in \mathcal{C}^2$: **closed** if
  $$\frac{\partial^2 F}{\partial t \partial x} = \frac{\partial^2 F}{\partial x \partial t}$$
  for all $(t, x) \in \Omega$.
  - Converse: also true if Ω: simply connected.
- Closed forms always have a potential (at least locally).

Some basics

- EXAMPLE:
  - Consider
  $$tx^2 + x - t\frac{\mathrm{d}x}{\mathrm{d}t} = 0.$$
  - $f(t,x) = tx^2 + x$ and $g(t,x) = -t$.
  - Not exact:
  $$\frac{\partial f}{\partial x} = 2xt + 1 \neq \frac{\partial g}{\partial t} = -1.$$
- EXAMPLE:
  - Consider
  $$t + \frac{1}{x} - \frac{t}{x^2}\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$
  - Exact equation with the potential function F:
  $$F(t,x) = \frac{t^2}{2} + \frac{t}{x} + C, \quad C \in \mathbb{R}.$$
  - $F(t,x) = 0$ implicitly defines the solutions (locally for $t \neq 0$ and $x \neq 0$ s.t. $\partial F/\partial x(t,x) \neq 0$).

Some basics

- Hamiltonian systems:
  - DEFINITION:
    - M: subset of $\mathbb{R}^d$ and $H : \mathbb{R}^d \times M \to \mathbb{R}$: $\mathcal{C}^1$ function.
    - Hamiltonian system with Hamiltonian H: first-order system of ODEs

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = -\dfrac{\partial H}{\partial q}(p, q), \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = \dfrac{\partial H}{\partial p}(p, q). \end{cases}$$

  - EXAMPLE:
    - Harmonic oscillator with Hamiltonian

$$H(p, q) = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kq^2;$$

      m and k: positive constants.
    - Given a potential V, Hamiltonian systems of the form:

$$H(p, q) = \frac{1}{2}p^\top M^{-1}p + V(q);$$

      M: symmetric positive definite matrix and $\top$: transpose.

Some basics

- Invariant for a system of ODEs:
  - DEFINITION:
    - $\Omega = I \times D$; $I \subset \mathbb{R}$ and $D \subset \mathbb{R}^d$.
    - Consider
$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x(t));$$
    - $f : \Omega \to \mathbb{R}^d$.
    - $F : D \to \mathbb{R}$: invariant if $F(x(t)) = Constant$.
    - $(t, x) \in I \times D$: stationary point if $f(t, x) = 0$.

Some basics

- Example:
  - Lotka-Volterra's ODEs:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u(v-2), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v(1-u). \end{cases}$$

  - Dynamics of biological systems in which two species interact: one as a predator and the other as prey.
  - Define

$$F(u,v) := \ln u - u + 2\ln v - v.$$

  - $F(u,v)$: invariant.
  - $(u,v)=(1,2)$ and $(u,v)=(0,0)$: stationary points.

Some basics

- Differentiation with respect to time,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F(u,v) &= \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}t} - \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{2}{v}\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= v - 2 - \frac{\mathrm{d}u}{\mathrm{d}t} + 2(1-u) - \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= (v-2) - u(v-2) + 2(1-u) + v(1-u) \\ &= (v-2)(1-u) + (2-v)(1-u) \\ &= 0.\end{aligned}$$

Some basics

- LEMMA:
  - Hamiltonian H: invariant of the associated Hamiltonian system.
- PROOF:
  - $$\frac{\mathrm{d}}{\mathrm{d}t} H(p(t), q(t)) = \frac{\partial H}{\partial p}(p(t), q(t)) \frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial H}{\partial q}(p(t), q(t)) \frac{\mathrm{d}q}{\mathrm{d}t}$$
    $$= -\frac{\partial H}{\partial p}(p(t), q(t)) \frac{\partial H}{\partial q}(p(t), q(t)) + \frac{\partial H}{\partial q}(p(t), q(t)) \frac{\partial H}{\partial p}(p(t), q(t))$$
    $$= 0.$$
  - $H(p, q)$: invariant of the associated system of equations.

Some basics

- EXAMPLE:
  - Consider

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = -\sin q, \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = p. \end{cases}$$

  - $H(p, q) = \frac{1}{2}p^2 - \cos q$:

$$\begin{cases} \dfrac{\partial H}{\partial q} = \sin q = -\dfrac{\mathrm{d}p}{\mathrm{d}t} \\ \dfrac{\partial H}{\partial p} = p = \dfrac{\mathrm{d}q}{\mathrm{d}t} \end{cases}$$

Some basics

- Equivalent expression for Hamiltonian systems:
  - $x = (p, q)^\top$ $(p, q \in \mathbb{R}^d)$;

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix};$$

    I: $d \times d$ identity matrix.
  - $J^{-1} = J^\top$.
  - Rewrite the Hamiltonian system in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = J^{-1} \nabla H(x).$$

Some basics

- DEFINITION Symplectic linear mapping
  - Matrix $A \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}$ (linear mapping from $\mathbb{R}^{2d}$ to $\mathbb{R}^{2d}$): symplectic if $A^\top J A = J$.
- DEFINITION Symplectic mapping
  - Differentiable map $g : U \to \mathbb{R}^{2n}$: symplectic if the **Jacobian matrix** $g'(p,q)$: everywhere symplectic, i.e., if

$$g'(p,q)^\top J g'(p,q) = J.$$

  - Taking the transpose of both sides of the above equation,

$$g'(p,q)^\top J^\top g'(p,q) = J^\top;$$

  - Or equivalently,

$$g'(p,q)^\top J^{-1} g'(p,q) = J^{-1}.$$

Some basics

- THEOREM:
  - If g: symplectic mapping, then it preserves the Hamiltonian form of the equation.

Some basics

- PROOF:
  - $x = (p, q)^\top$, $y = g(p, q)^\top$; $G(y) := H(x)$.
  - Chain rule $\Rightarrow$

$$\frac{\partial}{\partial x} H(x) = \frac{\partial}{\partial x} G(y) = \frac{\partial}{\partial y} G(y) \frac{\partial y}{\partial x}(x)$$
$$= (\nabla_y G(y))^\top g'^\top (p, q).$$

Some basics

- $\Rightarrow$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= g'^{\top}(p,q)\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= g'^{\top}(p,q)J^{-1}\left(\frac{\partial H(x)}{\partial x}\right)^{\top} \\ &= g'^{\top}J^{-1}g'\nabla_y G(y) \\ &= J^{-1}\nabla_y G(y).\end{aligned}$$

- $\Rightarrow$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = J^{-1}\nabla_y G(y).$$

Some basics

- DEFINITION:
  - Flow:
  $$\phi_t(p_0, q_0) = (p(t, p_0, q_0), q(t, p_0, q_0));$$
  - $\phi_t : U \to \mathbb{R}^{2d}$, $U \subset \mathbb{R}^{2d}$;
  - p_0 and q_0: initial data at $t = 0$.

Some basics

- THEOREM: Poincaré's theorem
  - H: twice differentiable.
  - Flow ϕ_t: symplectic transformation.

Some basics

- PROOF:
  - $y_0 = (p_0, q_0)$.
  - $$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{\partial\phi_t}{\partial y_0}\right)^\top J\left(\frac{\partial\phi_t}{\partial y_0}\right)\right)\\
=&\left(\frac{\partial\phi_t}{\partial y_0}\right)^{\prime\top} J\left(\frac{\partial\phi_t}{\partial y_0}\right)+\left(\frac{\partial\phi_t}{\partial y_0}\right)^{\top} J\left(\frac{\partial\phi_t}{\partial y_0}\right)^{\prime}\\
=&\left(\frac{\partial\phi_t}{\partial y_0}\right)^{\top}\nabla^2 H J^{-\top} J\left(\frac{\partial\phi_t}{\partial y_0}\right)+\left(\frac{\partial\phi_t}{\partial y_0}\right)^{\top} J J^{-1}\nabla^2 H\left(\frac{\partial\phi_t}{\partial y_0}\right)\\
=&0;
\end{aligned}$$
  - $\nabla^2 H$: Hessian matrix of $H(p,q)$ (symmetric).

Some basics

- $\partial \phi_t / \partial y_0$ at $t = 0$: identity map $\Rightarrow$
- $$\left(\frac{\partial \phi_t}{\partial y_0} \right)^\top J \left(\frac{\partial \phi_t}{\partial y_0} \right) = J$$

  for all t and all (p_0, q_0).

Some basics

- Symplecticity of the flow: characteristic property of the Hamiltonian system.
- THEOREM:
  - $f : U \to \mathbb{R}^{2n}$: continuously differentiable.
  - $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$: locally Hamiltonian iff $\phi_t(x)$: symplectic for all $x \in U$ and for all sufficiently small t.

Some basics

- PROOF:
  - Necessity $\Leftarrow$ Poincaré's Theorem.
  - Suppose that ϕ_t: symplectic; prove local existence of a Hamiltonian H s.t. $f(x) = J^{-1}\nabla H(s)$.
  - $\frac{\partial \phi_t}{\partial y_0}$: solution of
  $$\frac{\mathrm{d}y}{\mathrm{d}t} = f'(\phi_t(y_0))y;$$
  - $\Rightarrow$
  $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{\partial \phi_t}{\partial y_0}\right)^\top J\left(\frac{\partial \phi_t}{\partial y_0}\right)\right) = \left(\frac{\partial \phi_t}{\partial y_0}\right)^\top [f'(\phi_t(y_0))^\top J + Jf']\left(\frac{\partial \phi_t}{\partial y_0}\right)$$
  $$= 0.$$
  - Putting $t=0$; $J = -J^\top \Rightarrow Jf'(y_0)$: symmetric matrix for all y_0.
  - Integrability lemma $\Rightarrow Jf(y)$: can be written as the gradient of a function H.

Some basics

- LEMMA: Integrability lemma
    - $D \subset \mathbb{R}^d$: open set; $g : D \to \mathbb{R}^d \in \mathcal{C}^1$.
    - Suppose that the Jacobian $g'(y)$: symmetric for all $y \in D$.
    - For every $y_0 \in D$, there exists a neighborhood of y_0 and a function $H(y)$ s.t.
    $$g(y) = \nabla H(y)$$
    on this neighborhood.

Some basics

- PROOF:
  - Suppose that $y_0 = 0$, and consider a ball around y_0: contained in D.
  - Define
  $$H(y) = \int_0^1 y^\top g(ty)\mathrm{d}t.$$
  - Differentiation with respect to y_k, and symmetry assumption:
  $$\frac{\partial g_i}{\partial y_k} = \frac{\partial g_k}{\partial y_i}$$
  - $\Rightarrow$
  $$\begin{aligned}\frac{\partial H}{\partial y_k} &= \int_0^1 (g_k(ty) + y^\top \frac{\partial g}{\partial y_k}(ty)t)\mathrm{d}t \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(tg_k(ty))\mathrm{d}t = g_k(y)\end{aligned}$$
  - $\Rightarrow$
  $$\nabla H = g.$$

Some basics

- Gradient system:
  - $$\frac{\mathrm{d}x}{\mathrm{d}t} = -\nabla F(x);$$
  - F: potential function.
- LEMMA:
  - Hamiltonian system: gradient system iff H: harmonic.

Some basics

- PROOF:
  - Suppose that H: harmonic, i.e.,

$$\frac{\partial^2 H}{\partial p^2} + \frac{\partial^2 H}{\partial q^2} = 0.$$

  - Jacobian of $J^{-1}\nabla H$: symmetric

$$(J^{-1}\nabla H)' = \begin{pmatrix} -\frac{\partial^2 H}{\partial p \partial q} & -\frac{\partial^2 H}{\partial q^2} \\ \frac{\partial^2 H}{\partial p^2} & \frac{\partial^2 H}{\partial p \partial q} \end{pmatrix}$$

  - Integrability lemma $\Rightarrow$ there exists V s.t. $J^{-1}\nabla H = \nabla V \Rightarrow$ Hamiltonian system: gradient system.

Some basics

- Suppose that Hamiltonian system: gradient system.
- There exists V s.t.

$$\frac{\partial V}{\partial p} = \frac{\partial H}{\partial q} \quad \text{and} \quad \frac{\partial V}{\partial q} = -\frac{\partial H}{\partial p}.$$

- $\Rightarrow$

$$\Delta H := \frac{\partial^2 H}{\partial p^2} + \frac{\partial^2 H}{\partial q^2} = 0.$$

Some basics

- EXAMPLE:
  - Hamiltonian system with $H(p, q) = p^2 - q^2$: gradient system.