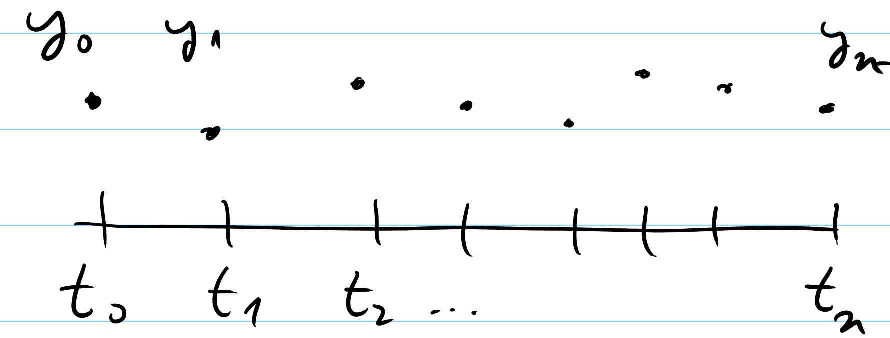
23.11.2023

1) Lagrange Interpolation Polynomials

1D



"without measurement errors"

use polynomials ; use trigonometric polynomials ; piecewise smooth functions

↳ splines, finite elements

$(t_0, \dots, t_{n-1})$, $(y_0, y_1 \dots y_{n-1}) \longmapsto f(t)$ $\quad f : [t_0, t_n] \to \mathbb{R}/\mathbb{C}$

$C[t_0, t_n] \supset V_n = \mathcal{P}_n$ = polynomials of degree $\leq n-1$ on $[t_0, t_n]$

$f \in C[t_0,t_n] \longmapsto f_n \in V_n$

$f_n(t_0) = y_0$

$\vdots$

$f_n(t_{n-1}) = y_{n-1}$

Given $y_0, \dots, y_{n-1} \mapsto$ unique $p \in \mathcal{P}_n$ s.t. $p(t_j) = y_j$, $j = 0, \dots, n-1$

Basis of monomials: $p_0(t) = 1, p_1(t) = t, \dots, p_{n-1}(t) = t^{n-1}$

$\Rightarrow$ Linear system for coefficient. $c_0, c_1, \dots, c_{n-1}$

$p = c_0 p_0 + c_1 p_1 + \dots + c_{n-1} p_{n-1}$

$$\begin{cases} p(t_j) = y_j \\ j = 0, \dots, n-1 \end{cases} \Rightarrow \begin{cases} c_0 p_0(t_j) + c_1 p_1(t_j) + \dots + c_{n-1} p_{n-1}(t_j) = y_j \\ j = 0, \dots, n-1 \end{cases}$$

Lagrange Polynomials = another basis in P_n

$$L_0(t) = \frac{(t-t_1)(t-t_2)\dots(t-t_{n-1})}{(t_0-t_1)(t_0-t_2)\dots(t_0-t_{n-1})}$$

$L_0(t_0) = 1$, $L_0(t_j) = 0$ for $j = 1, 2, \dots, n-1$

similarly $L_1(t), L_2(t), \dots, L_{n-1}(t)$

$\Rightarrow$ lin. independent n Polynomials in $P_n \Rightarrow$ basis

$$p_n(t) = y_0 L_0(t) + y_1 L_1(t) + \dots + y_{n-1} L_{n-1}(t)$$

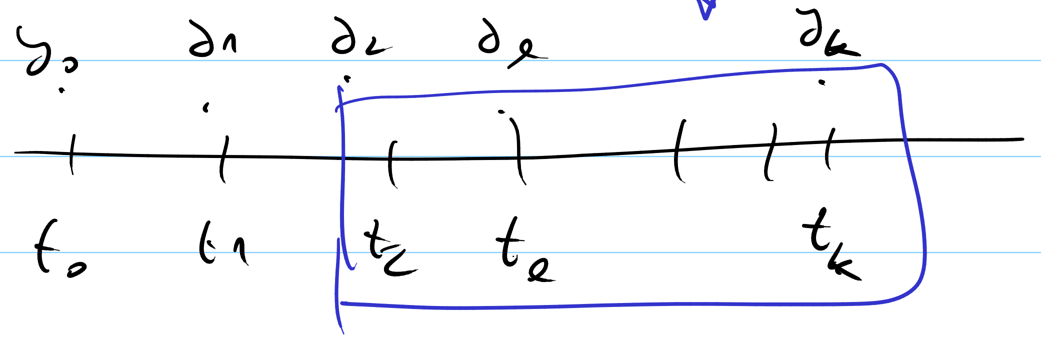
$$\Rightarrow p_n(t_j) = y_0 \cdot 0 + y_1 \cdot 0 + \dots + y_j \cdot 1 + \dots + y_{n-1} \cdot 0$$

$$= y_j$$

2) Partial polynomial interpolants

Aitken-Neville

uses only a subset of the points



disadvantage (1) – slower than with barycentric formula

– it can be used only for ONE SINGLE evaluation!

advantage: + new data can be easely added & processed

+ good for computing derivative values of the first divided differences are used for the Newton construction

together with the Horner scheme

$\Rightarrow$ stable way to construct & evaluate on interpolating polynomial at many points at once.

(*) Chebyshev I-kind

$$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_{n+1}(x) = 2x T_{n}(x) - T_{n-1}(x) \quad \text{for } n = 2, 3, \ldots \end{cases}$$

$\to$ polynomials of degree n $\downarrow$ build a basis in $\mathcal{P}_n$

orthogonal wrt to some skalar product(s)

$$T_n(x) = \cos(n \arccos x) \quad \text{for } x \in [-1, 1]$$

$\leftrightarrow$ Fourier!

(+) zeros of T_{n+1}: Chebyshev nodes on $[a, b]$

$$x_k = a + \frac{1}{2}(b-a)\left[1 + \cos\left(\frac{2k+1}{2(n+1)}\pi\right)\right] \quad k = 0, 1, \ldots, n$$

$\hookrightarrow$ optimal points for interpolation

$\hookrightarrow \|\cdot\|_\infty$ on $[a, b]$

(+) extrema of T_n (−1), +1

achieved between the Chebyshev nodes

$$x_k = a + \frac{1}{2}(b-a)\left[1 + \cos\left(\frac{k}{n}\pi\right)\right] \quad k = 0, 1, \ldots, n$$

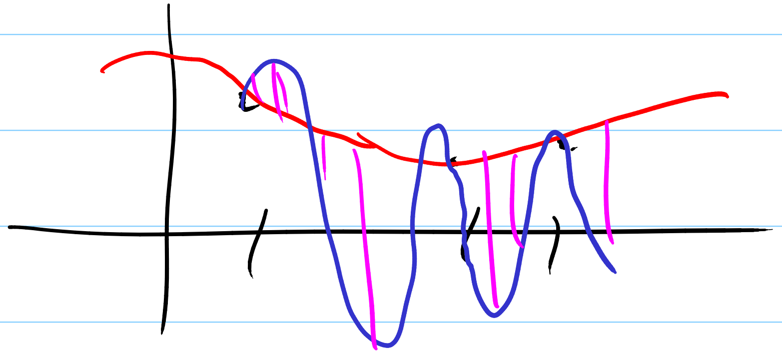
= zeros of T_{2n}

30.11.2023

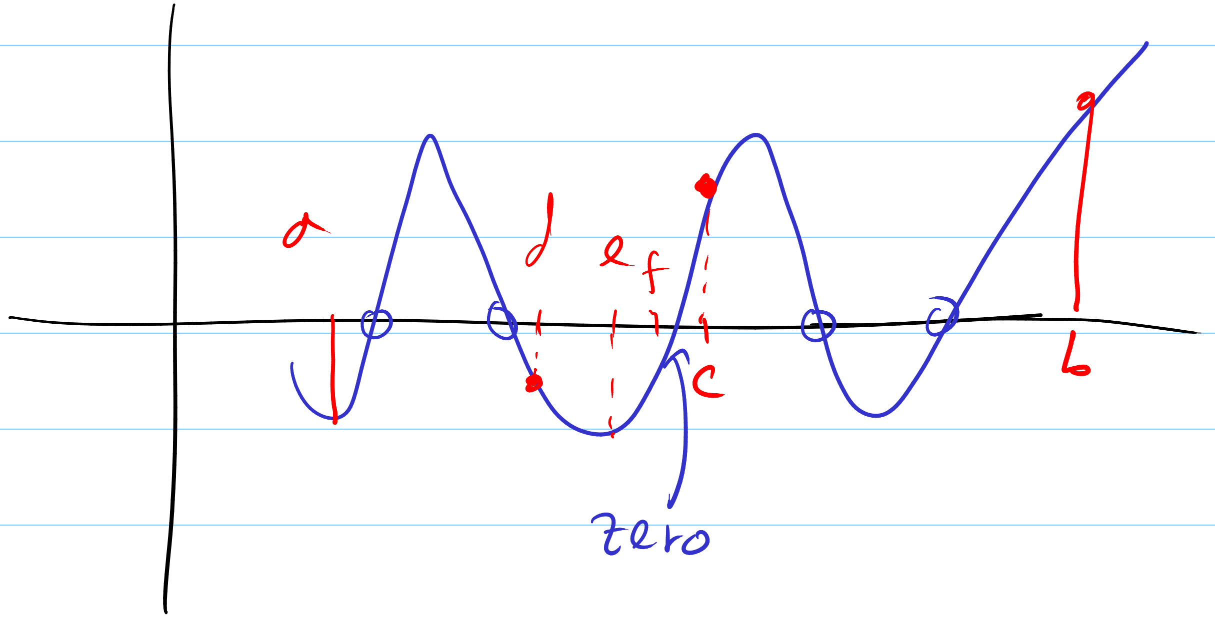
$f : I \to \mathbb{R}$ $\qquad f(t) \in \mathbb{R}$

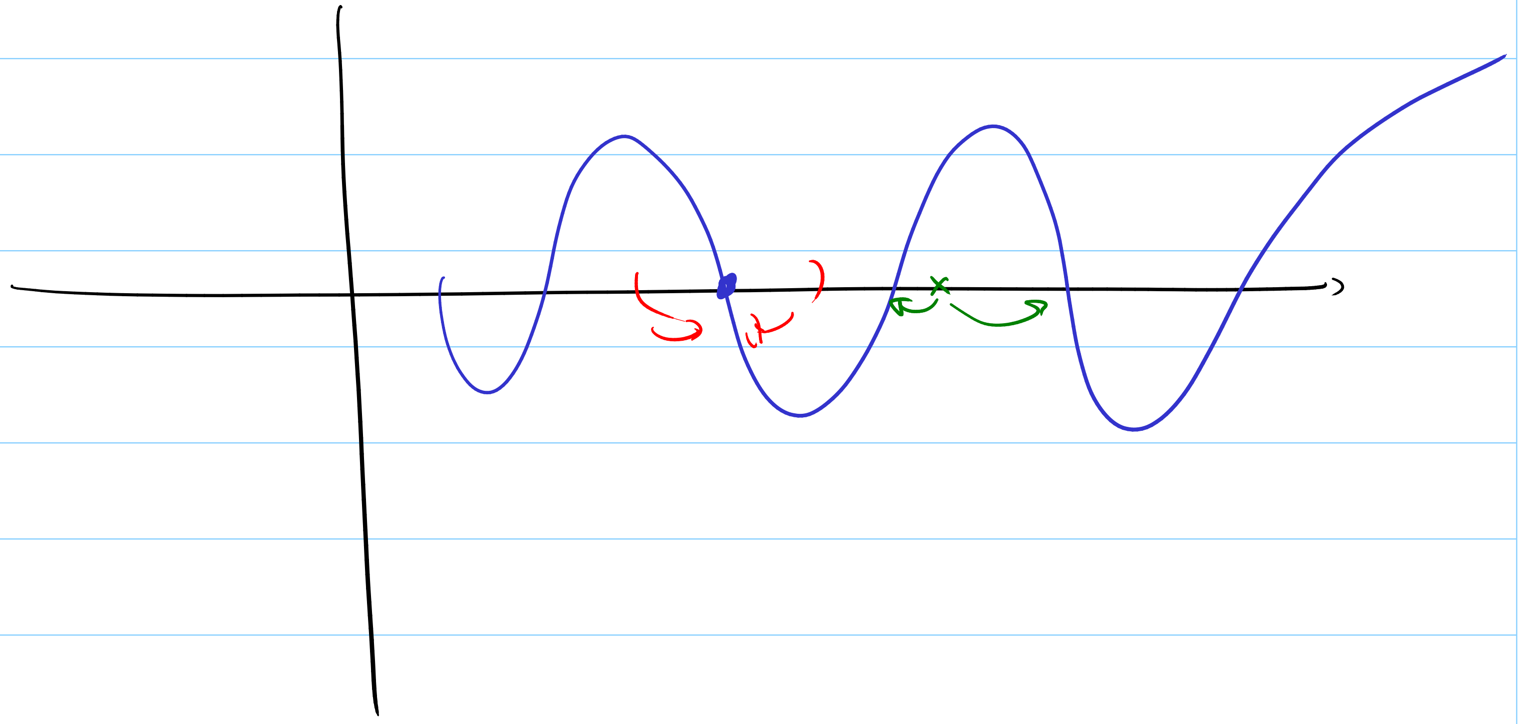
$I_J f : I \to \mathbb{R}$ $\qquad I_J(f)(t) \in \mathbb{R}$

$$\sup_{t \in I} \left| f(t) - I_J(f)(t) \right| =: \| f - I_J(f) \|_{L^\infty(I)}$$



Bisection:





Convergence:

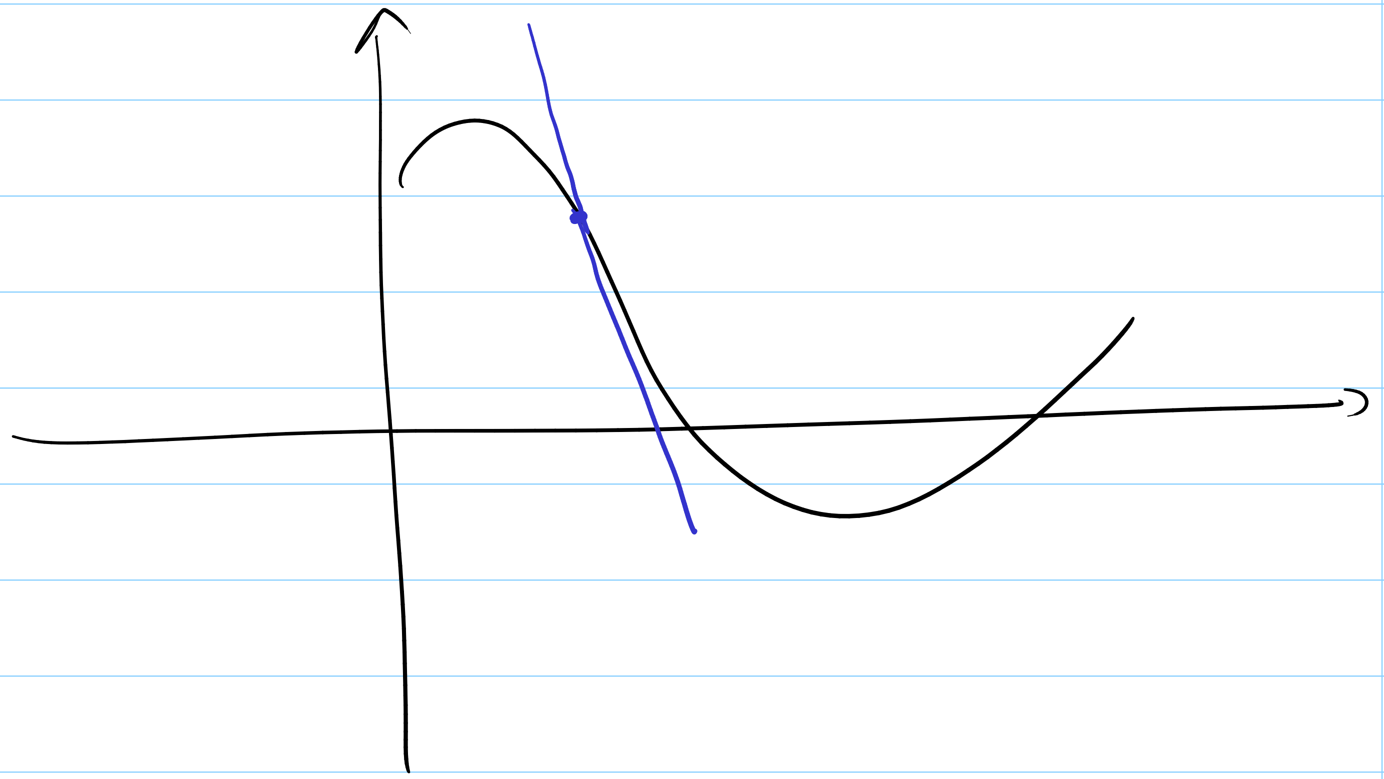
+ look at the digits! Do they stabilize?

+ make a linlog plot of error

+ compute approximation of p:

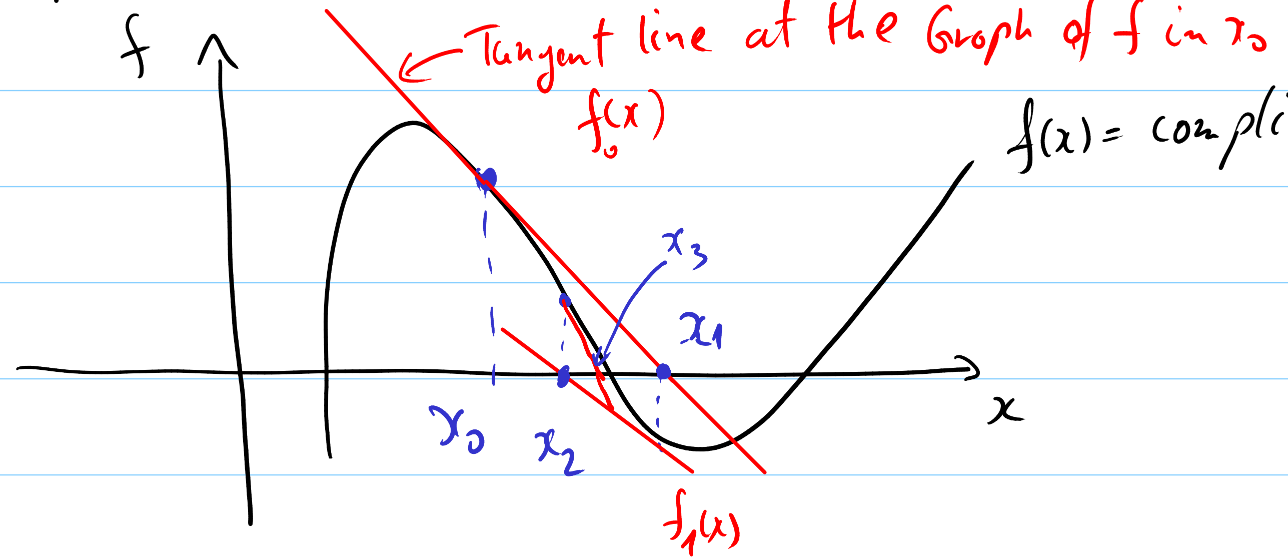
Remark 8.2.2.12 ⇒ "p"

+ if linear, estimate L from (8.2.2.7)



7.12.2023

1) Explain idea behind Newton.



$f(x)=$ complicated

Given x_0, propose a x_1 which is ideally, a better approximation to x^* with $f(x^*)=0$

Idea: replace the Graph of f by the the line tangent to the Graph at $(x_0, f(x_0))$

$$f(x) \approx f_0(x) = f(x_0) + f'(x_0)(x-x_0)$$

Find x_1 the solution of $f_0(x)=0$

Propose x_1 !

Newton in dD: Find $\underline{x}^* \in \mathbb{R}^d$ s.t.

$\underline{F}(\underline{x}^*)=\underline{0}$ $\qquad \underline{F}: \mathbb{R}^d \to \mathbb{R}^d$ smooth.

$$\underline{F}(\underline{x}) \approx \underline{F}(\underline{x}_0) + \underline{\underline{DF}}(\underline{x}_0)(\underline{x}-\underline{x}_0) = \underline{F}_0(\underline{x})$$

($\underline{\underline{DF}} \in \mathbb{R}^{d\times d}$, $\underline{x}_0 \in \mathbb{R}^d$)

solve $\underline{F}_0(\underline{x})=0 \Rightarrow \underline{x}_1$

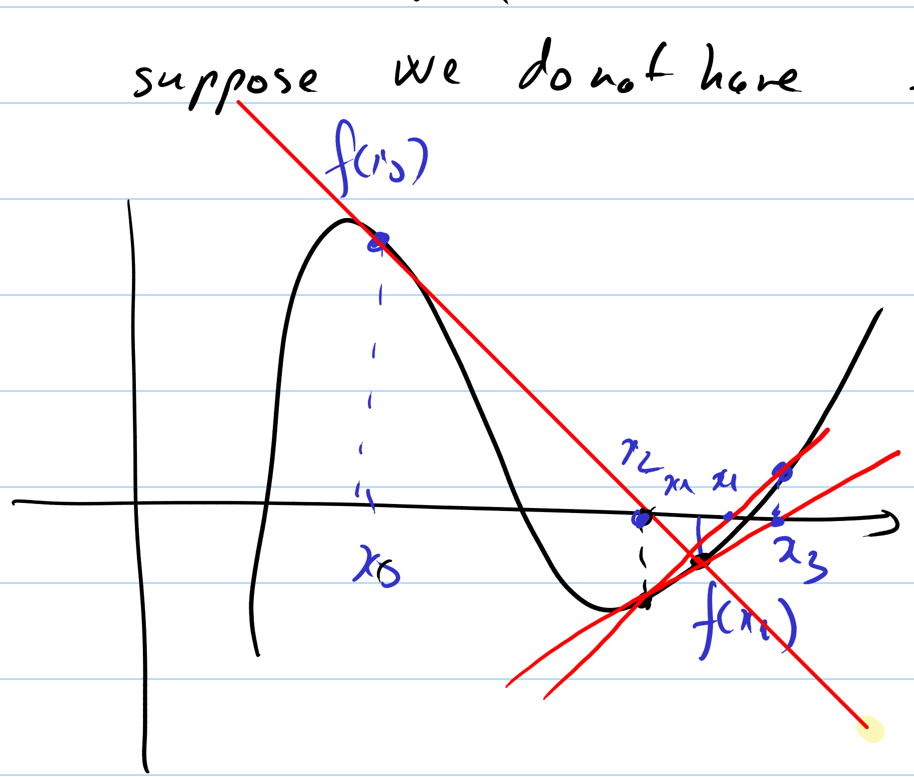
↑ linear algebraical equation!

Generalisations in 1D: + use another simple function (than linear)

+ use an approximation of f^{-1}.

2) Secant method

suppose we do not have f', $\underline{\underline{DF}}$



Use a line as approx. to the Graph of f

Use a second point on the Graph!

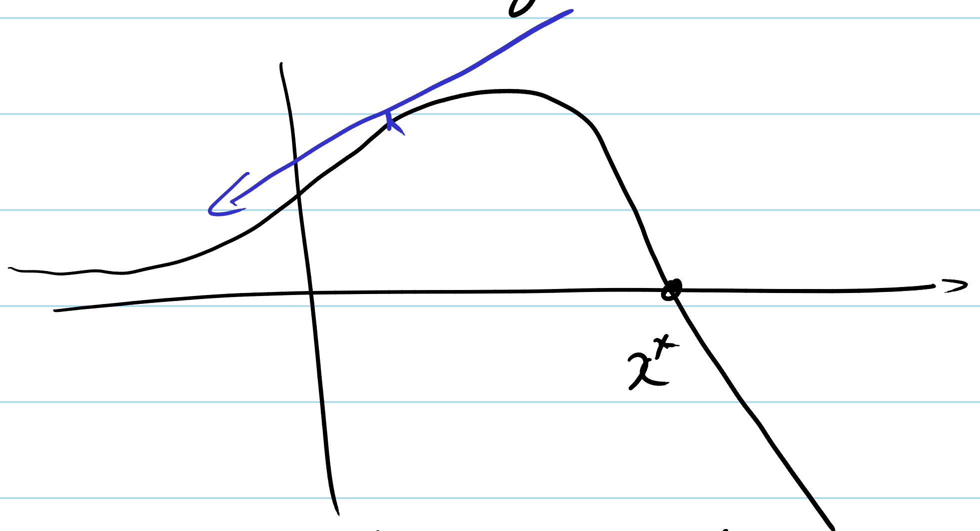
$\Rightarrow$ another linear approximation of f

its zero $\Rightarrow x_2$, etc.

<u>Note</u> The found zero depends on the starting point(s).

Question: when these methods fail?

- if $\underline{\underline{DF}}(x_k)$ is not invertible
- the tangent might point into the wrong direction



- corrections are too large

dD not so easy

$\rightarrow$ Broyden

$\rightarrow$ uses some very nice LA-tools ☺

$F(x) = x^3 + 3x^2 - 2x + 1$

$$\begin{array}{r} 25 \\ 5 \\ \hline 125 \end{array} \qquad \begin{array}{r} 1000 - \\ 321 \\ \hline 679 \end{array}$$

$F(x_0) = F(-10) = -1000 + 300 + 20 + 1 = -679$

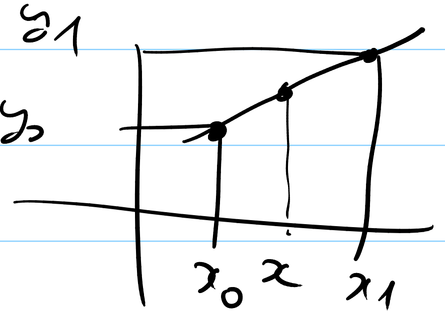
$F(x_1) = F(-5) = -5^3 + 3\cdot 25 + 10 + 1 = -125 + 86 = -29$

$-125 \qquad 75 \qquad \begin{array}{r} 125 - \\ 86 \\ \hline =29 \end{array}$

linear function through x_0: $(-10, -679)$

x_1: $(-5, -29)$

$$f(x) = \frac{(x-x_0)y_1}{x_1-x_0} + \frac{(x-x_1)y_0}{x_0-x_1}$$



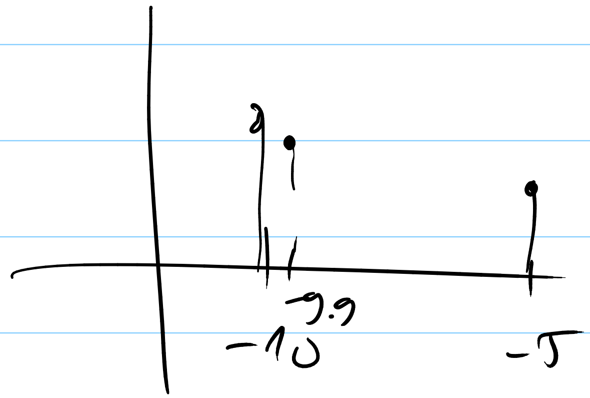
$$f(x) = \frac{(x+10)(-679)}{5} + \frac{(x+5)(-29)}{-15} =$$

$$= \frac{-3\cdot 679\, x - 30\cdot 679 - 29x - 5\cdot 29}{15} = 0$$

$$\begin{array}{r} 679 \\ 3 \\ \hline 2037+ \\ 29 \\ \hline 2066 \end{array} \qquad \begin{array}{r} 20370+ \\ 145 \\ \hline 20515 \end{array}$$

$-2066x - 20515 = 0$

$$x_2 = -\frac{20515}{2066} \approx -9.9 \ (?)$$



$$\frac{d}{dt}\left[f(g(t))\right] = f'(g(t))\, g'(t)$$

↑ chain rule

dD: Taylor for G, then Taylor for F ⇒

$$D(F\circ G)(x)\,\underline{h} = DF(G(x))\left(DG(x)\,\underline{h}\right)$$

Product rule

$$\left(f(t)g(t)\right)' = f'(t)g(t) + f(t)g'(t)$$

$b(u,v)$ is a bilinear form in u, v =>

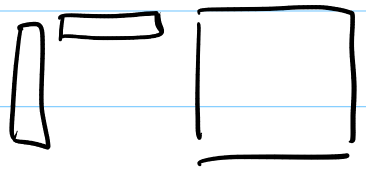
$$D\, b(F(\underline{x}), G(\underline{x})) = b(\underline{\underline{D}}F(\underline{x})\underline{h}, \underline{G}(\underline{x})) + b(\underline{F}(\underline{x}), \underline{\underline{D}}G(\underline{x})\underline{h})$$

Sherman-Morrison-Woodbury formula

$k = 1$

$\underline{\underline{\tilde{A}}}\,\underline{x} = \underline{b}$, $\underline{\underline{\tilde{A}}} = \underline{\underline{A}} + \underline{u}\,\underline{v}^H$ $\quad (\underline{\underline{A}} \in \mathbb{C}^{n\times n})$

$$\underline{x} = \underline{\underline{\tilde{A}}}^{-1}\underline{b} = \underline{\underline{A}}^{-1}\left(\underline{\underline{I}} - \frac{1}{1+\underline{v}^H \underline{\underline{A}}^{-1}\underline{u}}\,\underline{u}\,\underline{v}^H \underline{\underline{A}}^{-1}\right)$$

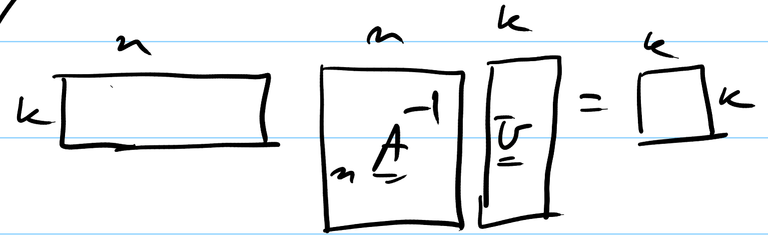


$$\left(\underline{\underline{A}}^{-1} - \frac{1}{1+\underline{v}^H \underline{\underline{A}}^{-1}\underline{u}}\,\underline{\underline{A}}^{-1}\underline{u}\,\underline{v}^H \underline{\underline{A}}^{-1}\right)\left(\underline{\underline{A}} + \underline{u}\,\underline{v}^H\right) =$$

$k \geq 2$ $\quad \underline{\underline{\tilde{A}}} = \underline{\underline{A}} + \underline{\underline{U}}\,\underline{\underline{V}}^H$

$$\underline{\underline{\tilde{A}}}^{-1} = \underline{\underline{A}}^{-1} - \underline{\underline{A}}^{-1}\underline{\underline{U}}\left(\underline{\underline{I}} + \underline{\underline{V}}^H \underline{\underline{A}}^{-1}\underline{\underline{U}}\right)^{-1}\underline{\underline{V}}^H \underline{\underline{A}}^{-1}$$

matrix!



$k \ll n$

easier to "invert"

reuse LU factorisation in an iterative algorithm!