Zurich Summer School 2007: Numerical Methods for High Frequency Scattering Problems

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Abstract

In these lecture notes, which interact with the notes/presentations of the other lecturers (Ralf Hiptmair, Markus Melenk, Eric Michielssen), we: introduce the Helmholtz equation and the precise formulation of various boundary value problems in Sobolev spaces; discuss the high frequency behaviour of solutions of the Helmholtz equation and say something about what is known from high frequency asymptotics; discuss the high frequency behaviour of standard boundary value problem and integral equation formulations. Finally, we discuss work in the last ten years on boundary element methods utilising basis functions which incorporate solutions of the Helmholtz equation in order to represent the highly oscillatory solution more effectively, leading to a reduced number of degrees of freedom.

Contents

Chapter 1

Boundary Value Problems for the Helmholtz Equation

1.1 The Physical Problem and its Mathematical Modelling

Many physicists and engineers are interested in the reliable simulation of processes in which acoustic waves are scattered by obstacles, with applications arising in areas as diverse as sonar (see figure 1.1), road, rail or aircraft noise, or building acoustics. Unless the geometry of the scattering object is

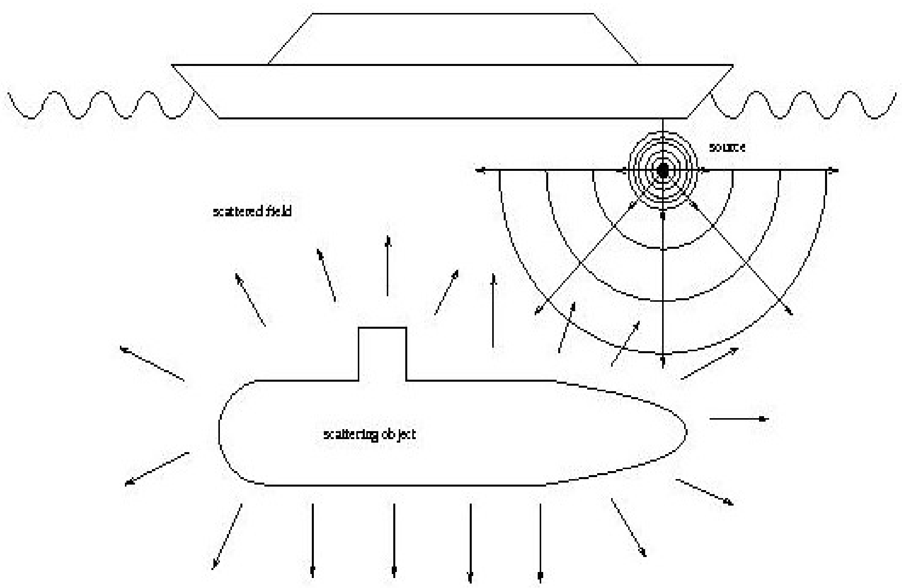

Figure 1.1: Typical acoustic scattering problem

particularly simple, the analytical solution of scattering problems is usually impossible, and hence numerical schemes are required.

Throughout these notes $P(x,t)$ will denote the pressure at time t at the point whose position vector is x. We will use Cartesian coordinates (Ox_1x_2 for 2D problems, $Ox_1x_2x_3$ for 3D problems). Thus, in 3D problems, x will be the vector $x = (x_1, x_2, x_3)$, with x_1, x_2, x_3 the three components of x. In 2D problems $x = (x_1, x_2)$ will have just two components.

In a homogeneous medium at rest the function P satisfies the wave equation

$$\Delta P - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = 0, \tag{1.1}$$

where c is the speed of sound and

$$\Delta = \nabla^2$$

is a shorthand for the Laplacian (e.g. $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ in 2D).

There are two approaches to solving the wave equation (1.1) numerically. The obvious approach is some form of direct numerical simulation, discretising the (hyperbolic) wave equation directly, e.g. by a finite difference em time domain method or by time domain integral equation methods (see the lectures of Michielssen). In these notes we will consider the alternative approach of working in the frequency domain, restricting attention to the (important) case of time harmonic (we assume throughout $\mathrm{e}^{-\mathrm{i}\omega t}$ time dependence for some $\omega > 0$[1]) acoustic propagation and scattering. Of course, having solved for harmonic time dependence for sufficiently many distinct frequencies ω, more general variations as a function of time can be obtained by Fourier synthesis, by combining harmonic time dependencies for different frequencies.

Assuming time harmonic time dependence the pressure is given by

$$P(x,t) = \Re\left(u(x)\mathrm{e}^{-\mathrm{i}\omega t}\right), \tag{1.2}$$

where $\omega = 2\pi f$ is the angular frequency, f the frequency (measured in Hz), $\mathrm{i} = \sqrt{-1}$, and $\Re$ denotes the real part. The function u, which is complex-valued in general, we will call the *complex acoustic pressure* (but often just the *pressure* for short). Note that we can write (1.2) more explicitly as

$$P(x,t) = A(x)\cos(\varphi(x) - \omega t), \tag{1.3}$$

where $A(x) = |u(x)|$, $\varphi(x) = \arg u(x)$, making clear the physical interpretation of u, that the modulus of $u(x)$, $|u(x)|$, is the amplitude of the time

[1] One has to be a little careful with the literature on time harmonic wave problems. Assuming time dependence $\mathrm{e}^{\mathrm{i}\omega t}$ with $\omega > 0$ is equally common and valid. For $\mathrm{e}^{\mathrm{i}\omega t}$ time dependence, used in some publications, all the formulas in these notes remain valid as long as one replaces all complex numbers by their complex conjugate, in particular replaces each i by $-$i.

harmonic pressure fluctuation at x, while $\arg u(x)$ determines the phase of the oscillation at x.

Frequently we are interested in Sound Pressure Level predictions. Since the root mean square of a time harmonic field is its amplitude divided by $\sqrt{2}$, the SPL at x is given by

$$SPL = 20 \log_{10} \left(\frac{|u(x)|}{\sqrt{2}\, u_{ref}} \right) \quad \text{dB},$$

where u_{ref} is the usual reference pressure. An important point for numerical calculation in general is that accurate prediction of SPL requires small *relative* errors in the computation of $|u|$. Of course, this then implies very small *absolute* errors at points x where $|u(x)|$ is small. For many applications such regions may be very important, for example if one is calculating the shielding performance of a noise barrier, when one is interested in accurate predictions (with small dB error and so small relative error) in the shadow zone. Thus very accurate numerical methods are of interest for a number of acoustic applications.

Substituting (1.2) into (1.1), we see that u satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0, \quad \text{in } \Omega \subset \mathbb{R}^d, \tag{1.4}$$

where $d = 1, 2$ or 3 is the dimension of the problem we are considering, Ω denotes the *domain of propagation*, the region in which the wave propagates, which is either a subset of the plane ($\mathbb{R}^2$) if we are solving a 2D problem, or is a subset of $\mathbb{R}^3$ if we are solving a fully 3D problem. (Occasionally, especially for instructional purposes, we wish to consider also 1D problems, in which case the domain Ω is a subset of $\mathbb{R}$, the real line, i.e. Ω is an interval of the form (a, b) with $a < b$.) The positive constant k is the *wave number*, given by

$$k := \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}.$$

Here we have introduced $\lambda = c/f$, the wavelength of plane waves of frequency f. Clearly, k is proportional to the frequency and inversely proportional to λ, with SI units m^{-1}.

A large part of the rest of these notes will discuss how to compute, by the boundary element method, solutions to (1.4) that also satisfy appropriate boundary conditions on the boundary of the domain of propagation. We denote the boundary of Ω by $\partial\Omega$ and will focus in these notes mainly on the *Dirchlet* boundary condition, that

$$u = g \quad \text{on } \partial\Omega. \tag{1.5}$$

This focus is made for the purpose of simplicity and because some of the developments that we will mention are not yet tested or the theory developed except in this boundary condition case. Sadly, the Dirichlet boundary condition is not very often physically relevant, at least in my experience of applied acoustics, except that the assumption that $u = 0$ is a reasonable approximation to reality at the sea-air interface in underwater acoustics.

While our emphasis will be more often on the Dirichlet boundary condition case, we will also consider the most commonly physically relevant boundary condition, namely the *impedance boundary condition*

$$\frac{\partial u}{\partial n} + \mathrm{i}k\beta u = g, \quad \text{on } \partial\Omega. \tag{1.6}$$

Let us spend a few moments explaining this boundary condition. First of all, in this equation, and throughout, $\partial/\partial n$ denotes the normal derivative on the boundary, i.e. the rate of increase in the direction n, where $n(x)$ denotes the unit normal at $x \in \partial\Omega$, **directed into**[2] Ω. Explicitly, in terms of the gradient of u,

$$\frac{\partial u}{\partial n}(x) = n(x) \cdot \nabla u(x), \tag{1.7}$$

i.e. the normal derivative is the scalar product of the gradient and the unit normal.

The function g on the right hand side of the equation is identically zero in acoustic scattering problems (problems where we are given an incident wave and a stationary scatterer and have to compute the resulting acoustic field), but is non-zero for radiation problems (where the motion of a radiating structure is given and we have to calculate the acoustic field radiated).

In (1.6) β is the relative surface admittance which, in general, is a function of position on the boundary (and also a function of frequency). The simplest case is when the boundary is *acoustically rigid* or *sound hard*. This is the case when no flow is possible across $\partial\Omega$ and $\beta = 0$ so that (1.6) simplifies to the so-called *sound hard* or *Neumann* boundary condition

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega. \tag{1.8}$$

More generally, β may be non-zero, its value at position x on $\partial\Omega$ given by

$$\beta(x) := \frac{Z_0}{Z_s(x)}$$

where $Z_0 = \rho c$ is the impedance of the medium of propagation (air, water, etc.), ρ its density, and $Z_s(x)$ is the *surface impedance* at x. The surface impedance is defined by the equation

$$Z_s(x) = \frac{u(x)}{-\mathbf{v}(x) \cdot n(x)}, \quad \text{for } x \in \partial\Omega,$$

[2] In writing about and coding the boundary element method one has to take great care about directions of normals. Many authors will take the unit normal in the opposite direction, which changes the sign of the normal derivative.

where $\mathbf{v}(x)$ is the velocity at x due to the acoustic field. Thus Z_s is the ratio on the surface of the pressure to the normal velocity into the surface. The impedance boundary condition is appropriate whenever, to a good approximation, this ratio is independent of the acoustic field. This is the case for sound hard surfaces (where the ratio is always ∞), for many naturally occurring surfaces in outdoor noise propagation, and for many man-made acoustically absorbing surfaces at lower frequencies.

In the case when the domain Ω is unbounded, the complete mathematical formulation of the problem has to include some condition which encapsulates, in a mathematical way, the idea that the acoustic field, or at least some part of it (e.g. the part which is reflected from the scattering obstacle), is travelling outwards, towards infinity. The usual conditions imposed are the so-called *Sommerfeld radiation conditions*, that

$$u(x) = O\left(r^{-(d-1)/2}\right), \tag{1.9}$$

$$\frac{\partial u}{\partial r}(x) - \mathrm{i}ku(x) = o\left(r^{-(d-1)/2}\right), \tag{1.10}$$

as $r \to \infty$, uniformly in $\hat{x} := x/r$. In this equation d is the dimension (2 or 3) and r is the radial direction, precisely $r = |x|$, the distance of x from the origin, so that, in terms of the gradient of u,

$$\frac{\partial u}{\partial r}(x) = \hat{x} \cdot \nabla u(x),$$

where $\hat{x} = x/|x|$ is the unit vector in the direction of x. Of course, the 'big O' and 'little o' notations in (1.9) and (1.10) have the following meanings: equation (1.9) says that the pressure, $u(x)$, must decrease, as we go to infinity, *at least as fast as* $r^{-(d-1)/2}$; equation (1.10) says that the left hand side of this equation, namely $\partial u/\partial r - iku$ must decrease *faster than* $r^{-(d-1)/2}$. Requiring that (1.9) and (1.10) hold *uniformly* in $\hat{x}$ means, of course, that

$$\max_{\hat{x}\in S} |u(x)| = O\left(r^{-(d-1)/2}\right),$$

$$\max_{\hat{x}\in S} \left|\frac{\partial u}{\partial r}(x) - \mathrm{i}ku(x)\right| = o\left(r^{-(d-1)/2}\right),$$

where $S := \{x \in \mathbb{R}^d : |x| = 1\}$.

The physical basis of the Sommerfeld radiation conditions is as follows. The first condition implies that $|u|^2$ decreases like r^{-1} in 2D, like r^{-2} in 3D. But this is exactly what one expects from energy considerations: the energy is spread over an ever larger and larger cylinder of circumference $2\pi r$ in 2D, is spread over the surface of a sphere of radius $4\pi r^2$ in 3D. The second condition says that $\partial u/\partial r - \mathrm{i}ku$ should be much smaller than $r^{-(d-1)/2}$, and so much smaller than u, when r is large. This makes sense as far away the wave travelling outwards appears locally like a plane wave travelling in the direction $\hat{x}$, i.e. it has the form

$$u(x) = Ae^{ikr}$$

where A is the local amplitude. But, for such an acoustics field it holds that $\partial u/\partial r - iku = 0$ exactly.

1.2 Precise Mathematical Formulations as Boundary Value Problems

In this section we will state precisely the mathematical formulations as a boundary value problem of the main scattering problems that we are going to consider. At this point in the notes I start to assume a greater mathematical sophistication and knowledge of basic mathematics of the numerical solution by finite element methods of PDEs, including some relevant knowledge of function spaces, relevant linear functional analysis, as contained in the notes of Ralf Hiptmair which have been circulated [6]. However, mindful of my own forgetfulness, I will include many reminders of the relevant definitions!

We start by making precise the types of domain that we wish to consider and introducing notations for sets of functions associated with these domains. Perhaps we should first note that, throughout, we understand the word *domain* in its technical, mathematical sense: a domain $\Omega \subset \mathbb{R}^d$ is a *connected, open* subset of $\mathbb{R}^d$. Given a set $\Gamma \subset \mathbb{R}^d$ we denote by $C(\Gamma)$ the set of continuous functions $f : \Gamma \to \mathbb{C}$. Given a domain $\Omega \subset \mathbb{R}^d$ and $n \in \mathbb{N}$ we denote by $C^n(\Omega)$ the set of all $u \in C(\Omega)$ that have well-defined partial derivatives of all orders less than or equal to n that are continuous in Ω.

For a scattering problem one needs an *incident wave.* Throughout u^i will denote the incident field; we always assume that u^i satisfies the Helmholtz equation, at least in a neighbourhood of the scattering obstacle. Most often we consider the simplest case of plane wave incidence, i.e. the case where, for some unit vector $\hat{d}$ (the direction of the plane wave),

$$u^i(x) = e^{ikx\cdot\hat{d}}, \quad x \in \mathbb{R}^d. \tag{1.11}$$

The *total field* in the presence of the obstacle (i.e. what one would measure in an experiment) we denote by u. Then $u^s := u - u^i$ is our notation for the *scattered field.*

The following is the simplest scattering problem that will serve as a model problem for much of these notes, namely acoustic scattering by a bounded sound soft obstacle. We start with the formulation in *classical* function spaces (i.e. where we look for a solution in spaces of continuous functions). When we are dealing with scattering problems we shall denote the domain *exterior* to the bounded scatterer by Ω^+, so that in all the scattering problems we consider the domain $\Omega^+ \subset \mathbb{R}^d$ ($d = 2$ or 3) is assumed to be unbounded, and to include the set $\{x : |x| > R\}$, for some $R > 0$. The complement of Ω^+, i.e. the set $\Upsilon := \mathbb{R}^d \setminus \Omega^+$, we call the *scattering object* or *scatterer* for short. We shall abbreviate $\partial\Omega^+$, the boundary of Ω^+ (which is also the boundary of Υ) by Γ (see Figure 1.2). Finally, $\Omega^- := \Upsilon \setminus \Gamma$ denotes the interior of the scatterer Υ. For much of these notes it will be the case that $\overline{\Omega^-} = \Upsilon$, i.e. Γ is also the boundary of Ω^-, but we do not assume this yet, in particular Ω^- may be the empty set.

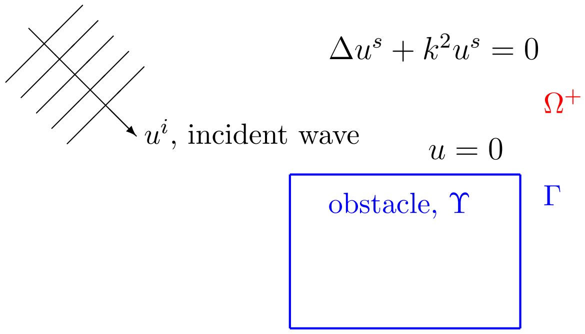

Figure 1.2: Schematic diagram of the Dirichlet scattering problem.

Problem 1.1 (The Dirichlet Scattering Problem - Version 1) *Given $k > 0$ and the incident field u^i, find $u^s \in C^2(\Omega^+) \cap C(\overline{\Omega^+})$ which satisfies the Helmholtz equation (1.4) in Ω^+ and the Sommerfeld radiation conditions (1.9)-(1.10), and is such that $u = u^i + u^s = 0$ on Γ.*

The above is the Dirichlet scattering problem formulated in a classical space setting (as in e.g. Colton and Kress [3, 4]). A very popular alternative is to formulate the scattering problem in a Sobolev space setting. For the moment the following Sobolev space is sufficient for our purpose (for more details of Sobolev spaces see [6] or [8]). Given a domain $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$), let $C_0^\infty(\Omega)$ denote the set of $u \in C^\infty(\Omega)$ whose support is a compact subset of Ω (which implies that u is identically zero in a neighbourhood of $\partial\Omega$). For every $u, v \in C_0^\infty(\Omega)$, the integral

$$(u, v)_{H^1(\Omega)} := \int_\Omega (u\bar{v} + \nabla u \cdot \nabla \bar{v})\, dx \tag{1.12}$$

is well-defined, in fact $(\cdot, \cdot)$ defines an *inner product* on the *vector space* $C_0^\infty(\Omega)$ so that, equipped with $(\cdot, \cdot)$, $C_0^\infty(\Omega)$ is an inner product space or pre-Hilbert space (see [6] for definitions). Of course, as an inner product space, $C_0^\infty(\Omega)$ is automatically also a *normed space*, with norm $\|\cdot\|_{H^1(\Omega)}$ defined by

$$\|u\|_{H^1(\Omega)} := (u, u)_{H^1(\Omega)}^{1/2} = \left\{ \int_\Omega \left(|\nabla u|^2 + |u|^2\right)\, dx \right\}^{1/2}, \tag{1.13}$$

and so also a *metric space*, with metric $d(\cdot,\cdot)$ defined by $d(u,v) := \|u - v\|_{H^1(\Omega)}$. Referring to the basic theory of metric spaces, we make the following definition.

Defintion 1.2 *We define the Sobolev space $H_0^1(\Omega)$ to be the* completion *of the metric space $C_0^\infty(\Omega)$. Extending the definition of the inner product (1.12) to $H_0^1(\Omega)$ in the natural way, $H_0^1(\Omega)$ is a Hilbert space.*

The attraction of this definition is that it is defined in terms of very basic concepts – the basic theory of metric spaces and the set $C_0^\infty(\Omega)$. The less attractive feature is that the elements of $H_0^1(\Omega)$ are elements of the completion of a metric space, i.e. are equivalence classes of Cauchy sequences, which are not as concrete as we would like for communicating with the widest audience!

The alternative definition of $H_0^1(D)$ avoids the notion of the completion of a metric space, but does not really win in terms of easy communication since it instead requires Lebesgue integration and measure and the idea of a weak derivative or the (closely related) idea of a distributional derivative. This alternative definition goes as follows. First, let $L^2(\Omega)$ denote the set of functions $u : \Omega \to \mathbb{C}$ which are Lebesgue measurable and for which

$$\|u\|_2 := \left\{ \int_\Omega |u|^2 \, dx \right\}^{1/2} < \infty. \tag{1.14}$$

It follows from simple properties of Lebesgue measure and fairly elementary arguments that $L^2(\Omega)$ is a normed space (with norm $\|\cdot\|_2$), in fact an inner product space with inner product $(\cdot,\cdot)_2$ defined by

$$(u,v)_2 := \int_\Omega u\bar{v} \, dx. \tag{1.15}$$

(To be precise, to make $\|\cdot\|_2$ a norm rather than a semi-norm we have to modify the definition of $L^2(\Omega)$ slightly, replacing $L^2(\Omega)$ by a new version $\tilde{L}^2(\Omega)$, the elements of $\tilde{L}^2(\Omega)$ being subsets of $L^2(\Omega)$, precisely the equivalence classes of $L^2(\Omega)$ under the equivalence relation of equality almost everywhere, so that two function in $L^2(\Omega)$ are regarded as equivalent if they are equal except on a set of Lebesgue measure zero. However, noting this important detail, for the sake of simplicity we do not hereafter distinguish notationally between $L^2(\Omega)$ and $\tilde{L}^2(\Omega)$.) Slightly deeper arguments (see e.g. [2]) show that $L^2(\Omega)$ is a Hilbert space, i.e. that $L^2(\Omega)$ is a complete metric space with respect to the metric generated by the norm $\|\cdot\|_2$.

Next, note that if $u \in C^1(\Omega)$ and $v \in C_0^\infty(\Omega)$, it holds (e.g. by the divergence theorem or a simple integration by parts) that

$$\int_\Omega u \frac{\partial v}{\partial x_j} \, dx = -\int_\Omega \frac{\partial u}{\partial x_j} v \, dx.$$

Inspired by this equation, let us say that $u \in L^2(\Omega)$ has a partial derivative $\partial_j u = \frac{\partial u}{\partial x_j}$ in a weak sense if there exists a locally Lebesgue integrable function w[3] such that

$$\int_\Omega u \frac{\partial v}{\partial x_j}\, dx = -\int_\Omega wv\, dx, \quad \forall v \in C_0^\infty(\Omega), \tag{1.16}$$

and, if this is the case, let us denote this weak derivative w by $\partial_j u$. It is not difficult (if you know about Lebesgue integration) to show that if w_1 and w_2 are both weak derivatives in this sense then they are equal almost everywhere. It follows that if $u \in C^1(\Omega)$ then a weak derivative $\partial_j u$ of u is equal, almost everywhere, to the classical partial derivative $\frac{\partial u}{\partial x_j}$. Finally, let us say that u has a weak gradient ∇u if u has a weak partial derivative $\partial_j u$, for $j = 1, \ldots, d$.

With these preliminaries we can define both the Sobolev space $H^1(\Omega)$ and $H_0^1(\Omega)$ as a subspace of $H^1(\Omega)$.

Defintion 1.3 *Let the Sobolev space $H^1(\Omega) \subset L^2(\Omega)$ denote the set of those $u \in L^2(\Omega)$ that have a weak gradient ∇u which is also in $L^2(\Omega)$. Then $H^1(\Omega)$ is a Hilbert space with the inner product and norms (1.12) and (1.13). Let $H_0^1(D)$ denote the closure of $C_0^\infty(\Omega) \subset H^1(D)$.*

Of course the above definition implies that, if $u \in H_0^1(\Omega)$ then $\|u\|_{H^1(\Omega)} < \infty$ and there exists a sequence $(u_n) \subset C_0^\infty(\Omega)$ such that $\|u - u_n\|_{H^1(\Omega)} \to 0$ as $n \to \infty$. If Ω is sufficiently regular (e.g. Lipschitz will do) then a similar density result holds for $H^1(\Omega)$ (see [8]), that, if $u \in H^1(\Omega)$, then there exists a sequence $(u_n) \subset C^\infty(\mathbb{R}^d)$ such that $\|u - u_n\|_{H^1(\Omega)} \to 0$ as $n \to \infty$.

The two definitions that we have given above of $H_0^1(\Omega)$ coincide in an appropriate sense.

Finally, to formulate the scattering problem, we need to introduce the vector space $H_0^{1,\mathrm{loc}}(\Omega^+)$. Let $H_0^{1,\mathrm{loc}}(\Omega^+)$ denote the set of functions $u : \Omega \to \mathbb{C}$ such that the product uv is in $H_0^1(\Omega^+)$ for every $v \in C_0^\infty(\mathbb{R}^d)$.

Here is our second, Sobolev space version, of the scattering problem. For convenience, in this second formulation, we restrict our attention to the case when u^i is the incident plane wave given by (1.11).

Problem 1.4 (The Dirichlet Scattering Problem - Version 2) *Given $k > 0$ and the unit vector $\hat{d}$ (the direction of the incident plane wave u^i), find $u \in C^2(\Omega^+) \cap H_0^{1,\mathrm{loc}}(\Omega^+)$ such that u^s satisfies the Helmholtz equation (1.4) in Ω^+ and the Sommerfeld radiation conditions (1.9)-(1.10).*

[3] I.e. a function $w : \Omega \to \mathbb{C}$ which is Lebesgue measurable and for which $\int_S |w|\, dx < \infty$ for every compact set $S \subset \Omega$, this requirement ensuring that the integral on the right hand side of (1.16) exists for all $v \in C_0^\infty(\Omega)$.

Note that in this version of the scattering problem the requirement that $u = 0$ on Γ is imposed in a weak sense, through the requirement that $u \in H_0^{1,\mathrm{loc}}(\Omega^+)$. Regarding the choice of the two formulations and which is correct from a physical point of view we note that:

1. Version 1 requires that $u^s \in C(\overline{\Omega^+})$, so that the acoustic pressure is continuous up to the boundary.

2. Let $\Omega_R^+ := \{x \in \Omega^+ : |x| < R\}$, for $R \geq R_0 := \max_{x\in\Upsilon} |x|$. Then Version 2 requires that $\int_{\Omega_R^+}(|\nabla u|^2 + |u|^2)\, dx < \infty$ for every $R > R_0$, which is a requirement that the acoustic field have locally finite energy.

3. The choice of which version to use turns out, in the end, to be to a large extent irrelevant, in the sense that the problems are almost equivalent in the sense of having the same solutions. Precisely, it can be shown that every solution of Version 1 is also a solution of Version 2 (the arguments to do this are contained in the proof that Version 1 has at most one solution, for which see e.g. [4]). To the best of my knowledge the converse statement may not be true (though I do not know a counter-example). However, it is true that if Ω^+ is a *regular domain* in the sense of e.g. [7], in particular if Ω^+ is a Lipschitz domain in the sense of e.g. [6] or [8], then, if u is a solution of Version 2 then u is equal almost everywhere to a solution $\tilde{u} \in C(\overline{\Omega^+})$ of Version 1.

Proving the last statement is quite tricky (the so-called *regularity arguments* needed are discussed in Chapter 1 of [7] or see [5]). But in fact, while uniqueness and existence of solution for Version 1 can be established fairly easily by integral equation methods in the case when Ω^+ is a fairly smooth (C^2) domain (see [3]), establishing existence of solution for Version 1 for non-smooth, Lipschitz domains requires this indirect route of first establising existence of solution for Version 2 and then arguing that the solution to Version 2 also satisfies Version 1. In the next few sections we will sketch uniqueness and existence of solution, in fact give the solution explicitly in the case when Υ is a ball, focusing on Version 2 of the scattering problem.

To achieve this goal we will need to know a little more about Sobolev spaces, and a little about trace operators.

Defintion 1.5 *For $u \in C^\infty(\bar{\Omega})$ (defined as the set of restrictions to $\bar{\Omega}$ of functions in $C^\infty(\mathbb{R}^d)$) define the* trace *of u on $\partial\Omega$ (denoted γu) to be the restriction of u to $\partial\Omega$. The mapping $\gamma : C^\infty(\bar{\Omega}) \to C^\infty(\partial\Omega)$, which takes u to γu, is called the* trace operator*.*

It is well known that, if the domain Ω is Lipschitz, then there exists a constant $C > 0$ such that[4]

$$\|\gamma u\|_{L^2(\partial\Omega)} \le C\|u\|_{H^1(\Omega)}, \quad \forall u \in C^\infty(\bar{\Omega}). \tag{1.17}$$

Here

$$\|\gamma u\|_{L^2(\partial\Omega)} = \left\{\int_{\partial\Omega} |u|^2\, ds\right\}^{1/2}.$$

The inequality (1.17) implies that exists a unique extension of γ to $H^1(\Omega)$ in such a way that the resulting operator is a bounded linear operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$, satisfying (1.17) for $u \in H^1(\Omega)$.

Remark 1.6 *We remark that requiring that (1.9) hold in the above scattering problems is not strictly necessary. Precisely, we have the following result (e.g. [3] or [4]): if $u \in C^2(\Omega^+)$ satisfies the Helmholtz equation in Ω^+ and the Sommerfeld radiation condition (1.10), then*

$$\begin{aligned} u(x) &= O\left(r^{-(d-1)/2}\right), && (1.18)\\ \frac{\partial u}{\partial r}(x) &= O\left(r^{-(d-1)/2}\right), && (1.19)\end{aligned}$$

as $r \to \infty$, uniformly in $\hat{x} := x/r$.

1.3 Uniqueness of Solution

In this section we sketch a proof of uniqueness of solution for the Dirichlet scattering problem (Version 2). Uniqueness of solution for problems of time harmonic scattering by bounded obstacles depends on the following lemma due to Rellich. (For a proof see e.g. [3].) For $R > 0$ let

$$\Gamma_R := \{x \in \mathbb{R}^d : |x| = R\}.$$

Lemma 1.7 (Rellich's lemma) *If $u \in C^2(\Omega^+)$ satisfies the Helmholtz equation in Ω^+ and*

$$\int_{\Gamma_R} |u|^2\, ds \to 0$$

as $R \to \infty$ then $u = 0$ in Ω^+.

It is an appropriate point to remind the reader of the divergence theorem and Green's first theorem. In the statement of these theorems and later, by $C^m(\bar{\Omega})$ we denote the set of restrictions to $\bar{\Omega}$ of functions in $C^m(\mathbb{R}^d)$.

[4] In fact a stronger result holds, with the left hand side replaced by $\|\gamma u\|_{H^{1/2}(\partial\Omega)}$, a boundary fractional Sobolev space norm.

Theorem 1.8 (Divergence Theorem) *If Ω is a bounded Lipschitz domain and $F : \mathbb{R}^d \to \mathbb{C}^d$ is a C^1 vector field, then*

$$\int_\Omega \nabla \cdot F \, dx = -\int_{\partial\Omega} F \cdot n \, ds,$$

where $n(x)$ is the unit normal at $x \in \partial\Omega$ directed into Ω (which is well-defined almost everywhere on $\partial\Omega$ if Ω is Lipschitz).

Proof. For a proof in the case of a Lipschitz domain, see e.g. [8]. ∎

Theorem 1.9 (Green's first theorem) *If Ω is a bounded Lipschitz domain and $u \in C^2(\bar{\Omega})$, $v \in C^1(\bar{\Omega})$, then*

$$\int_\Omega (v\Delta u + \nabla u \cdot \nabla v) \, dx + \int_{\partial\Omega} v \frac{\partial u}{\partial n} ds = 0.$$

Proof. Apply the divergence theorem to $u\nabla v$. ∎

To make use of Rellich's lemma we prove the following result.

Lemma 1.10 *If $u \in C^2(\Omega^+)$ satisfies the Helmholtz equation in Ω^+ and the Sommerfeld radiation condition (1.10) and if, for some $R > R_0$,*

$$\Im \int_{\Gamma_R} u \frac{\partial \bar{u}}{\partial r} \, ds \geq 0,$$

then $u = 0$ in Ω^+.

Proof. Applying Green's first theorem to u and $\bar{u}$ in $\{x : R < |x| < R_1\}$, with $R_1 > R$, we find that

$$\int_\Omega (|\nabla u|^2 - k^2 |u|^2) \, dx + \int_{\Gamma_R} u \frac{\partial \bar{u}}{\partial r} ds - \int_{\Gamma_{R_1}} u \frac{\partial \bar{u}}{\partial r} ds = 0,$$

so that

$$\Im \int_{\Gamma_{R_1}} u \frac{\partial \bar{u}}{\partial r} \, ds = \Im \int_{\Gamma_R} u \frac{\partial \bar{u}}{\partial r} \, ds \geq 0.$$

Thus, and by the Sommerfeld radiation condition,

$$\begin{aligned} \int_{\Gamma_{R_1}} \left\{ \left| \frac{\partial u}{\partial r} \right|^2 + k^2 |u|^2 \right\} ds &\leq \int_{\Gamma_{R_1}} \left\{ \left| \frac{\partial u}{\partial r} \right|^2 + k^2 |u|^2 + 2k \Im \left(u \frac{\partial \bar{u}}{\partial r} \right) \right\} ds \\ &= \int_{\Gamma_{R_1}} \left| \frac{\partial \bar{u}}{\partial r} - \mathrm{i} k u \right|^2 ds \to 0 \end{aligned}$$

as $R_1 \to \infty$. Applying Rellich's lemma we see that $u = 0$ in Ω^+. ∎

To make use of this result we show the following lemma which is key to establishing both uniqueness and existence. Let

$$R_0 := \max_{x \in \Upsilon} |x|$$

and

$$\Omega_R^+ := \{x \in \Omega^+ : |x| < R\}, \text{ for } R \geq R_0.$$

Lemma 1.11 *If u satisfies Version 2 of the Dirichlet scattering problem, and $v \in H_0^{1,\mathrm{loc}}(D)$, then, for every $R \geq R_0$,*

$$\int_{\Omega_R^+} \left(\nabla u \cdot \nabla \bar{v} - k^2 u \bar{v} \right) \, ds - \int_{\Gamma_R} \gamma \bar{v} \frac{\partial u}{\partial r} \, ds = 0. \tag{1.20}$$

(Here γ is the (bounded) trace operator from $H^1(\Omega_R)$ to $L^2(\Gamma_R)$, where $\Omega_R := \{x : |x| > R\}$.)

Proof. In the case when $v \in C_0^\infty(\Omega^+)$, equation (1.20) follows immediately from Green's theorem (cf. proof of Lemma 1.10). The general case follows by first replacing v by a function in $w \in H_0^1(\Omega^+)$ which coincides with v in $\overline{\Omega_R^+}$. Next one approximates w by $w_n \in C_0^\infty(\Omega^+)$, with $\|w - w_n\|_{H^1(\Omega^+)} \to 0$ as $n \to \infty$ (which implies that $\|\gamma w - \gamma w_n\|_{L^2(\Gamma_R)} \to 0$ as $n \to \infty$), notes that (1.20) holds with v replaced by w_n, and takes the limit as $n \to \infty$. ■

Corollary 1.12 *Version 2 of the Dirichlet scattering problem has at most one solution.*

Proof. Suppose there are two solutions, u_1 and u_2, and let $u := u_1 - u_2$. Then, by Lemma 1.11, (1.20) holds with $v = u$. Taking imaginary parts, we see that

$$\Im \int_{\Gamma_R} u \frac{\partial \bar{u}}{\partial r} \, ds = 0,$$

for all $R > 0$. The result follows from Lemma 1.10. ■

1.4 Scattering by a Circle or Sphere

In this section we give explicitly the solution to the Dirichlet scattering problem (Versions 1 or 2) in the case when

$$\Omega^+ = \Omega_R := \{x : |x| > R\},$$

for some $R > 0$. These solutions are given explicitly in terms of cylindrical and spherical Bessel functions.

For $\nu \geq 0$ let J_ν and Y_ν denote the usual Bessel functions of the first and second kind of order ν (see e.g. [1] for definitions) and let

$$H_\nu^{(1)} := J_\nu + \mathrm{i}Y_\nu$$

denote the corresponding Hankel function of the first kind of order ν. Of course, where C_ν denotes any linear combination of J_ν and Y_ν, it holds that C_ν is a solution of Bessel's equation of order ν, i.e.

$$z^2 C_\nu''(z) + zC_\nu'(z) + (z^2 - \nu^2)C_\nu(z) = 0. \tag{1.21}$$

In the 3D case it is convenient to work also with the spherical Bessel functions j_m, y_m, and

$$h_m^{(1)} := j_m + \mathrm{i}y_m,$$

for $m = 0, 1, \ldots$. These can be defined directly (see e.g. Nédélec [9]) by recurrence relations which imply that

$$h_m^{(1)}(z) = \mathrm{e}^{\mathrm{i}z} p_m(z^{-1}) z^{-1},$$

where p_m is a polynomial of degree m with $p_m(0) = 1$. Alternatively, the spherical Bessel functions can be defined in terms of the usual Bessel functions via the relations

$$j_m(z) = \sqrt{\frac{\pi}{2z}} J_{m+1/2}(z), \quad y_m(z) = \sqrt{\frac{\pi}{2z}} Y_{m+1/2}(z). \tag{1.22}$$

It is convenient also to introduce the notations

$$M_\nu(z) := |H_\nu^{(1)}(z)|, \quad N_\nu(z) := |H_\nu^{(1)\prime}(z)|.$$

The arguments we will make in the next section depend on the fact that $M_\nu(z)$ is decreasing on the positive real axis for $\nu \geq 0$.

Suppose first that $d = 2$ (the scatterer is a circle). Introducing standard cylindrical polar coordinates, we expand u^i on Γ_R as the Fourier series

$$u^i(x) = \sum_{m \in \mathbb{Z}} a_m \mathrm{e}^{\mathrm{i}m\theta},$$

where (R, θ) are the polar coordinates of x. Explicitly, in the case when u^i is the plane wave $u^i(x) = \mathrm{e}^{\mathrm{i}kx_1}$, it is holds that $a_m = \mathrm{i}^m$ ([4, equation (3.66)]), and explicit expressions can be given for a_m also in the case when u^i is the incident field due to a point source ([4, equation (3.65)]), indeed this expression will be key to Ralf's first implementation of the high frequency fast multipole method.

Since $u^i \in C^\infty(\Gamma_R)$ it holds that the series is rapidly converging, i.e. that $a_m = o(|m|^{-p})$ as $|m| \to \infty$, for every $p > 0$. It follows from separation of variables arguments (e.g. see Colton and Kress [4] or Nédélec [9]), and since

$u^s = -u^i$ on Γ_R, that the corresponding Fourier series representation of u^s in G_{R_1} is

$$u^s(x) = -\sum_{m\in\mathbb{Z}} a_m e^{im\theta} \frac{H^{(1)}_{|m|}(kr)}{H^{(1)}_{|m|}(kR)}, \tag{1.23}$$

where (r,θ) are now the polar coordinates of x. Further, properties of the Bessel functions imply that this series, and all its partial derivatives with respect to r and θ, converge absolutely and uniformly in compact subsets of $\overline{\Omega_R}$. It is easy then to check that u^s satisfies the Helmholtz eqution, and that u^s satisfies the Sommerfeld radiation condition can be shown as in [4, Theorem 2.14]. Differentiating with respect to r, we note that

$$\frac{\partial u^s}{\partial r}(x) = -\sum_{m\in\mathbb{Z}} k\, a_m e^{im\theta} \frac{{H^{(1)}}'_{|m|}(kr)}{H^{(1)}_{|m|}(kR)}. \tag{1.24}$$

We turn now to the 3D case $d = 3$. Introducing standard spherical polar coordinates (r,θ,ϕ), we expand u^i on Γ_R as the spherical harmonic expansion

$$u^i(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta,\phi), \tag{1.25}$$

where (R,θ,ϕ) are the spherical polar coordinates of x and the functions Y_ℓ^m, $m = -\ell,\ldots,\ell$, are the standard spherical harmonics of order ℓ (see, for example, [9, Theorem 2.4.4], [4]). We recall (e.g. [9]) that $\{Y_\ell^m : \ell = 0,1,\ldots,\ m = -\ell,\ldots,\ell\}$ is a complete orthonormal sequence in $L^2(S)$, where $S := \{x : |x| = 1\}$ is the unit sphere, and an orthogonal sequence in $H^1(S)$. Since $v \in C^\infty(\Gamma_R)$, it holds that the series is rapidly converging, i.e. that $a_\ell^m = o(|\ell|^{-p})$ as $|\ell| \to \infty$, for every $p > 0$ [9].

The solution of the Dirichlet problem for the Helmholtz equation in the exterior of a sphere is discussed in detail in [9]. It follows from (1.25) and [9, (2.6.55)] that, for $x \in \Omega_R$,

$$u^s(x) = -\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta,\phi) \frac{h_\ell^{(1)}(kr)}{h_\ell^{(1)}(kR)}, \tag{1.26}$$

where (r,θ,ϕ) are now the polar coordinates of x, and hence that [9, (2.6.70)-(2.6.74)]

$$\frac{\partial u^s}{\partial r}(x) = -k \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta,\phi) \frac{{h_\ell^{(1)}}'(kr)}{h_\ell^{(1)}(kR)}. \tag{1.27}$$

1.5 The Dirichlet to Neumann Map for a Circle/Sphere

The above formulae solve the Dirichlet scattering problem for a ball in 2D and 3D by separation of variables, expanding u^s, which satisfies the Helmholtz equation and Sommerfeld radiation condition, as a linear combination of separable solutions of the Helmholtz equation, each of which satisfies the radiation condition, and choosing the coefficients in the series so as to satisfy the boundary condition $u^s = -u^i$ on Γ.

More generally, given any boundary data $\phi \in C^\infty(\Gamma_R)$, the above formulae give a prescription for solving the Dirichlet boundary value problem: find $u \in C^\infty(\overline{\Omega_R})$ such that u satisfies the Helmholtz equation in Ω_R and the Sommerfeld radiation conditions, and $u = \phi$ on Γ_R. Further, the formulae above tell us how to compute $\partial u/\partial r$, in particular the normal derivative $\partial u/\partial r$ on Γ_R.

We will call the operator $T_R : C^\infty(\Gamma_R) \to C^\infty(\Gamma_R)$ which maps the *Dirichlet data* ϕ on Γ_R to the corresponding *Neumann data*, the trace of $\frac{\partial u}{\partial r}$ on Γ_R, where u is the solution to the Dirichlet problem with data ϕ, the *Dirichlet to Neumann map*. Explicitly, in the 2D case $d = 2$, if ϕ has the Fourier series expansion

$$\phi(x) = \sum_{m\in\mathbb{Z}} a_m \mathrm{e}^{\mathrm{i}m\theta},$$

where (R, θ) are the polar coordinates of x, then (see (1.23) and 1.24)) the solution to the Dirichlet problem is

$$u(x) = \sum_{m\in\mathbb{Z}} a_m \mathrm{e}^{\mathrm{i}m\theta} \frac{H^{(1)}_{|m|}(kr)}{H^{(1)}_{|m|}(kR)}, \tag{1.28}$$

and

$$T_R\phi(x) = k \sum_{m\in\mathbb{Z}} a_m \mathrm{e}^{\mathrm{i}m\theta} \frac{H^{(1)'}_{|m|}(kR)}{H^{(1)}_{|m|}(kR)}, \quad x \in \Gamma_R. \tag{1.29}$$

Similarly, in the 3D case, if ϕ has the spherical harmonics expansion

$$\phi(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta, \phi),$$

where (R, θ, ϕ) are the spherical polar coordinates of x, then (see (1.26) and (1.27) or [9])

$$u(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta, \phi) \frac{h_\ell^{(1)}(kr)}{h_\ell^{(1)}(kR)}, \tag{1.30}$$

and

$$T_R\phi(x) = k\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m Y_\ell^m(\theta,\phi)\frac{h_\ell^{(1)'}(kR)}{h_\ell^{(1)}(kR)}, \quad x \in \Gamma_R. \tag{1.31}$$

Where $H^s(\Gamma_R)$ is the surface Sobolev space of order s, it will be significant shortly that (see e.g. [9]) there exists a unique extension of T_R to the Sobolev space $H^s(\Gamma_R)$, for any $s \in \mathbb{R}$, and that it holds that

$$T_R : H^s(\Gamma_R) \to H^{s-1}(\Gamma_R) \tag{1.32}$$

and is bounded [9]. The following properties of the Dirchlet to Neumann map will also be important.

Lemma 1.13 *For all $R > 0$ and all $\phi \in H^{1/2}(\Gamma_R)$ it holds that*

$$\Re\int_{\Gamma_R}\bar{\phi}T_R\phi\, ds \le 0 \text{ and } \Im\int_{\Gamma_R}\bar{\phi}T_R\phi\, ds \ge 0.$$

Proof. In view of the mapping property (1.32) with $s = 1/2$, it is enough to show the lemma for the case $\phi \in C^\infty(\Gamma_R)$ to deduce the result for the general case. So assume that $\phi \in C^\infty(\Gamma_R)$.

In the 2D case, where $T_R\phi$ is given by (1.29), defining $c_0 := |a_0|^2/|H_0^{(1)}(kR)|^2$ and, for $m \in \mathbb{N}$, $c_m := (|a_m|^2 + |a_{-m}|^2)/|H_m^{(1)}(kR)|^2$, and $\rho := kR$, and using the orthogonality of $\{\mathrm{e}^{im\theta} : m \in \mathbb{Z}\}$, we see that

$$\begin{aligned}\int_{\Gamma_R}\bar{\phi}T_R\phi\, ds &= 2\pi\rho\sum_{m\in\mathbb{Z}}|a_m|^2\frac{\overline{H_{|m|}^{(1)}(\rho)}H_{|m|}^{(1)'}(\rho)}{|H_{|m|}^{(1)}(kR)|^2}\\ &= 2\pi\rho\sum_{m=0}^{\infty}c_m\left(\Re\left(\overline{H_m^{(1)}(\rho)}\,H_m^{(1)'}(\rho)\right) + \mathrm{i}\left(J_m(\rho)Y_m'(\rho) - J_m'(\rho)Y_m(\rho)\right)\right)\\ &= \sum_{m=0}^{\infty}c_m\left(\pi\rho\frac{d}{d\rho}\left(M_m^2(\rho)\right) + 4\mathrm{i}\right),\end{aligned} \tag{1.33}$$

where in the last step we have used the Wronskian formula [1, (9.1.16)] that

$$\pi\rho(J_\nu(\rho)Y_\nu'(\rho) - J_\nu'(\rho)Y_\nu(\rho)) = 2. \tag{1.34}$$

Since $M_m(\rho)$ is decreasing on $(0,\infty)$ we see that the lemma holds in the 2D case.

In the 3D case with $T_R\phi$ given by (1.31), using the orthonormality in $L^2(S)$ of the spherical harmonics Y_ℓ^m, we see that, where $c_\ell := |h_\ell^{(1)}(kR_1)|^{-2}\sum_{m=-\ell}^{\ell}|a_\ell^m|^2$ and $\rho := kR$,

$$\begin{aligned}\int_{\Gamma_R}\bar{\phi}T_R\phi\, ds &= R^2\int_S\bar{v}(R\hat{x})\frac{\partial v}{\partial r}(R\hat{x})\, ds(\hat{x}) = R\rho\sum_{\ell=0}^{\infty}c_\ell\,\overline{h_\ell^{(1)}(\rho)}h_\ell^{(1)'}(\rho)\\ &= R\sum_{\ell=0}^{\infty}c_\ell\left(\frac{\rho}{2}\frac{d}{d\rho}\left(|h_\ell^{(1)}(\rho)|^2\right) + \frac{\mathrm{i}}{\rho}\right),\end{aligned} \tag{1.35}$$

where in the last step we have used (1.22) and (1.34). Recalling that $|h_\ell^{(1)}(\rho)| = \sqrt{\pi/(2\rho)}M_{\ell+1/2}(\rho)$ is decreasing on $(0,\infty)$, we see that the lemma holds in the 3D case. ■

1.6 The Weak or Variational Formulation

We now have the ingredients to make a *weak* or *variational* formulation of Version 2 of the Dirichlet scattering problem.

First, from Lemma 1.11, we have that, if u satisfies Version 2 of the Dirichlet scattering problem, and $v \in H_0^{1,\mathrm{loc}}(D)$, then, for every $R > R_0 = \max_{x\in\Upsilon}|x|$,

$$\int_{\Omega_R^+} \left(\nabla u \cdot \nabla \bar{v} - k^2 u\bar{v}\right)\, ds - \int_{\Gamma_R} \gamma\bar{v}\frac{\partial u}{\partial r}\, ds = 0. \tag{1.36}$$

Next we have that on Γ_R, $\frac{\partial u^s}{\partial r} = T_R\gamma u^s$. Combining these equations, and defining, for $R > R_0$,

$$V_R := \{v|_{\Omega_R^+} : v \in H_0^1(\Omega^+)\} \subset H^1(\Omega_R^+)$$

and the sesquilinear form $a(\cdot,\cdot)$ and anti-linear functional f on V_R by

$$a(u,v) := \int_{\Omega_R^+} (\nabla u \cdot \nabla\bar{v} - k^2 u\bar{v})\, dx - \int_{\Gamma_R} \gamma\bar{v}T_R\gamma u\, ds, \tag{1.37}$$

and

$$f(v) := \int_{\Gamma_R} \bar{v}\left(\frac{\partial u^i}{\partial r} - T_R u^i\right)\, ds, \tag{1.38}$$

we see that we have shown that $u|_{\Omega_R^+}$ satisfies the variational problem: find $u \in V_R$ such that

$$a(u,v) = f(v), \quad v \in V_R. \tag{1.39}$$

The arguments leading to the variational problem can also be reversed. Thus, altogether, we have the following theorem.

Theorem 1.14 *If u is a solution to Version 2 of the Dirchlet scattering problem then $u|_{\Omega_R^+} \in V_R$ satisfies (1.39). Conversely, suppose $u \in V_R$ satisfies (1.39), let $F_R := \gamma u^s$ be the trace of $u^s = u - u^i$ on Γ_R, and extend the definition of $u = u^i + u^s$ to Ω^+ by setting $u^s|_{\Omega_R}$ to be the solution of the Dirichlet problem in Ω_R, with data F_R on Γ_R (this solution given explicitly by (1.28) and (1.30), in the cases $d = 2$ and $d = 3$, respectively). Then this extended function satisfies Version 2 of the Dirichlet scattering problem.*

1.7 Existence of Solution to the Dirichlet Scattering Problem

Our approach to showing existence of solution to Version 2 of the Dirichlet scattering problem is to use Theorem 1.14 and show existence of solution to the equivalent variational problem (1.39). To do this we use the general theory of linear variational problems (e.g. [6]).

First we split a into a V_R-elliptic part and a compact part, a_0 and k, respectively, defined by

$$a_0(u,v) := \int_{\Omega_R^+} (\nabla u \cdot \nabla \bar{v} + k^2 u\bar{v})\, dx - \int_{\Gamma_R} \gamma\bar{v} T_R \gamma u\, ds, \tag{1.40}$$

$$k(u,v) := -2k^2 \int_{\Omega_R^+} u\bar{v}\, dx. \tag{1.41}$$

Both a_0 and k are bounded sesquilinear forms (the boundedness of a_0 depends on the boundedness of $T_R : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ and of the trace operator $\gamma : V_R \to H^{1/2}(\Gamma_R)$). Indeed k is compact (i.e. the associated linear operator [6] $K : V_R \to V_R'$ (V_R' the dual space of V_R) is compact), which follows from the compactness of the imbedding operator $V_R \to L^2(\Omega_R^+)$. Finally, for some $\gamma > 0$,

$$\Re a_0(u,u) \geq \gamma \|u\|^2_{H^1(\Omega_R^+)},$$

which follows from Lemma 1.13. Thus $a = a_0 + k$ is *coercive.* Since the weak problem is equivalent to Version 2 of the Direct scattering problem (Theorem 1.14) and the Direct scattering problem has at most one solution (Corollary 1.12), it follows that the weak problem (1.39) and Version 2 of the Dirichlet scattering problem have exactly one solution.

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