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High frequency homogenization of bubbly crystals*

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Abstract

This paper is concerned with the high frequency homogenization of bubbly phononic crystals. It is a follow-up of the work [H. Ammari et al., Subwavelength phononic bandgap opening in bubbly media, J. Diff. Eq., 263 (2017), 5610–5629] which shows the existence of a subwavelength band gap in such media. We show that the first Bloch eigenvalue achievs its maximum at the corner of the Brillouin zone. By computing the asymptotic of the Bloch eigenfunctions in the periodic structure near that critical frequency, we demonstrate that these eigenfunctions can be decomposed into two parts: one part is slowly varying and satisfies a homogenized equation, while the other is periodic across each elementary crystal cell and is varying. This is very different from the usual homogenization where the second part is constant. This homogenization theory is termed high frequency homogenization in the sense that it is concerned with the asymptotic of wave fields near the critical frequency where a subwavelength band gap opens rather than the zero frequency. Our results shed light into the wave propagation theory in metamaterials. In particular, they rigorously justify, in the nondilute case, the observed superfocusing of acoustic waves in bubbly crystals near and below the maximum of the first Bloch eigenvalue.

Mathematics Subject Classification (MSC2000). 35R30, 35C20.

Keywords. bubble, phononic crystal, band gap, Bloch theory, homogenization, metamaterial

1 Introduction

This paper is devoted to the understanding of wave propagation in metamaterials (see for instance [17]) which consists of subwavelength resonators arranged periodically in a background medium. These metamaterials differ from the usual photonic/phononic

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crystals (see for instance [20]) in the sense that their periods are much smaller than the free space wavelength of the functioning frequency of the materials. Note that in the later case the periods are comparable to the wavelengths. Because of the subwavelength scale of the period, a homogenization theory is possible to describe the marcoscopic behavior of the materials, and this results in effective media having negative parameters such as negative mass, negative bulk modulus, negative electric permittivity, negative magnetic permeability, or double negative refractive index, or high contrast. The study of these metamaterials has drawn increasing interest nowadays because of their many important applications in fields such as superresolution, cloaking, and novel optic and phononic devices [32, 31].

There are many interesting mathematical works related to the homogenization theory for metamaterials, see for instance [13, 22, 26, 11]. Nevertheless, there is still much to understand about the wave propagation in these materials. Compared to the classic homogenization theory [12, 16, 1, 30], the local resonance or cell resonance [27, 28] in the quasi-static regime produces strong interactions among the cells in the periodic structure and this can induce rich physics on the subwavelength scale which cannot be understood by the standard homogenization theory. Especially, one is interested in the high frequency regime when the frequency is near the critical frequency where a subwavelength band gap of the periodic structure opens. Because this gap opens in the quasi-static regime (corresponding to a subwavelength scale), a homogenization theory is possible for such a structure. We note that the critical frequency usually occurs at the corner (or edge in the two dimensional case) of the Brillouin zone in three dimensions where one has typically anti-periodic Bloch eigenfunctions. In this frequency regime, the Bloch eigenfunctions vary on the microscale (the scale of the elementary crystal cell), and thus, a homogenization theory which describes the macroscopic behavior of the wave field seems impossible at first glance. This also makes the interpretation of the possible homogenization theory a perplexing task. On the other hand, the standard homogenization theory is applicable to the Bloch eigenfunctions which are near the center of the Brillouin zone and hence a much lower frequency regime. For this reason, the homogenization theory developed in this work is termed a high frequency homogenization.

The bubbly media, because of the simplicity of the constituent resonant structure, the air bubbles, become a natural model for such studies. It is known that a single bubble in the water possesses a quasi-static resonance which is called the Minnaert resonance [29, 4]. This resonance makes the air bubble an ideal subwavelength resonator (the bubble can be two order of magnitude smaller than the wavelength at the resonant frequency) and hence a basic building block for metamaterials. We refer to [24, 23, 25, 33] for the experiments which motivated our series of studies of bubbles [4, 2, 10, 9]. In [9], using the fact that a single bubble can be well approximated by a monopole, and the point interaction approximation, we derived an effective medium theory for bubbly media consisting of dilute bubbles which may not be arranged periodically in a bounded domain. Our results show that, in the dilute case, near and below the Minnaert resonant frequency, the effective medium has high refractive index, which explains the superfocusing phenomenon observed in the experiment reported in [23]; while near and above the Minnaert resonant frequency, the effective medium is dissipative.

Motivated by this work, we investigated the band structure of a bubbly phononic crystal which is made of periodically arranged bubbles in a homogeneous fluid [2]. We showed that there exists a subwavelength band gap in such a structure. This subwavelength band gap is mainly due to the cell resonance of the bubbles in the quasi-static regime and is quite different from the usual band gaps in photonic/phononic crystals where the gap opens at wavelength which is comparable to the period of the structure [19, 3, 5]. We refer to [10] for the related work on bubbly metasurfaces which is a homogenization theory for a thin layer of periodically arranged bubbles mounted on a perfect reflection surface. We also refer to [14, 15] and the references therein for the other interesting related works on wave propagation in bubbly media.

In this paper, based on the previous two works: effective medium theory for bubbly media in the dilute regime and the existence of a subwavelength band gap in the bubbly crystals, we further investigate the homogenization theory of the bubbly crystal near the frequency where the band gap opens. Our main approach is based on rigorous asymptotic analysis and layer potential techniques [5] which enables us to derive explicit formulas for the Bloch eigenfunctions. It is these formulas which make both the homogenization theory and the justification of the superfocusing phenomenon in the nondilute case possible.

We remark that our paper is related to the works [18, 11]. There are three major differences. (i) The homogenization in [18, 11] is concerned with the perturbation of the standing waves which are the Bloch eigenfunction at the edge of the Brillouin zone (in the two dimensional case). Our work is concerned with Bloch eigenfunctions near which the subwavelength band gap opens. We show that the band gap opens at the corner (edge in two dimensions) of the Brillouin zone; (ii) The main approach in [18, 11] is based on a two-scale analysis, while our work relies on layer potential techniques; (iii) The theory in [18, 11] may be restricted to the two dimensional case for certain structures, while our theory is applicable to any metamaterial where a subwavelength band gap exists. To sum up, this work complements the results of [18, 11].

The paper is organized in the following way. In Section 2, we state the high frequency homogenization problem for the bubbly crystal, which we are interested in. Then in Section 3, we introduce some preliminaries on the layer potentials and quasi-periodic layer potential techniques. Next in Section 4, we consider the normalized unit cell problem. We show that the first Bloch eigenvalue attains its maximum at the corner of the Brillouin zone. We also derive its asymptotic near that corner point. Finally in Section 5, we derive the high frequency homogenization theory by analyzing the asymptotic of the Bloch eigenfunctions when the frequency is near the critical frequency which is the maximum of the first Bloch eigenvalue and where a band gap opens.

2 Problem setup

We first describe the bubble phononic crystal under consideration. Let Y be the unit cell $[-1/2, 1/2]^3$ in $\mathbb{R}^3$, and let D be a bounded and simply connected smooth domain contained in Y. The bubbles are periodically arranged with period $s > 0$ in each direction. More precisely, let $D_s = sD$ be the domain occupied by the bubble in the unit cell $Y_s = [-\frac{s}{2}, \frac{s}{2}]^3$. Then the bubbles occupy the domain $\cup_{n\in\mathbb{Z}^d}(sD + n)$. We denote by ρ_b and κ_b the density and the bulk modulus of the air inside the bubbles, respectively, and by ρ and κ the corresponding parameters for the background medium. Let $B_s = [-\frac{\pi}{s}, \frac{\pi}{s}]^3$ be the Brillouin zone corresponding to the periodic structure. The Bloch eigenvalues and eigenfunctions are solutions to the following α-periodic equations in the cell Y_s for each $\alpha \in B_s$:

$$\begin{cases} \nabla \cdot \dfrac{1}{\rho} \nabla u + \dfrac{\omega^2}{\kappa} u = 0 \quad \text{in} \quad Y_s \backslash D_s, \\ \nabla \cdot \dfrac{1}{\rho_b} \nabla u + \dfrac{\omega^2}{\kappa_b} u = 0 \quad \text{in} \quad D_s, \\ u_+ - u_- = 0 \quad \text{on} \quad \partial D_s, \\ \dfrac{1}{\rho} \dfrac{\partial u}{\partial \nu}\Big|_+ - \dfrac{1}{\rho_b} \dfrac{\partial u}{\partial \nu}\Big|_- = 0 \quad \text{on} \quad \partial D_s, \\ e^{-i\alpha\cdot x} u \ \text{ is periodic.} \end{cases} \tag{2.1}$$

Here, $\partial/\partial\nu$ denotes the outward normal derivative and $|_\pm$ denote the limits from outside and inside D.

Let

$$v = \sqrt{\frac{\kappa}{\rho}}, \quad v_b = \sqrt{\frac{\kappa_b}{\rho_b}}, \quad k = \frac{\omega}{v} \quad \text{and} \quad k_b = \frac{\omega}{v_b}$$

be respectively the speed of sound outside and inside the bubbles, and the wavenumber outside and inside the bubbles. We also introduce the dimensionless contrast parameter

$$\delta = \frac{\rho_b}{\rho}.$$

For bubbly media, we assume that $\delta \ll 1$, justifying the high contrast nature of the media. In realistic setup, δ may be of the order of 10^{-3}. On the other hand, we assume that

$$\frac{k_b}{k} = \frac{v}{v_b} = \sqrt{\frac{\rho_b \kappa}{\rho \kappa_b}} = O(1),$$

i.e., the wave numbers inside and outside the bubbles are comparable.

It is known that (2.1) has nontrivial solutions for discrete values of ω which are called the Bloch eigenvalues. These eigenvalues can be arranged in the following increasing manner (see [5]):

$$0 \le \omega_{1,s}^\alpha \le \omega_{2,s}^\alpha \le \cdots .$$

We denote the Bloch eigenfunction corresponding to the eigenvalue $\omega_{j,s}^\alpha$ by $u_{j,s}^\alpha$.

We have the following band structure of propagating frequencies for the given periodic structure:

$$[0, \max_\alpha \omega_{1,s}^\alpha] \cup [\min_\alpha \omega_{2,s}^\alpha, \max_\alpha \omega_{2,s}^\alpha] \cup [\min_\alpha \omega_{3,s}^\alpha, \max_\alpha \omega_{3,s}^\alpha] \cup \cdots .$$

In [2], it is shown that there is a subwavelength band gap in the above band structure for fixed s (say $s = 1$) and sufficiently small δ. More precisely, one has

$$\omega_*^s := \max_\alpha \omega_{1,s}^\alpha = O(\delta^{\frac{1}{2}}) < \min_\alpha \omega_{2,s}^\alpha = O(1).$$

In this paper, we investigate the asymptotic properties of the Bloch eigenfunctions in the bubbly crystal when the frequency ω is near the critical frequency ω_*^s where the subwavelength band gap opens for $s \ll 1$. By a scaling argument, it can be shown that

$$\omega_*^s = \frac{1}{s}\omega_*^1.$$

In order to fix the critical frequency in the limit when s tends to zero, we rescale the contrast parameter δ as follows:

$$\delta = O(s^2).$$

Then the critical frequency remains of order one in the limiting process when s tends to zero. Thus we are in the situation when the wavelength (of the free space) is of order one and the cell size is of order $s \ll 1$. As a result, one can develop a homogenization theory for such a structure. Indeed, in what follows, we shall show that the Bloch eigenfunctions $u_{1,s}^\alpha$, when the frequency is near the critical frequency ω_*^s, can be decomposed into two parts: one part is slowly varying and satisfies a homogenized equation, while the other is periodic across each elementary crystal cell and is rapidly varying. This is very different from the usual homogenization where the second part is constant. Our main approach is the layer potential techniques which we shall introduce in the next section. It is also worth emphasizing that the rapid variations of the second part of the solution justifies the superfocusing of acoustic waves in bubbly crystals experimentally observed in [21].

3 Preliminaries

We collect notations and some results regarding the Green function and the quasi-periodic Green's function for the Helmholtz equation in three dimensions. We refer to [5] and the references therein for the details.

We introduce the single layer potential $\mathcal{S}_D^k : L^2(\partial D) \to H^1(\partial D), H^1_{loc}(\mathbb{R}^3)$ associated with D and the wavenumber k defined by, $\forall \mathbf{x} \in \mathbb{R}^3$,

$$\mathcal{S}_D^k[\psi](\mathbf{x}) := \int_{\partial D} G^k(\mathbf{x}, \mathbf{y})\psi(\mathbf{y})\mathrm{d}\sigma(\mathbf{y}),$$

where

$$G^k(\mathbf{x}, \mathbf{y}) := -\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|},$$

is the Green function of the Helmholtz equation in $\mathbb{R}^3$, subject to the Sommerfeld radiation condition. Here, $L^2(\partial D)$ is the space of square integrable functions and $H^1(\partial D)$ is the standard Sobolev space.

We also define the boundary integral operator $(\mathcal{K}_D^k)^* : L^2(\partial D) \to L^2(\partial D)$ by

$$(\mathcal{K}_D^k)^*[\psi](\mathbf{x}) := \text{p.v.} \int_{\partial D} \frac{\partial G_k(\mathbf{x}, \mathbf{y})}{\partial \nu(\mathbf{x})} \psi(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}), \quad \forall \mathbf{x} \in \partial D.$$

Here p.v. stands for the Cauchy principal value. We use the notation $\frac{\partial}{\partial \nu}\Big|_{\pm}$ indicating

$$\frac{\partial u}{\partial \nu}\Big|_{\pm}(\mathbf{x}) = \lim_{t \to 0^+} \langle \nabla u(\mathbf{x} \pm t\nu(\mathbf{x})), \nu(\mathbf{x}) \rangle,$$

with ν being the outward unit normal vector to ∂D. Then the following jump formula holds:

$$\frac{\partial}{\partial \nu}\Big|_{\pm} \mathcal{S}_D^k[\phi](\mathbf{x}) = \left(\pm \frac{1}{2} I + (\mathcal{K}_D^k)^* \right) [\phi](\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \partial D.$$

We now define quasi-periodic layer potentials. Let $Y = Y_1$ be the unit cell $[-1/2, 1/2]^3$. For $\alpha \in [-\pi, \pi]^3$, the function $G^{\alpha,k}$ is defined to satisfy

$$(\triangle_{\mathbf{x}} + k^2) G^{\alpha,k}(\mathbf{x}, \mathbf{y}) = \sum_{n \in \mathbb{R}^3} \delta(\mathbf{x} - \mathbf{y} - n) e^{\mathrm{i} n \cdot \alpha},$$

where δ is the Dirac delta function and $G^{\alpha,k}$ is α-quasi-periodic, i.e., $e^{-\mathrm{i}\alpha \cdot \mathbf{x}} G^{\alpha,k}(\mathbf{x}, \mathbf{y})$ is periodic in $\mathbf{x}$ with respect to Y. It is known that $G^{\alpha,k}$ can be written as

$$G^{\alpha,k}(\mathbf{x}, \mathbf{y}) = \sum_{n \in \mathbb{Z}^3} \frac{e^{\mathrm{i}(2\pi n + \alpha) \cdot (\mathbf{x} - \mathbf{y})}}{k^2 - |2\pi n + \alpha|^2},$$

if $k \neq |2\pi n + \alpha|$ for any $n \in \mathbb{Z}^3$. We remark that

$$G^{\alpha,k}(\mathbf{x}, \mathbf{y}) = G^{\alpha,0} + \sum_{\ell=1}^{\infty} k^{2\ell} G_\ell^{\alpha,\#} := G^{\alpha,0}(\mathbf{x}, \mathbf{y}) - \sum_{\ell=1}^{\infty} k^{2\ell} \sum_{n \in \mathbb{Z}^d} \frac{e^{i(2\pi n + \alpha) \cdot (\mathbf{x} - \mathbf{y})}}{|2\pi n + \alpha|^{2(\ell+1)}} \tag{3.1}$$

when $\alpha \neq 0$, and $k \to 0$.

We are ready to define the quasi-periodic single layer potential $\mathcal{S}_D^{\alpha,k}$:

$$\mathcal{S}_D^{\alpha,k}[\phi](\mathbf{x}) = \int_{\partial D} G^{\alpha,k}(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) d\sigma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3.$$

Then $\mathcal{S}^{\alpha,k}[\phi]$ is an α-quasi-periodic function satisfying the Helmholtz equation $(\triangle + k^2)u = 0$. In addition, one has the jump formula:

$$\frac{\partial}{\partial \nu}\Big|_{\pm} \mathcal{S}_D^{\alpha,k}[\phi](\mathbf{x}) = \left(\pm \frac{1}{2} I + (\mathcal{K}_D^{-\alpha,k})^* \right) [\phi](\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \partial D,$$

where $(\mathcal{K}_D^{-\alpha,k})^*$ is the operator given by

$$(\mathcal{K}_D^{-\alpha,k})^*[\phi](\mathbf{x}) = \text{p.v.} \int_{\partial D} \frac{\partial}{\partial \nu(x)} G^{\alpha,k}(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) d\sigma(\mathbf{y}).$$

We remark that $\mathcal{S}_D^0$, $\mathcal{S}_D^{\alpha,0} : L^2(\partial D) \to H^1(\partial D)$ are invertible for $\alpha \neq 0$; see [5]. Moreover, the following decomposition holds for the layer potential $\mathcal{S}_D^{\alpha,k}$:

$$\mathcal{S}_D^{\alpha,k} = \mathcal{S}_D^{\alpha,0} + \sum_{\ell=1}^{\infty} k^{2\ell} \mathcal{S}_{D,\ell}^{\alpha} \quad \text{with} \quad \mathcal{S}_{D,\ell}^{\alpha}[\psi] := \int_{\partial D} G_\ell^{\alpha,\#}(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{3.2}$$

where the convergence holds in $\mathcal{B}(L^2(\partial D), H^1(\partial D))$, the set of linear bounded operators from $L^2(\partial D)$ onto $H^1(\partial D)$.

Finally, we introduce the α-quasi capacity of D, denoted by $\mathrm{Cap}_{D,\alpha}$,

$$\mathrm{Cap}_{D,\alpha} := \int_{Y \setminus D} |\nabla u|^2,$$

where u is the α-periodic harmonic function in $Y \setminus \bar{D}$ with $u = 1$ on ∂D. For $\alpha \neq 0$, we have $u(x) = \mathcal{S}_D^{\alpha,0} \left(\mathcal{S}_D^{\alpha,0}\right)^{-1}[1](x)$ and

$$\mathrm{Cap}_{D,\alpha} := - \int_{\partial D} \left(\mathcal{S}_D^{\alpha,0}\right)^{-1}[1](y) d\sigma(y).$$

Moreover, we have a variational definition of $\mathrm{Cap}_{D,\alpha}$. Indeed, let $C_\alpha^\infty(Y)$ be the set of C^∞ functions in Y which can be extended to C^∞ α-periodic functions in $\mathbb{R}^3$. Let $\mathcal{H}_\alpha$ be the closure of the set $C_\alpha^\infty(Y)$ in H^1, and let $\mathcal{V}_\alpha := \{v \in \mathcal{H}_\alpha : v = 1 \text{ on } \partial D\}$. Then we can show that

$$\mathrm{Cap}_{D,\alpha} = \min_{v \in \mathcal{V}_\alpha} \int_{Y \setminus D} |\nabla v|^2.$$

4 The Bloch eigenvalue in the unit period case

We consider the bubbly phononic crystal when the period $s = 1$ in this section. For ease of notation, we write $\omega_{1,1}^\alpha = \omega_1^\alpha$ and $u_{1,1}^\alpha = u_1^\alpha$. We are interested in the point in the Brillouin zone where the maximum of ω_1^α is achieved. It is clear that due to time reversal symmetry, we have

$$\omega_1^\alpha = \omega_1^{-\alpha}.$$

On the other hand,

$$\omega_1^\alpha = \omega_1^{\alpha+(2\pi,2\pi,2\pi)}.$$

Therefore,

$$\omega_1^\alpha = \omega_1^{-\alpha+(2\pi,2\pi,2\pi)}.$$

It then follows that (π, π, π) is a critical point of ω_1^α. In what follows, we prove that (π, π, π) is a maximum point with the following symmetry assumption on D.

Assumption 4.1 *D is symmetric with respect to planes $\{(x_1, x_2, x_3) : x_j = 0\}$, $j = 1, 2, 3$.*

Proposition 4.2 *Suppose that Assumption 4.1 holds. Let $\alpha^* := (\pi, \pi, \pi)$. Then $Cap_{D,\alpha}$ and ω_1^α attain their maxima at $\alpha = \alpha^*$.*

Proof. By the variational principle, we have the following characterization of $\mathrm{Cap}_{D,\alpha}$:

$$\mathrm{Cap}_{D,\alpha} = \min_{v \in \mathcal{V}_\alpha} \int_{Y \setminus D} |\nabla v|^2.$$

Let $\mathcal{V}_{\alpha,0} := \{v \in \mathcal{H}_\alpha : v = 1 \text{ on } \partial D, \quad v = 0 \text{ on } \partial Y\}$. Then $\mathcal{V}_{\alpha,0} \subset \mathcal{V}_\alpha$ and

$$\mathrm{Cap}_{D,\alpha} = \min_{v \in \mathcal{V}_\alpha} \int_{Y \setminus D} |\nabla v|^2 \leq \min_{v \in \mathcal{V}_{\alpha,0}} \int_{Y \setminus D} |\nabla v|^2.$$

Now, let v_0 be a harmonic function satisfying

$$\begin{cases} v_0 = 1 & \text{on } \partial D, \\ v_0 = 0 & \text{on } \partial Y. \end{cases} \tag{4.1}$$

Since D is symmetric, v_0 is symmetric with respect to the planes $x_j = 0$, $j = 1, 2, 3$, and hence v_0 can be extended to an α^*-quasi-periodic function (anti-periodic in each direction). By the characterization of $\mathrm{Cap}_{D,\alpha^*}$, we have

$$\int_{Y \setminus D} |\nabla v_0|^2 = \mathrm{Cap}_{D,\alpha^*}.$$

It follows that

$$\mathrm{Cap}_{D,\alpha} \leq \min_{v \in \mathcal{Y}} \int_{Y \setminus D} |\nabla v|^2 \leq \int_{Y \setminus D} |\nabla v_0|^2 = \mathrm{Cap}_{D,\alpha^*}.$$

Thus $\mathrm{Cap}_{D,\alpha}$ attains its maximum at $\alpha = \alpha^*$.

The argument for ω_1^α is similar to the one for $\mathrm{Cap}_{D,\alpha}$. By symmetry, we see that $u_1^{\alpha^*}$ is symmetric with respect to the planes $\{x_j = 0\}$, $j = 1, 2, 3$. This combined with anti-periodicity yields that $u_1^{\alpha^*}$ is zero on ∂Y. Let

$$\mathcal{H}_{\alpha,0} := \{u \in \mathcal{H}_\alpha : \ u = 0 \text{ on } \partial Y\}.$$

Then

$$(\omega_1^\alpha)^2 = \min_{u \in \mathcal{H}_\alpha} \frac{\int_Y \rho^{-1} |\nabla v|^2}{\int_Y \kappa^{-1} |v|^2} \leq \min_{u \in \mathcal{H}_{\alpha,0}} \frac{\int_Y \rho^{-1} |\nabla v|^2}{\int_Y \kappa^{-1} |v|^2} \leq \frac{\int_Y \rho^{-1} |\nabla u_1^{\alpha^*}|^2}{\int_Y \kappa^{-1} |u_1^{\alpha^*}|^2} = (\omega_1^{\alpha^*})^2.$$

This completes the proof. □

Remark 1 *We conjecture that the maximum of the first Bloch eigenvalue is achieved at the corner of the Brillouin zone for general periodic metamaterials under a similar symmetry assumption as Assumption 4.1 on the cell structure.*

In the sequel, we suppose that Assumption 4.1 holds. To simplify the calculations and the presentation, we also assume

Assumption 4.3 *The wave speed inside the bubble is equal to the one outside, i.e., $v = v_b$.*

Let u_1^α be the α-quasi-periodic propagating wave mode corresponding to ω_1^α, which is given in [2] by

$$u_1^\alpha = \begin{cases} \mathcal{S}_D^{\alpha,\omega_1^\alpha/v}\left(\mathcal{S}_D^{\alpha,0}\right)^{-1}[1] + O(\delta^{1/2}) & \text{in } Y \setminus D, \\ \mathcal{S}_D^{\omega_1^\alpha/v_b}\left(\mathcal{S}_D^{0}\right)^{-1}[1] + O(\delta^{1/2}) & \text{in } D. \end{cases} \tag{4.2}$$

Since we have

$$\mathcal{S}_D^{\omega_1^\alpha/v_b}\left(\mathcal{S}_D^{0}\right)^{-1}[1] \approx \mathcal{S}_D^{0}\left(\mathcal{S}_D^{0}\right)^{-1}[1] = \mathcal{S}_D^{\alpha,0}\left(\mathcal{S}_D^{\alpha,0}\right)^{-1}[1] \approx \mathcal{S}_D^{\alpha,\omega_1^\alpha/v}\left(\mathcal{S}_D^{\alpha,0}\right)^{-1}[1]$$

in D up to a remainder of order of $O(\delta^{1/2})$, u_1^α can be approximated in Y by

$$u_1^\alpha = \mathcal{S}_D^{\alpha,\omega_1^\alpha/v}\left(\mathcal{S}_D^{\alpha,0}\right)^{-1}[1] + O(\delta^{1/2}). \tag{4.3}$$

We shall investigate the behavior of ω_1^α near $\alpha = \alpha^*$. We first introduce some notation. Let

$$G^{\tilde{\alpha},1}(\mathbf{x}) = \sum_{n \in \mathbb{Z}^3} \frac{e^{i(2\pi n + \alpha^*)\cdot \mathbf{x}}}{k^2 - |2\pi n + \alpha^*|^2}\left(\frac{|\tilde{\alpha}|^2}{k^2 - |2\pi n + \alpha^*|^2} + \frac{4((2\pi n + \alpha^*)\cdot\tilde{\alpha})^2}{(k^2 - |2\pi n + \alpha^*|^2)^2}\right),$$

and define the boundary integral operator

$$\mathcal{S}_1^{\tilde{\alpha}}[\phi](\mathbf{x}) := \int_{\partial D} G_1^{\tilde{\alpha}}(\mathbf{x} - \mathbf{y})\phi(\mathbf{y})d\sigma(\mathbf{y}).$$

Lemma 4.4 *The following holds*

$$\mathcal{S}_D^{\alpha^* + \epsilon\tilde{\alpha},0} = e^{i\epsilon\tilde{\alpha}\cdot\mathbf{x}}\left(\mathcal{S}_D^{\alpha^*,0} + \epsilon^2\mathcal{S}_1^{\tilde{\alpha}} + O(\epsilon^4)\right),$$

where the $O(\epsilon^4)$ term is an operator from $L^2(\partial D)$ to $H^1(\partial D)$ whose operator norm is of order ϵ^4.

Proof. Since

$$\begin{aligned} &\frac{1}{k^2 - |2\pi n + \alpha^* + \epsilon\tilde{\alpha}|^2} \\ &= \frac{1}{k^2 - |2\pi n + \alpha^*|^2}\left(1 + \frac{\epsilon^2|\tilde{\alpha}|^2}{k^2 - |2\pi n + \alpha^*|^2} + \frac{4\epsilon^2((2\pi n + \alpha^*)\cdot\tilde{\alpha})^2}{(k^2 - |2\pi n + \alpha^*|^2)^2}\right) + O(\epsilon^3|\tilde{\alpha}|^3), \end{aligned}$$

using the symmetry in the summation, we can derive that

$$
\begin{aligned}
&G^{\alpha^*+\epsilon\tilde{\alpha},k}(\mathbf{x}) = e^{i\epsilon\tilde{\alpha}\cdot\mathbf{x}} \sum_{n\in\mathbb{Z}^3} \frac{e^{i(2\pi n+\alpha^*)\cdot\mathbf{x}}}{k^2-|2\pi n+\alpha^*+\epsilon\tilde{\alpha}|^2} \\
&= e^{i\epsilon\tilde{\alpha}\cdot\mathbf{x}} \sum_{n\in\mathbb{Z}^3} \frac{e^{i(2\pi n+\alpha^*)\cdot\mathbf{x}}}{k^2-|2\pi n+\alpha^*|^2} \left(1+\frac{\epsilon^2|\tilde{\alpha}|^2}{k^2-|2\pi n+\alpha^*|^2}+\frac{4\epsilon^2((2\pi n+\alpha^*)\cdot\tilde{\alpha})^2}{(k^2-|2\pi n+\alpha^*|^2)^2}\right)+O(\epsilon^4|\tilde{\alpha}|^4) \\
&= e^{i\epsilon\tilde{\alpha}\cdot\mathbf{x}}\left(G^{\alpha^*,k}(\mathbf{x})+\epsilon^2 G_1^{\tilde{\alpha}}(\mathbf{x})\right)+O(\epsilon^4|\tilde{\alpha}|^4).
\end{aligned}
$$

The lemma follows. □

Let $\Lambda_D^{\tilde{\alpha}}$ be a quadratic function in $\tilde{\alpha}$ defined by

$$
\Lambda_D^{\tilde{\alpha}} := \frac{1}{2}\int_{\partial D} (\tilde{\alpha}\cdot\mathbf{y})^2 \left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}[1]+\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}[(\tilde{\alpha}\cdot\mathbf{x})^2]-2\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}\mathcal{S}_1^{\tilde{\alpha}}\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}[1]d\sigma(\mathbf{y}).
$$

Lemma 4.5 *For every small $\epsilon > 0$, it holds that*

$$
Cap_{D,\alpha^*+\epsilon\tilde{\alpha}} = Cap_{D,\alpha^*} + \epsilon^2\Lambda_D^{\tilde{\alpha}} + O(\epsilon^4).
$$

Moreover, $\Lambda_D^{\tilde{\alpha}}$ is a negative semidefinite quadratic function of $\tilde{\alpha}$.

Proof. Recall that

$$
\mathrm{Cap}_{D,\alpha^*+\epsilon\tilde{\alpha}} = -\int_{\partial D}\left(\mathcal{S}_D^{\alpha^*+\epsilon\tilde{\alpha},0}\right)^{-1}[1](\mathbf{y})d\sigma(\mathbf{y}).
$$

We solve $1 = \mathcal{S}_D^{\alpha^*+\epsilon\tilde{\alpha},0}[\phi](\mathbf{x})$ for ϕ. Since $\mathcal{S}_D^{\alpha^*+\epsilon\tilde{\alpha},0} = e^{i\epsilon\tilde{\alpha}\cdot\mathbf{x}}\left(\mathcal{S}_D^{\alpha^*,0}+\epsilon^2\mathcal{S}_1^{\tilde{\alpha}}+O(\epsilon^4)\right)$, we get

$$
\begin{aligned}
\phi(\mathbf{y}) &= e^{i\tilde{\alpha}\cdot\epsilon\mathbf{y}}\left(\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}-\epsilon^2\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}\mathcal{S}_1^{\alpha}\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}\right)[e^{-i\tilde{\alpha}\cdot\epsilon\mathbf{x}}](\mathbf{y})+O(\epsilon^4) \\
&= e^{i\tilde{\alpha}\cdot\epsilon\mathbf{y}}\left[\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}[1]-\epsilon^2\left(\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}[(\tilde{\alpha}\cdot\mathbf{x})^2/2]-\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}\mathcal{S}_1^{\tilde{\alpha}}\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}[1]\right)\right](\mathbf{y}) \\
&\quad + O(\epsilon^4).
\end{aligned}
$$

Using the expansion

$$
e^{i\tilde{\alpha}\cdot\epsilon\mathbf{y}} = 1+i\epsilon\tilde{\alpha}\cdot\mathbf{y}-\frac{1}{2}\epsilon^2(\tilde{\alpha}\cdot\mathbf{y})^2+\frac{-i\epsilon^3}{6}(\alpha\cdot\mathbf{y})^3+O(\epsilon^4),
$$

and the symmetry in the integration, we have

$$
\begin{aligned}
\mathrm{Cap}_{D,\alpha^*+\epsilon\tilde{\alpha}} &= -\int_{\partial D}\phi(\mathbf{y})d\sigma(\mathbf{y}) \\
&= \mathrm{Cap}_{D,\alpha^*}+\epsilon^2\Lambda_D^{\tilde{\alpha}}+O(\epsilon^4).
\end{aligned}
$$

Finally, since $\mathrm{Cap}_{D,\alpha}$ attains the maximum at $\alpha = \alpha^*$, $\Lambda_D^{\tilde{\alpha}}$ is a negative semidefinite quadratic function of $\tilde{\alpha}$. This completes the proof. $\square$

We now introduce a matrix $\{\lambda_{ij}\}$ such that

$$\frac{v_b^2}{|D|}\Lambda_D^{\tilde{\alpha}} := -\sum_{1\le i,j\le 3} \lambda_{ij}\tilde{\alpha}_i\tilde{\alpha}_j. \tag{4.4}$$

It is clear that the matrix $\{\lambda_{ij}\}$ is symmetric and positive semidefinite.

Lemma 4.6 *We have*

$$(\omega_1^{\alpha^*+\epsilon\tilde{\alpha}}(\delta))^2 = \delta\omega_{M,\alpha^*}^2 - \epsilon^2\delta \sum_{1\le i,j\le 3} \lambda_{ij}\tilde{\alpha}_j\tilde{\alpha}_j + \lambda_0(\alpha^*)\delta^2 + O(\epsilon^4\delta + \epsilon^2\delta^2 + \delta^3).$$

Proof. Recall that the following asymptotic formula of ω_1^α in [2] holds:

$$(\omega_1^\alpha(\delta))^2 = \frac{\delta v_b^2 \mathrm{Cap}_{D,\alpha}}{|D|} + O(\delta^2) := \delta(\omega_{M,\alpha})^2 + O(\delta^2), \tag{4.5}$$

where $\omega_{M,\alpha} == \sqrt{\frac{v_b^2\mathrm{Cap}_{D,\alpha}}{|D|}}$, and the $O(\delta^2)$ term is uniformly of order δ^2 with respect to α away from 0. A further asymptotic expansion implies that the $O(\delta^2)$ term can be further decomposed as

$$O(\delta^2) = \lambda_0(\alpha)\delta^2 + O(\delta^3),$$

where $O(\delta^3)$ denotes some uniformly bounded remaining term. Since α^* is a critical point of ω_1^α regardless of δ, we have $\lambda_0(\alpha) = \lambda_0(\alpha^*) + O(|\alpha - \alpha^*|^2)$.

Finally, we obtain that

$$(\omega_{M,\,\alpha^*+\epsilon\tilde{\alpha}})^2 = \frac{v_b^2\mathrm{Cap}_{D,\alpha^*+\epsilon\tilde{\alpha}}}{|D|} = (\omega_{M,\alpha^*})^2 + \frac{\epsilon^2 v_b^2}{|D|}\Lambda_D^{\tilde{\alpha}} + O(\epsilon^4).$$

This completes the proof. $\square$

5 Homogenization of the Bloch eigenfunction

We now consider the asymptotic behaviors of the Bloch eigenfunction in the bubbly crystal with period s. We first present the relation between the Bloch eigenvalues and eigenfunctions at different scales, which follows from a scaling argument.

Lemma 5.1 *Let ω_1^α and u_1^α be the Bloch eigenvalue and eigenfunction of the bubbly crystal with period 1, then the bubbly crystal with period s has Bloch eigenvalue*

$$\omega_{1,s}^{\alpha/s} = \frac{1}{s}\omega_1^\alpha$$

and eigenfunction

$$u_{1,s}^{\alpha/s}(\mathbf{x}) = u_1^\alpha\left(\frac{\mathbf{x}}{s}\right).$$

As a consequence, we see that

$$\omega_*^s = \max_\alpha \omega_{1,s}^\alpha = \frac{1}{s} \max_\alpha \omega_1^\alpha = \frac{1}{s}\omega_1^{\alpha^*}.$$

Moreover, the maximum is attained at the corner point $\frac{1}{s}\alpha^*$ of the Brillouin zone B_s. We are interested in the behavior of the Bloch eigenfunction $u_{1,s}^\alpha$ for α near α^*/s.

Lemma 5.2

$$u_{1,s}^{\alpha^*/s+\tilde{\alpha}}(\mathbf{x}) = e^{i\tilde{\alpha}\cdot\mathbf{x}} S\left(\frac{\mathbf{x}}{s}\right) + O(s^2 + \delta^{1/2}), \tag{5.1}$$

where the function S is defined by

$$S(\mathbf{x}) = \mathcal{S}_D^{\alpha^*,0}\left[\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}[1]\right](\mathbf{x}). \tag{5.2}$$

Proof. Note that $u_{1,s}^{\alpha^*/s+\tilde{\alpha}}(\mathbf{x}) = u_1^{\alpha^*+s\tilde{\alpha}}(\mathbf{x}/s)$. By (4.3), we have

$$u_1^{\alpha^*+s\tilde{\alpha}}(\mathbf{x}/s) = \mathcal{S}_D^{\alpha^*+s\tilde{\alpha},v^{-1}\omega_1^{\alpha^*+s\tilde{\alpha}}}\left[\left(\mathcal{S}_D^{\alpha^*+s\tilde{\alpha},0}\right)^{-1}[1]\right]\left(\frac{\mathbf{x}}{s}\right) + O(\delta^{1/2}).$$

Since $\omega_1^{\alpha^*+s\tilde{\alpha}} = O(\delta^{1/2})$, using (3.2), we further obtain

$$u_1^{\alpha^*+s\tilde{\alpha}}(\mathbf{x}/s) = \mathcal{S}_D^{\alpha^*+s\tilde{\alpha},0}\left[\left(\mathcal{S}_D^{\alpha^*+s\tilde{\alpha},0}\right)^{-1}[1]\right]\left(\frac{\mathbf{x}}{s}\right) + O(\delta^{1/2}). \tag{5.3}$$

On the other hand, in the proof of Lemma 4.5, we showed that

$$\left(\mathcal{S}_D^{\alpha^*+s\tilde{\alpha},0}\right)^{-1}[1]\left(\frac{\mathbf{y}}{s}\right) = e^{i\tilde{\alpha}\cdot\mathbf{y}}\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}[1]\left(\frac{\mathbf{y}}{s}\right) + O(s^2). \tag{5.4}$$

It follows from (5.3) and (5.4) that

$$\begin{aligned} u_1^{\alpha^*+s\tilde{\alpha}}(\mathbf{x}/s) &= \int_{s\partial D} e^{i\tilde{\alpha}\cdot(\mathbf{x}-\mathbf{y})} G^{\alpha^*,0}\left(\frac{\mathbf{x}-\mathbf{y}}{s}\right) e^{i\tilde{\alpha}\cdot\mathbf{y}}\left(\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}[1]\right)\left(\frac{\mathbf{y}}{s}\right) s^{-2} d\sigma(\mathbf{y}) \\ &\quad + O(s^2+\delta^{1/2}) \\ &= e^{i\tilde{\alpha}\cdot\mathbf{x}} \mathcal{S}_D^{\alpha^*,0}\left[\left(\mathcal{S}_D^{\alpha^*,0}\right)^{-1}[1]\right]\left(\frac{\mathbf{x}}{s}\right) + O(s^2+\delta^{1/2}). \end{aligned}$$

Then the lemma follows. □

We note that the function S defined by (5.2) is a piecewise harmonic function with $S = 1$ on ∂D and $S = 0$ on ∂Y. $S\left(\frac{\mathbf{x}}{s}\right)$ varies on the small scale s and describes the microscopic behavior of the $u_{1,s}^{\alpha^*/s+\tilde{\alpha}}$. On the other hand, the function $e^{i\tilde{\alpha}\cdot\mathbf{x}}$ represents the macroscopic behavior of $u_{1,s}^{\alpha^*/s+\tilde{\alpha}}$. We now derive a homogenized equation for this macroscopic field near the critical frequency ω_*^s.

We first recall from [2] that

$$\omega_1^{\alpha^*} = C\sqrt{\delta} + O(\delta^{3/2})$$

for some positive constant C of order one which depends on D, v_b and α^*. In order to keep the critical frequency ω_*^s on a fixed order when the cell size s tends to zero, we assume that

Assumption 5.3 *$\delta = \mu s^2$ for some positive constant μ.*

As a result, we have

$$\max_{\alpha} \omega_{1,s}^{\alpha} = \omega_*^s = \frac{1}{s}\omega_1^{\alpha^*} = O(1).$$

Now, suppose that ω is near ω_*^s. We need to find the corresponding Bloch eigenfunctions, or $\tilde{\alpha}$ so that

$$\omega^2 = \omega_{1,s}^{\alpha^*/s+\tilde{\alpha}}.$$

Using Lemmas 4.6 and 5.1, we have

$$\omega_*^2 - \omega^2 = \delta \sum_{1\leq i,j\leq 3} \lambda_{ij}\tilde{\alpha}_i\tilde{\alpha}_j + O(s^4), \tag{5.5}$$

where $\omega_* := \omega_*^s$.

Let

$$\omega_*^2 - \omega^2 = \beta\delta$$

for some constant β. Then we have

$$\sum_{1\leq i,j\leq 3} \lambda_{ij}\tilde{\alpha}_i\tilde{\alpha}_j = \beta + O(s^2),$$

which shows that all the solutions $\tilde{\alpha}$ lie approximately on the ellipsoid defined by

$$\{\alpha = (\alpha_1, \alpha_2, \alpha_3) : \sum_{1\leq i,j\leq 3} \lambda_{ij}\alpha_i\alpha_j = \beta\}.$$

It also implies that the plane wave $u(\mathbf{x}) := e^{i\tilde{\alpha}\cdot\mathbf{x}}$ satisfies to a leading order term

$$\sum_{1\leq i,j\leq 3} \lambda_{ij}\partial_i\partial_j\hat{u}(\mathbf{x}) + \beta\hat{u}(\mathbf{x}) = 0. \tag{5.6}$$

We now consider two cases. Case I: $\beta > 0$ and Case II: $\beta < 0$.

In the first case, $\tilde{\alpha}$ is a well-defined vector in $\mathbb{R}^3$, which justifies the existence of true Bloch eigenfunctions. However, in the second case $\tilde{\alpha}$ is a pure imaginary vector, and hence is not a Bloch eigenfunction. The associated function $e^{i\tilde{\alpha}\cdot\mathbf{x}}$ either grows or decays exponentially along certain direction. This justifies that a band gap opens at the critical frequency ω_*.

To conclude, we have the following main result on the homogenization theory for the bubbly crystals.

Theorem 5.4 *Under Assumption 4.1, 4.3 and 5.3, for frequencies in a small neighborhood of maximum of the first Bloch eigenvalue, say, $\omega_*^2 - \omega^2 = O(s^2)$, the following asymptotic of Bloch eigenfunction $u_{1,s}^{\alpha^*/s+\tilde{\alpha}}$ holds:*

$$u_{1,s}^{\alpha^*/s+\tilde{\alpha}}(\mathbf{x}) = e^{i\tilde{\alpha}\cdot\mathbf{x}} S\left(\frac{\mathbf{x}}{s}\right) + O(s),$$

where the macroscopic field $e^{i\tilde{\alpha}\cdot\mathbf{x}}$ satisfies the following equation

$$\sum_{1\le i,j\le 3} \lambda_{ij}\partial_i\partial_j\hat{u}(\mathbf{x}) + \frac{\omega_*^2 - \omega^2}{\delta}\hat{u}(\mathbf{x}) = 0. \tag{5.7}$$

which can be viewed as the homogenized equation for the bubbly phononic crystal, while the microscopic field is periodic and varies on the scale of s.

It is clear that the homogenized medium is very dispersive below and near the critical frequency ω_*. Moreover, the microscopic oscillations of the field at the period of the crystal justify the superfocusing phenomenon [21]. This mechanism differs from the one based on the high contrast effective medium theory derived in [9], which is valid only in the dilute case.

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