Second-Kind Boundary Integral Equations for Electromagnetic Scattering at Composite Objects

X. Claeys and R. Hiptmair and E. Spindler

Research Report No. 2016-43
September 2016

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Funding SNF: 137873/1
Second-Kind Boundary Integral Equations for Electromagnetic Scattering at Composite Objects

Xavier Claeys*, Ralf Hiptmair†, and Elke Spindler†

September 30, 2016

Abstract We consider electromagnetic scattering of time-harmonic fields in $\mathbb{R}^3$ at objects composed of several linear, homogeneous, and isotropic materials. Adapting earlier work on acoustic scattering [X. Claeys, R. Hiptmair, and E. Spindler, A second-kind Galerkin boundary element method for scattering at composite objects, BIT 55(1):33–57, 2015] we develop a novel second-kind direct boundary integral formulation for this scattering problem, extending the so-called Müller formulation for a homogeneous scatterer to composite objects. The new formulation is amenable to Galerkin boundary element discretization by means of discontinuous tangential surface vector-fields. Numerical tests demonstrate competitive accuracy of the new approach compared with a widely used direct Galerkin boundary element method based on a first-kind boundary integral formulation. For piecewise constant approximation our experiments also confirm fast convergence of GMRES iterations independently of mesh resolution.

Keywords Electromagnetic scattering, second-kind boundary integral equations, Galerkin boundary element methods

Mathematics Subject Classification (2000) 65N12, 65N38, 65R20

* Sorbonne Universités, UPMC Univ Paris 06, CNRS, INRIA, UMR 7598, Laboratoire Jacques-Louis Lions, équipe Alpines, F-75005, Paris, France. The work of X. Claeys was partially supported the French National Research Agency under grant
† Seminar for Applied Mathematics, Swiss Federal Institute of Technology, Zurich, Switzerland, emails: {hiptmair,elke.spindler}@sam.math.ethz.ch. The work of E. Spindler was partially supported by SNF under grant 20021_137873/1.
1 Introduction

1.1 Model Problem for a Composite Scatterer

We consider scattering of an incident electromagnetic wave at an obstacle occupying the bounded region of space \( \Omega_* \subset \mathbb{R}^3 \). The obstacle is composed of several parts corresponding to open subdomains \( \Omega_i \subset \Omega_* \), \( i = 1, \ldots, L \), that form a partition of \( \Omega_* \) in the sense that \( \Omega_i \cap \Omega_j = \emptyset \) for \( j \neq i \) and \( \Omega_* = \bigcup_{i=1}^{L} \overline{\Omega}_i \). Both \( \Omega_* \) and all \( \Omega_i \) are supposed to be connected curvilinear Lipschitz polyhedra. This is also true of the unbounded complement \( \Omega_0 := \mathbb{R}^3 \setminus \overline{\Omega}_* \). The generic situation that we have in mind is sketched in Figure 1.1. There, \( \mathbf{n}_i \) denotes the exterior unit normal vector field for \( \Omega_i \), \( i = 1, \ldots, L \). We denote the common interface of \( \Omega_i \) and \( \Omega_j \) by \( \Gamma_{ij} = \partial \Omega_i \cap \partial \Omega_j \), \( i \neq j \), \( i, j \in \{1, \ldots, L\} \). We call skeleton \( \Sigma \) the union of all interfaces \( \Gamma_{ij} \); \( \Sigma := \bigcup_{0 \leq j < i \leq L} \Gamma_{ij} = \bigcup_{i=0}^{L} \partial \Omega_i \).

The (lossless) materials of the parts of the scatterer and in \( \Omega_0 \) possess different dielectric properties, reflected through different constant wave numbers \( \kappa_i \in \mathbb{R} \), \( i = 1, \ldots, L \), prevailing in \( \Omega_i \). Defining a piecewise constant coefficient function \( \kappa \in L^\infty(\mathbb{R}^3) \) by \( \kappa|_{\Omega_i} := \kappa_i \), the total electric field \( \mathbf{E} \) will solve \(^1\) (in a weak sense)

\[
\text{curl curl } \mathbf{E} - \kappa(\mathbf{x})^2 \mathbf{E} = \mathbf{0} \quad \text{in } \mathbb{R}^3 .
\]

\(^1\) When referring to the composite, piecewise constant wave number, we are going write \( \kappa(\mathbf{x}) \) to emphasize the dependence of the wave number \( \kappa \) on \( \mathbf{x} \in \mathbb{R}^d \).
In addition, the scattered field \( E_{\text{sc}} = E - E_{\text{inc}} \) must fulfill the Silver-Müller radiation conditions

\[
\lim_{r \to \infty} \int_{\{x \in \mathbb{R}^3 : ||x|| = r\}} \left( \frac{\mathbf{curl} E_{\text{sc}}(x) \times x}{r} - i\kappa_0 E_{\text{sc}}(x) \right) \cdot dS(x) = 0, \tag{1.1b}
\]

where \( E_{\text{inc}} \) denotes a given incident field satisfying (1.1a) for \( \kappa(x) \equiv \kappa_0 \in \mathbb{R}^3 \).

Existence and uniqueness of solutions of (1.1) is well established \[23\].

We point out that (1.1a) implies the following transmission conditions in the sense of distributions on \( \Gamma_{ij} \) for all \( i, j \in \{0, \ldots, L\}, i \neq j \):

\[
\left( \mathbf{curl} E_j \right)|_{\partial \Omega_{ij}} \times n_j + \left( \mathbf{curl} E_i \right)|_{\partial \Omega_{ji}} \times n_i = 0, \tag{1.2}
\]

where we used the notation \( E_j := E|_{\Omega_j}, j = 0, \ldots, L \), and \( n_i \) is the exterior unit normal for \( \Omega_i \), see Figure 1.1.

Remark 1.1 An important aspect in the setting of the composite scatterer is the presence of material junction points and edges (marked as ⬤ in Figure 1.1), i.e. points and edges where three or more subdomains abut. In those regions some subdomains \( \Omega_i \) will inevitably have non-smooth boundaries.

1.2 Boundary Integral Equations

To provide the foundation for a numerical treatment of the electromagnetic transmission problem (1.1) we transform it into Boundary Integral Equations (BIEs). Among others, a rationale is that BIEs can easily accommodate unbounded domains, as long as the scatterer is bounded. Then we are going to use the Galerkin Boundary Element Method (BEM) to discretize the BIEs in variational form.

We will aim for a specific type of BIEs in variational form, so-called single-trace formulations (STF). The term single-trace refers to the class of single-trace spaces, on which the formulation is posed. Single-trace spaces are constructed in such a way that they contain the skeleton traces of all functions satisfying the transmission conditions (1.2). They will be defined precisely in Section 3.

The first-kind single-trace formulation is widely used and also known as Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) scheme \[11, 33, 39\]. The formulation for a homogeneous scatterer \( (L = 1) \) was analyzed in \[9, 10\]. Its extension to composite obstacles is straightforward and is covered in \[5\]. Low-order Galerkin BEM for first-kind STFs yield ill-conditioned linear systems on fine meshes.

Nowadays, matrix compression techniques like Adaptive Cross Approximation (ACA) \[2\] or Fast Multipole Methods (FMM) \[18, 34\] are indispensable for competitive implementation of the BEM. They entail the use of iterative solvers. As a consequence, slow convergence of iterative solvers for
ill-conditioned Galerkin matrices becomes a major concern and preconditioning of the linear Galerkin systems arising from the first-kind STF becomes mandatory. Apparently, for composite scatterers with junction points/edges, the widely used so-called Calderón preconditioning techniques [25,37] cannot be applied [14, Section 4], [15, Section 1]. It takes profound modifications of the integral equations in the form of so-called multi-trace formulations [14,15], to pave the way for Calderón preconditioning.

In this article we pursue a different policy, the so-called second-kind single-trace formulation. In contrast to first-kind BIEs, low-order Galerkin BEM for second-kind boundary integral equations generally yield linear systems for which iterative solvers converge fast, rendering preconditioning unnecessary. For electromagnetic scattering at a homogeneous object \( (L = 1) \), second-kind BIEs are well-known, see [31,38,40], and they are sometimes called Müller formulation. For a long time it remained unclear how to extend them to composite scatterers, i.e. geometric arrangements that feature edges or points where more than two materials abut. Only recently, in [12,17] and the parallel work [22], a breakthrough was achieved for acoustic transmission problems.

1.3 Novelty and Outline

Based on our work for acoustic scattering covered in [12,16,17] and the results for a homogeneous scatterer \( (L = 1) \) from [31,38,40], we extend the second-kind STF to composite electromagnetic scattering. In Section 2, we give a brief introduction to the relevant boundary integral equations. Appropriate spaces for boundary integral equations in the context of a composite scatterer, namely single- and multi-trace spaces, are explained in Subsection 3. In Subsection 4.2, we derive the second-kind STF based on a so-called multi-potential representation formula for the scattering solution, see Corollary 4.1.

Instead of working in classical energy trace spaces, in Subsection 4.3 we consider the second-kind STF in the space \( L^2_t(\Sigma) \) of square-integrable tangential vector fields on \( \Sigma \). This is possible, because all the involved operators are continuous in \( L^2_t(\Sigma) \). The advantage of working in \( L^2_t(\Sigma) \) is, that it paves the way for switching to an interface-based variational formulation, see Section 4.4. It also provides more flexibility in choosing the trial and test spaces for Galerkin discretization, for instance, one may use simple piecewise constant vector fields on a given quasi-uniform and shape regular mesh of \( \Sigma \), see Section 5.

Section 6 concludes the article with a presentation of numerical results, which indicate that the accuracy of the second-kind Galerkin solution can compete with the classical first-kind results. We observe that the spectra of the second-kind Galerkin matrices cluster around 1 and fast, almost mesh-independent convergence of GMRES.
2 Traces, Potentials and Boundary Integral Operators

On the boundary of \( \Omega_i \), we define the following space of tangential, square integrable vector fields:

\[
L^2_i(\partial \Omega_i) := \{ u \in L^2(\partial \Omega_i) : u \cdot n_i = 0 \text{ a.e. on } \partial \Omega_i \}. \tag{2.1}
\]

It is endowed with the inner product \([u, v]_{\partial \Omega_i} := \langle u, v \rangle_{\partial \Omega_i} := \int_{\partial \Omega_i} u \cdot v dS\).

Next we introduce local trace operators and potentials associated with a subdomain \( \Omega_i \), \( i \in \{ *, 0, 1, \ldots, L \} \) and appropriate spaces on which the operators are well-defined. Writing \( \partial (\partial \Omega_i) := C^\infty_{\text{comp}}(\mathbb{R}^3) |_{\partial \Omega_i} \) we define the local (interior) tangential trace operators

\[
\gamma^i_\times : \left( \mathcal{D}(\partial \Omega_i) \right)^3 \to L^2_i(\partial \Omega_i), \quad U \mapsto U \times n_i|_{\partial \Omega_i},
\]

\[
\gamma^i_\gamma : \left( \mathcal{D}(\partial \Omega_i) \right)^3 \to \left( \mathcal{D}(\partial \Omega_i) \right), \quad U \mapsto n_i \times (U \times n_i)|_{\partial \Omega_i}.
\]

The presentation of BIEs for electromagnetic fields entails understanding the meaning of traces acting on functions in the “electromagnetic energy spaces” \(^2\)

\[
H_{\text{loc}}(\text{curl}^2, \Omega) := \left\{ U \in H_{\text{loc}}(\text{curl}, \Omega) \left| \text{ curl } \text{ curl } U \in L^2_{\text{loc}}(\Omega) \right. \right\},
\]

\[
H_{\text{loc}}(\text{curl}, \Omega) := \left\{ U \in L^2_{\text{loc}}(\Omega) \left| \text{ curl } U \in L^2_{\text{loc}}(\Omega) \right. \right\}. \tag{2.2}
\]

We are not going to give details and just recall some of the result from [4], borrowing all notations from that paper, among them the spaces

\[
H^{-\frac{1}{2}}(\text{curl}_R, \partial \Omega_i) := \left\{ u \in H^{-\frac{1}{2}}(\partial \Omega_i) : \text{ curl}_R u \in H^{-\frac{1}{2}}(\partial \Omega_i) \right\},
\]

\[
H^{-\frac{1}{2}}(\text{div}_R, \partial \Omega_i) := \left\{ u \in H^{-\frac{1}{2}}(\partial \Omega_i) : \text{ div}_R u \in H^{-\frac{1}{2}}(\partial \Omega_i) \right\},
\]

both endowed with the respective graph norms. We call them electric and magnetic trace spaces and use the shorthand notations \( \mathcal{T}_{\text{el}}(\partial \Omega_i) := H^{-\frac{1}{2}}(\text{curl}_R, \partial \Omega_i) \) and \( \mathcal{T}_{\text{m}}(\partial \Omega_i) := H^{-\frac{1}{2}}(\text{div}_R, \partial \Omega_i) \).

These two spaces are dual to each other when taking the extended the bilinear form \( \langle \cdot, \cdot \rangle_{\partial \Omega_i} \) as duality pairing \([6, \text{ p. 43}]\). For more information about the surface divergence \( \text{ div }_R \) and the surface curl \( \text{ curl}_R \), we also refer to [8]. We summarize [6, Thm. 5.4], [8, Thm. 4.1] in the following trace theorem.

**Lemma 2.1 (Electric and Magnetic Trace Operators)** The electric trace operator \( \gamma^i_{\text{el}} := \gamma^i_\times \) and magnetic trace operator \( \gamma^i_{\text{m}} := \gamma^i_\gamma \circ \text{ curl} \) can be extended to bounded surjective mappings

\[
\gamma^i_{\text{el}} : H_{\text{loc}}(\text{curl}, \Omega_i) \to \mathcal{T}_{\text{el}}(\partial \Omega_i), \quad \gamma^i_{\text{m}} : H_{\text{loc}}(\text{curl}^2, \Omega_i) \to \mathcal{T}_{\text{m}}(\partial \Omega_i).
\]

\(^2\) The spaces \( H(\text{curl}^2, \Omega), H(\text{curl}, \Omega) \) are defined analogously by dropping the subscript \( \text{ loc} \) everywhere in (2.2).
As a compact notation we use the total energy trace space
\[ \mathcal{T}(\partial \Omega_i) := \mathcal{T}_{el}(\partial \Omega_i) \times \mathcal{T}_{m}(\partial \Omega_i), \] (2.3)
which is self-dual with respect to the skew-symmetric pairing\(^3\)
\[ \langle (u, \varphi) \rangle_{\mathcal{T}(\partial \Omega_i)} := \langle u, \varphi \rangle_{\mathcal{T}_{el}(\partial \Omega_i)} - \langle v, \nu \rangle_{\partial \Omega_i}, \quad u := \begin{pmatrix} u \\ \nu \end{pmatrix}, \quad v := \begin{pmatrix} v \\ \varphi \end{pmatrix} \in \mathcal{T}(\partial \Omega_i). \]

A related compact notation is the total trace operator \(\gamma_{\text{tot}}\)
\[ \gamma_{\text{tot}}^i : H_{\text{loc}}(\text{curl}^2, \Omega_i) \to \mathcal{T}(\partial \Omega_i), \quad \gamma_{\text{tot}}^i U := \begin{pmatrix} \gamma_{\text{el}}^i U \\ \gamma_{\text{m}}^i U \end{pmatrix}. \] (2.4)

Next, we introduce local potentials associated with a subdomain \(\Omega_i, i \in \{\ast, 0, \ldots, L\}\), see [9, Section 4], [10, Section 3.3], [27, Section 4.3]. For sufficiently regular scalar and vector valued functions on \(\partial \Omega_i\), \(\varphi\) and \(\varphi\), respectively, the scalar and vector single layer potentials are given by the improper integrals
\[ \mathcal{V}_i[k](\varphi) = \int_{\partial \Omega_i} \Phi_k(\|x - y\|)\varphi(y)dS(y), \quad x \in \mathbb{R}^3 \setminus \partial \Omega_i, \]
\[ \mathcal{V}_i[k](\varphi) = \int_{\partial \Omega_i} \Phi_k(\|x - y\|)\varphi(y)dS(y), \quad x \in \mathbb{R}^3 \setminus \partial \Omega_i, \] (2.5)
based on the fundamental solution for the Helmholtz operator [35, Rem. 3.1.3]
\[ \Phi_k(\xi) = \frac{\exp(i \kappa \xi)}{4\pi \xi}, \quad \xi \in \mathbb{R} \setminus \{0\}. \] (2.6)

For any \(s \in (-\frac{1}{2}, \frac{1}{2}]\), they provide continuous linear operators [35, Sect. 3.1.2]
\[ \mathcal{V}_i[k] : H^{s-\frac{1}{2}}(\partial \Omega_i) \to H^{1+s}_{\text{loc}}(\mathbb{R}^3 \setminus \partial \Omega_i), \]
\[ \mathcal{V}_i[k] : H^{s-\frac{1}{2}}(\partial \Omega_i) \to H^{1+s}_{\text{loc}}(\mathbb{R}^3 \setminus \partial \Omega_i). \]

These potentials enter the definition of the Maxwell single and double layer potentials
\[ \mathcal{S}_i[k](\varphi) := \mathcal{V}_i[k](\varphi) + \frac{1}{\kappa^2} \text{grad} \mathcal{V}_i[k](\text{div} \varphi), \] (2.7)
\[ \mathcal{D}_i[k](v) := \text{curl} \mathcal{V}_i[k](n_i \times v). \] (2.8)

which give rise to continuous operators [9, Thm. 5]
\[ \mathcal{S}_i[k] : H^{-\frac{1}{2}}(\text{div} \Gamma, \partial \Omega_i) \to H_{\text{loc}}(\text{curl}^2, \mathbb{R}^3 \setminus \partial \Omega_i) \cap H_{\text{loc}}(\text{div} 0, \mathbb{R}^3 \setminus \partial \Omega_i), \]
\[ \mathcal{D}_i[k] : H^{-\frac{1}{2}}(\text{curl} \Gamma, \partial \Omega_i) \to H_{\text{loc}}(\text{curl}^2, \mathbb{R}^3 \setminus \partial \Omega_i) \cap H_{\text{loc}}(\text{div} 0, \mathbb{R}^3 \setminus \partial \Omega_i). \]

\(^3\) Fraktur font is used to designate functions in the total energy trace space, whereas Roman typeface is reserved for electric traces, and Greek symbols for magnetic traces.
where
\[ H_{\text{loc}}(\text{div} \ 0, \mathbb{R}^3 \setminus \partial \Omega_i) := \{ \mathbf{V} \in L^2_{\text{loc}}(\mathbb{R}^3) \mid \text{div} \ \mathbf{V} \equiv 0 \ \text{on} \ \Omega_i \cup \mathbb{R}^3 \setminus \overline{\Omega}_i \} . \]

Moreover, the electric Maxwell single and double layer potentials supply solutions of the homogeneous Maxwell equation (1.1a) with constant wave number \( \kappa \) on \( \Omega_i \cup \mathbb{R}^3 \setminus \overline{\Omega}_i \). They also satisfy the Silver-Müller radiation conditions (1.1b).

These potentials permit us to state the fundamental Stratton-Chu representation formula, which is the starting point for boundary integral equations for electromagnetic scattering, see [9, Theorem 6], [32, Theorem 5.5.1], [19, Section 6.2], [30, Section 9.2]:

**Lemma 2.2 (Local Stratton-Chu Representation Formula)**

Every solution \( \mathbf{E} \in H_{\text{loc}}(\text{curl}^2, \Omega_i) \cap H_{\text{loc}}(\text{div} \ 0, \Omega_i) \) of \( \text{curl} \ \mathbf{E} - \kappa^2 \mathbf{E} = 0 \) in \( \Omega_i \) (that satisfies the Silver-Müller radiation conditions, if \( i = 0 \),) fulfills

\[ \mathbf{G}_i[\kappa](\gamma_{\text{tot}}^i \mathbf{E}) = \begin{cases} \mathbf{E} & \text{on } \Omega_i, \\ 0 & \text{on } \mathbb{R}^3 \setminus \overline{\Omega}_i, \end{cases} \]

where the local total potential is defined by

\[ \mathbf{G}_i[\kappa](\mathbf{u}) := -\mathbf{D}_i[\kappa](\mathbf{u}) + \mathbf{S}_i[\kappa](\nu), \quad \mathbf{u} := \begin{pmatrix} u \\ \nu \end{pmatrix} \in \mathcal{T}(\partial \Omega_i). \]

Off \( \partial \Omega_i \) the potentials are regular enough to accommodate total traces from both sides of \( \partial \Omega_i \). They satisfy the jump relations [9, Thm. 7]

\[ (\gamma_{\text{tot}}^{i,c} - \gamma_{\text{tot}}^i) \mathbf{G}_i[\kappa](\mathbf{u}) = -\mathbf{u} \quad \text{for all } \mathbf{u} \in \mathcal{T}(\partial \Omega_i), \quad (2.9) \]

where \( \gamma_{\text{tot}}^{i,c} \) means the trace onto \( \partial \Omega_i \) with normal vector field \( \mathbf{n}_i \) but taken from outside \( \Omega_i \) (from inside its complement).

### 3 Multi- and Single-Trace Spaces

#### 3.1 Energy Trace Spaces

The space containing all local traces of functions in \( H_{\text{loc}}(\text{curl}^2, \Omega_i), \ i \in \{0, \ldots, L\} \) is given by the skeleton multi-trace space, see also [13, Sections 3&4].

**Definition 3.1 (Multi-Trace Space)** The skeleton multi-trace space \( \mathcal{M}\mathcal{T}(\Sigma) \) is defined as the product of local total energy trace spaces from (2.3):

\[ \mathcal{M}\mathcal{T}(\Sigma) := \bigotimes_{i=0}^L \mathcal{T}(\partial \Omega_i) \cong \bigotimes_{i=0}^L \mathcal{T}_{el}(\partial \Omega_i) \times \bigotimes_{i=0}^L \mathcal{T}_m(\partial \Omega_i). \]
It owes its name to the fact that on each interface $\Gamma_{ij}$ a function $u \in \mathcal{MT}(\Sigma)$ comprises two pairs of electric and magnetic traces, one contributed by the subdomain on either side.

Next, we define the skeleton multi-trace operator $\gamma^\Sigma_{\text{tot}}$. It collects the total traces from (2.4) for all $i \in \{0, \ldots, L\}$. Hence, it maps $H_{\text{loc}}(\mathbf{curl}, \mathbb{R}^3 \setminus \Sigma)$ into the multi-trace space $\mathcal{MT}(\Sigma)$ and is given by

$$\gamma^\Sigma_{\text{tot}} : \left\{ H_{\text{loc}}(\mathbf{curl}, \mathbb{R}^3 \setminus \Sigma) \rightarrow \mathcal{MT}(\Sigma), \right.$$}

$$\left. \mathbf{E} \mapsto \gamma^\Sigma_{\text{tot}} \mathbf{E} := (\gamma^0_{\text{tot}} \mathbf{E}, \ldots, \gamma^L_{\text{tot}} \mathbf{E}) \right\}. \tag{3.1}$$

Self-duality of $\mathcal{MT}(\Sigma)$ is induced by the $L^2_t$-type skew-symmetric bilinear pairing

$$\langle \langle u, v \rangle \rangle := \sum_{i=0}^L \langle \langle u_i, v_i \rangle \rangle_{T(\partial \Omega_i)}, \quad u = (u_0, \ldots, u_L), \quad v = (v_0, \ldots, v_L) \in \mathcal{MT}(\Sigma). \tag{3.2}$$

For sufficiently smooth functions we can rewrite (3.2) using the fact that each interface is visited twice when summing integrals over all subdomain boundaries:

$$\langle \langle u, v \rangle \rangle = \sum_{0 \leq j < i \leq L} \int_{\Gamma_{ij}} u_i \cdot \varphi_i - \nu_i \cdot \nu_i + u_j \cdot \varphi_j - \nu_j \cdot \nu_j \, dS, \tag{3.3}$$

where we used the notation $u_i = (u_i, \nu_i), v_i = (v_i, \varphi_i)$ for $u_i, v_i \in T(\partial \Omega_i)$ from (3.2).

We introduce the subspace of $\mathcal{MT}(\Sigma)$ compatible with the transmission conditions (1.2).

**Definition 3.2 (Skeleton Single-Trace Space)** The skeleton single-trace space $\mathcal{ST}(\Sigma)$ is defined as

$$\mathcal{ST}(\Sigma) := \left\{ ((u_0, \nu_0), \ldots, (u_L, \nu_L)) \in \mathcal{MT}(\Sigma) \mid \exists \mathbf{U} \in H(\mathbf{curl}, \mathbb{R}^3), \quad u_i = \gamma^i_t \mathbf{U}, \quad \exists \mathbf{V} \in H(\mathbf{curl}, \mathbb{R}^3), \quad \nu_i = \gamma^{\times}_i \mathbf{V}, \quad \forall i \in \{0, \ldots, L\} \right\}. \tag{3.4}$$

The skeleton single-trace space collects all possible skeleton multi-traces of solutions of (1.1a) in $\mathcal{MT}(\Sigma)$:

$$\mathbf{E} \text{ solves } (1.1) \Rightarrow \gamma^\Sigma_{\text{tot}} \mathbf{E} \in \mathcal{ST}(\Sigma).$$

The following polar set characterization of $\mathcal{ST}(\Sigma)$ as a subspace of $\mathcal{MT}(\Sigma)$, see also [13, Prop. 3.1], [14, Thm. 3.1], will play an important role when deriving the single-trace formulations.

**Lemma 3.1 (Polarity Property)** The single-trace space $\mathcal{ST}(\Sigma)$ is a Lagrangian subspace of $\mathcal{MT}(\Sigma)$

$$\mathcal{ST}(\Sigma) = \{ \mathbf{u} \in \mathcal{MT}(\Sigma) : \langle \langle \mathbf{u}, \mathbf{v} \rangle \rangle = 0, \forall \mathbf{v} \in \mathcal{ST}(\Sigma) \}.$$ 

The proof uses the same ideas as used to verify [12, Proposition 2.1] in the acoustic setting. As an immediate corollary, the single-trace space is closed in $\mathcal{MT}(\Sigma)$.
3.2 $L^2_t$ trace spaces

Since we aim for formulations in $L^2_t$, we need $L^2_t$-counterparts of the spaces from the previous subsection.

**Definition 3.3 ($L^2_t$-Multi-Trace Space)** The skeleton $L^2_t$-multi-trace space $\mathcal{ML}_t(\Sigma)$ is the Hilbert space

$$\mathcal{ML}_t(\Sigma) := \bigoplus_{i=0}^{L} (L^2_t(\partial \Omega_i) \times L^2_t(\partial \Omega_i)) \cong \bigotimes_{i=0}^{L} (\bigotimes_{i=0}^{L} (L^2_t(\partial \Omega_i))).$$

The bilinear pairing from (3.2) extends naturally to the $L^2_t$-setting. Moreover, in contrast to the classical energy trace setting, the restriction operator to an interface $\Gamma_{ij}$ is well defined in $\mathcal{ML}_t(\Sigma)$. Hence, on $\mathcal{ML}_t(\Sigma)$ we can redefine (3.2) as follows.

$$\langle \langle u, v \rangle \rangle = \sum_{j<i} \int_{\Gamma_{ij}} u_i \cdot \varphi_i - \nu_i \cdot v_i + u_j \cdot \varphi_j - \nu_j \cdot v_j \, dS,$$

with $u = (u_0, \ldots, u_L), u_i = (u_i, \nu_i) \in L^2_t(\partial \Omega_i) \times L^2_t(\partial \Omega_i)$ and $v = (v_0, \ldots, v_L), v_i = (v_i, \varphi_i) \in L^2_t(\partial \Omega_i) \times L^2_t(\partial \Omega_i), i \in \{0, \ldots, L\}$.

The subspace of functions in $\mathcal{ML}_t(\Sigma)$ complying with the transmission conditions (1.2), i.e. the analogue to the classical single-trace space $ST(\Sigma)$, can be defined as follows:

**Definition 3.4 ($L^2_t$-Single-Trace Space)** The $L^2_t$-single-trace space $\mathcal{SL}_t(\Sigma)$ is given by

$$\mathcal{SL}_t(\Sigma) := \left\{ \left( \begin{array}{c} u_0 \\ \nu_0 \\ \vdots \\ u_L \\ \nu_L \end{array} \right) \in \mathcal{ML}_t(\Sigma) \left| \right. u_j |_{\Gamma_{ij}} = u_i |_{\Gamma_{ij}}, \nu_j |_{\Gamma_{ij}} = -\nu_i |_{\Gamma_{ij}} \forall j < i \in \{0, \ldots, L\} \right\}.$$  

Note that the constraints in Definition 3.4 encode the transmission conditions (1.2).

As in the case of the classical single-trace space $ST(\Sigma)$, there is a Lagrangian subspace characterization of $\mathcal{SL}_t(\Sigma)$ as a subspace of $\mathcal{ML}_t(\Sigma)$:

**Lemma 3.2 (Polarity Property)** The $L^2_t$-single-trace space $\mathcal{SL}_t(\Sigma)$ can be identified as its own polar set in the $L^2_t$-multi-trace space $\mathcal{ML}_t(\Sigma)$.

$$\mathcal{SL}_t(\Sigma) = \{ u \in \mathcal{ML}_t(\Sigma) : \langle \langle u, v \rangle \rangle = 0, \forall v \in \mathcal{SL}_t(\Sigma) \}.$$  

The proof is analogous to the one from [17, Lemma 4.1], see also [36, Lem. 6.2.7]. Again, it is immediate that $\mathcal{SL}_t(\Sigma) \subset \mathcal{ML}_t(\Sigma)$ is closed.
4 Second-Kind Single-Trace Formulation

4.1 Preface: Homogeneous Scatterer $L = 1$

For the case $L = 1$, that is, a simple homogeneous scatterer, there is only a single interface $\Gamma_{01} = \partial\Omega_0 = \partial\Omega_1$. In this setting the second-kind Müller formulation is well known [31, §23], but we recall it to fix ideas and guide the reader gently towards our generalization derived in the next two subsections. Lemma 2.2 provides the following potential representation for the solution $E$ of (1.1):

$$
E = \begin{cases}
E_{\text{inc}} - D_0[\kappa_0](\gamma_0^E(E - E_{\text{inc}})) + S_0[\kappa_0](\gamma_m^E(E - E_{\text{inc}})) & \text{in } \Omega_0, \\
-D_1[\kappa_1](\gamma_1^E) + S_1[\kappa_1](\gamma_m^E) & \text{in } \Omega_1.
\end{cases}
$$

(4.1)

To streamline notations, we write $\{\gamma\}_F$ for the arithmetic average of interior and exterior traces onto $F := \Gamma_{01}$ for some trace operator $\gamma$. For instance, $\{\gamma_{el}\}_F$ returns the average of the electric trace for a vectorfield defined on both sides of $F$. Moreover, we write $\gamma_x$ for the pointwise restriction of a function to $F$. Taking electric and magnetic traces of $E$ onto $F$ from both sides, using the definition (2.7) of $S_i[\kappa_i]$, the identity $\gamma_m^E D_i[\kappa_i](\nu_i \times \nu) = \kappa^2 \gamma_x^E V_i[\kappa_i](\nu_i \times \nu) - \text{curl}_F V_i[\kappa_i](\text{curl}_F \nu)$ [24, Lemma 5.3], and applying the jump relations (2.9) yields boundary integral equations:

$$
\begin{align*}
\gamma_{el}^1 E &= - (\gamma_{el})_F D_1[\kappa_1](\gamma_{el}^1 E) + \frac{1}{2} \gamma_{el}^1 E + \{\gamma_{el}\}_F V_1[\kappa_1](\gamma_m^1 E) \\
&\quad + \kappa_1^{-2} \text{grad}_F (\gamma_x)_F V_1[\kappa_1](\text{div}_F (\gamma_m^1 E)), \\
\gamma_0^1 E &= +\text{curl}_F (\gamma_x)_F V_1[\kappa_1](\text{curl}_F (\gamma_{el}^1 E)) + \kappa_1^{-2} \{\gamma_{el}\}_F V_1[\kappa_1](\gamma_{el}^1 E \times \nu_1) \\
&\quad + \{\gamma_0^1\}_F V_1[\kappa_1](\gamma_0^1 E) + \frac{1}{2} \gamma_0^1 E, \\
\gamma_{el}^0 E &= - (\gamma_{el})_F D_0[\kappa_0](\gamma_{el}^0 E) + \frac{1}{2} \gamma_{el}^0 E + \{\gamma_{el}\}_F V_0[\kappa_0](\gamma_m^0 E) \\
&\quad + \kappa_0^{-2} \text{grad}_F (\gamma_x)_F V_0[\kappa_0](\text{div}_F (\gamma_m^0 E)) + \ldots, \\
\gamma_m^0 E &= +\text{curl}_F (\gamma_x)_F V_0[\kappa_0](\text{curl}_F (\gamma_{el}^0 E)) + \kappa_0^{-2} \{\gamma_x\}_F V_0[\kappa_0](\gamma_{el}^0 E \times \nu_0) \\
&\quad + \{\gamma_m^0\}_F V_0[\kappa_0](\gamma_0^0 E) + \frac{1}{2} \gamma_m^0 E + \ldots.
\end{align*}
$$

Here the ellipsis $\ldots$ stands for terms involving $E_{\text{inc}}$, which we do not state for the sake of clarity. They will enter the right hand sides of the boundary integral equations. Below we specify them for the general formulation. Notice that by the transmission conditions $\gamma_{el}^1 E = \gamma_{el}^0 E$ and $\gamma_m^1 E = -\gamma_m^0 E$. We use this fact and add the electric traces, weighted with $\kappa_1^2$:

$$
\begin{align*}
\frac{1}{2} (\kappa_1^2 + \kappa_0^2) \gamma_{el}^1 E &= - (\gamma_{el})_F \left( \kappa_1^2 D_1[\kappa_1] - \kappa_0^2 D_1[\kappa_0] \right) (\gamma_{el}^1 E) \\
&\quad + (\gamma_{el})_F \left( \kappa_1^2 V_1[\kappa_1] - \kappa_0^2 V_1[\kappa_0] \right) (\gamma_m^1 E) \\
&\quad + \text{grad}_F (\gamma_x)_F \left( V_1[\kappa_1] - V_1[\kappa_0] \right) (\text{div}_F (\gamma_m^1 E)) + \ldots.
\end{align*}
$$

(4.2a)
Note that $\mathbf{D}_0[\kappa] = -\mathbf{D}_1[\kappa]$ due to the opposite orientation of normals. We can also subtract the magnetic traces and get

$$
\gamma_m^1 \mathbf{E} = + \text{curl}_\Gamma (\gamma_x^1 R (\mathbf{V}_1[\kappa_1] - \mathbf{V}_1[\kappa_0])) (\text{curl}_\Gamma (\gamma_m^1 \mathbf{E})) + \{ \gamma_x^1 R (\kappa_2^2 \mathbf{V}_1[\kappa_1] - \kappa_2^2 \mathbf{V}_1[\kappa_0]) (\gamma_m^1 \mathbf{E} \times \mathbf{n}_1) + \{ \gamma_m^1 R (\mathbf{V}_1[\kappa_1] - \mathbf{V}_1[\kappa_0]) (\gamma_m^1 \mathbf{E}) + \ldots .
$$

(4.2b)

Thus, we arrive at two boundary integral equations for the two unknown traces $\gamma_m^1 \mathbf{E}$ and $\gamma_m^1 \mathbf{E}$. Yet, the key benefit of the above special linear combinations is that they lead to differences of single layer potentials that differ in wave number only, marked with red in (4.2). There cancellation of the leading singularities of their integral kernels will occur. Thus, the difference potentials will be strongly regularizing. More precisely, according to [35, Lemma 3.9.8], they map continuously $H^{-1}(\partial \Omega_i) \to H^2_{loc}(\mathbb{R}^3 \setminus \partial \Omega_i)$ in both the scalar and vectorial case.

We conclude that the boundary integral operators in the third line of (4.2a) and the first and third line of (4.2b) will be bounded and even compact in $L^2(\Gamma)$. The same holds true for the vectorial single layer boundary integral operators highlighted in green in (4.2). Moreover, as explained in [9, Lemma 11], for any $\kappa$ the boundary integral operators $\{ \gamma_i^1 \}_{\Gamma} \mathbf{D}_1[\kappa]$ (printed in blue in (4.2)) feature Cauchy-singular kernels and are continuous in $L^2(\Gamma)$. If $\Gamma$ is smooth, they will even turn out to be compact in that space. Summing up, (4.2) can be rescaled to the form $(\text{Id} - T)(\gamma_m^1 \mathbf{E}) + \ldots$ with $T$ continuous on $L^2(\Gamma) \times L^2(\Gamma)$, a typical 2nd-kind boundary integral equation.

**Remark 4.1 (Fredholmness of Index Zero)** The integral operator $T$ underlying (4.2) is Fredholm of index zero in $L^2(\Gamma) \times L^2(\Gamma)$, see [38, Lemma 4.1], [38, Lemma 4.6] (for the case of a $C^1$-domain), and [28, p. 155ff., “An alternative proof of xix in Theorem 5.1”] or [29, Theorem 1.1] for merely Lipschitz $\Gamma$. Since uniqueness of solutions of (4.2) can be established by standard arguments, existence of a solution in $L^2(\Gamma) \times L^2(\Gamma)$ is guaranteed and it will supply the traces $(\gamma_m^1 \mathbf{E}, \gamma_m^1 \mathbf{E})$.

4.2 Second-Kind Formulation in Energy Trace Spaces

For $L > 1$ the key idea underlying the derivation of the second-kind single-trace formulation is a multi-potential representation formula. For acoustic scattering this was given in [17, Def. 3.3]. For electromagnetics an additional judicious scaling is needed in order to cope with the division by $\kappa^2$ in the Maxwell single layer potential $\mathbf{S}[\kappa]$ from (2.7).

**Definition 4.1 (Scaled Multi-Potential)** The *Maxwell multi-potential* is defined as the sum of all (possibly scaled) local potentials $\mathbf{G}_i[\kappa_i]$, $i = 0, \ldots, L$:

$$
\mathbf{M}[\theta, \alpha] : \left\{ \begin{array}{l}
\mathcal{M} \mathcal{T}(\Sigma) \rightarrow H_{bc}(\text{curl}^2, \mathbb{R}^3 \setminus \Sigma), \\
u = (u_0, \ldots, u_L) \rightarrow \sum_{i=0}^{L} \alpha_i \mathbf{G}_i[\theta_i](u_i),
\end{array} \right.
$$
with $\theta := (\theta_0, \ldots, \theta_L) \in \mathbb{R}^{L+1}_+$ representing a wave number vector and $\alpha = (\alpha_0, \ldots, \alpha_L) \in \mathbb{R}^{L+1}$ a scaling parameter vector.

The special wave number vector $\kappa$ collects the given wave numbers prevailing in the subdomains: $\kappa = (\kappa_0, \ldots, \kappa_L)$. Then, an immediate consequence of the local Stratton-Chu representation formula from Lemma 2.2 is the following.

**Corollary 4.1 (Global Representation Formula)** A solution $E$ of the transmission problem (1.1) satisfies for any $\alpha \in \mathbb{R}^{L+1}_+\!$:

$$
\sum_{i=0}^{L} \alpha_i^2 \chi_{\Omega_i} E - \alpha_0^2 \chi_{\Omega_0} E_{\text{inc}} = M[\kappa, \alpha] \gamma_\text{tot}(E - \chi_{\Omega_0} E_{\text{inc}}),
$$

where $\chi_{\Omega}$ stands for the characteristic function of a domain $\Omega$.

Corollary 4.1 provides the starting point for the derivation of the second-kind BIE for composite scatterers. Taking the cue from the linear combination used in Section 4.1, we fix the scaling parameter vector to be $\alpha = \kappa$ when taking the electric trace and use $\alpha = 1 := (1, \ldots, 1)$ in the case of the magnetic trace. To keep notations short, we introduce the global compound boundary integral operator $M[\theta]$ for any $\theta \in \mathbb{R}^{L+1}_+$ by

$$
M[\theta] : \mathcal{M}\mathcal{T}(\Sigma) \to \mathcal{M}\mathcal{T}(\Sigma),
$$

$$
u \mapsto \begin{pmatrix}
\gamma_{el}^\Sigma M[\theta, \theta](\nu) \\
\gamma_{m}^\Sigma M[\theta, 1](\nu)
\end{pmatrix},
$$

where $\gamma_{el}^\Sigma U := (\gamma_{el}^0 U, \ldots, \gamma_{el}^L U)$, $\gamma_{m}^\Sigma U := (\gamma_{m}^0 U, \ldots, \gamma_{m}^L U)$. Taking the skeleton multi-trace $\gamma_{\text{tot}}^\Sigma$ on both sides of the global representation formula from Corollary 4.1, we obtain

$$
(Id_\kappa - M[\kappa]) \gamma_{\text{tot}}^\Sigma E = (Id_\kappa - M[\kappa]) e_{\text{inc},0},
$$

where the scaling operator $Id_\kappa : \mathcal{M}\mathcal{T}(\Sigma) \to \mathcal{M}\mathcal{T}(\Sigma)$ is defined as

$$
Id_\kappa \begin{pmatrix}
(v_0) \\
(\phi_0)
\end{pmatrix}, \ldots, \begin{pmatrix}
(v_L) \\
(\phi_L)
\end{pmatrix} := \begin{pmatrix}
\kappa_0^2 v_0 \\
\phi_0
\end{pmatrix}, \ldots, \begin{pmatrix}
\kappa_L^2 v_L \\
\phi_L
\end{pmatrix}
$$

and $e_{\text{inc},0} := (\gamma_{\text{tot}}^0 E_{\text{inc}}, 0, \ldots, 0)$.

**Remark 4.2 (Notation)** in the sequel, for the sake of brevity, we write $\mathbf{M}$ and $\mathbf{M}[\theta, \kappa]$ instead of $M[\kappa, \kappa]$ and $M[\kappa]$. When we refer to a setting where all $L+1$ wave numbers are equal to $\theta \in \mathbb{R}_+$ and all scaling coefficients agree with $\alpha \in \mathbb{R}$, we write $\mathbf{M}[\theta, \alpha] = \mathbf{M}[(\theta, \ldots, \theta), (\alpha, \ldots, \alpha)]$ and $\mathbf{M}[\theta] = \mathbf{M}[(\theta, \ldots, \theta)]$. 

Remark 4.3 (Right-hand Side) By assumption, $E_{\text{inc}}$ solves the electromagnetic scattering problem in $\Omega = \mathbb{R}^3 \setminus \Omega_0$ for $\kappa = \kappa_0$. Hence, we obtain from the Stratton-Chu local representation formula from Lemma 2.2 that

$$G[\kappa_0]\{\gamma^\ast_{\text{tot}} E_{\text{inc}}\} = \begin{cases} E_{\text{inc}} & \text{on } \Omega = \mathbb{R}^3 \setminus \Omega_0, \\ 0 & \text{on } \Omega_0. \end{cases} \quad (4.7)$$

Since $E_{\text{inc}} \in H_{\text{loc}}(\text{curl}^2, \mathbb{R}^3)$, it holds that $(\gamma_{\text{tot}}^0 - \gamma_{\text{tot}}^0)E_{\text{inc}} = 0$, which means, since $\Omega^* = \mathbb{R}^3 \setminus \Omega_0$,

$$\gamma^*_{\text{el}} E_{\text{inc}} = \gamma_{\text{el}}^0 E_{\text{inc}}, \quad \gamma^*_{\text{m}} E_{\text{inc}} = -\gamma_{\text{m}}^0 E_{\text{inc}}. \quad (4.8)$$

Hence we can rewrite (4.7) as

$$G_0[\kappa_0]\{\gamma^0_{\text{tot}} E_{\text{inc}}\} = \begin{cases} -E_{\text{inc}} & \text{on } \Omega = \mathbb{R}^3 \setminus \Omega_0, \\ 0 & \text{on } \Omega_0. \end{cases} \quad (4.9)$$

Using this intermediate result, we can now simplify the right hand side of (4.5) to

$$(\text{Id}_\kappa - M[\kappa]) \epsilon_{\text{inc},0} - \text{Id}_\kappa \epsilon_{\text{inc},0} - \text{Id}_\kappa \gamma_{\text{tot}}^0 G_0[\kappa_0]\{\gamma^0_{\text{tot}} E_{\text{inc}}\}$$

$$(\ast) \quad \text{Id}_\kappa \epsilon_{\text{inc},0} - \text{Id}_\kappa \gamma_{\text{tot}}^0 E_{\text{inc}} = \text{Id}_\kappa \gamma_{\text{tot}}^0 E_{\text{inc}},$$

where $\epsilon_{\text{inc},0} = (\gamma^0_{\text{tot}} E_{\text{inc}}, 0, \ldots, 0)$ and for $(\ast)$ we applied (4.9) together with $\Omega_i \subset \Omega$ for all $i \in \{1, \ldots, L\}$. Hence, a simplified version of (4.5) reads

$$(\text{Id}_\kappa - M[\kappa]) \gamma_{\text{tot}}^0 E = \text{Id}_\kappa \gamma_{\text{tot}}^0 E_{\text{inc}}. \quad (4.10)$$

We choose the skeleton trace $\epsilon = \gamma_{\text{tot}}^0 E \in ST(\Sigma)$ of the total field as unknown, insert (4.10) into the bilinear form $\langle \langle \cdot, \cdot \rangle \rangle$ from (3.2) and test with $v \in MT(\Sigma)$. This yields the following BIE in variational form $4$:

**Formulation 4.1** Seek $\epsilon \in ST(\Sigma)$ such that

$$\langle \langle \text{Id}_\kappa - M[\kappa], \epsilon, v \rangle \rangle = \langle \langle \text{Id}_\kappa, \epsilon_{\text{inc}}, v \rangle \rangle, \quad \forall v \in MT(\Sigma), \quad (4.11)$$

where $\epsilon_{\text{inc}} := \gamma_{\text{tot}}^0 E_{\text{inc}}$.

The next lemma paves the way for a regularized version of Formulation 4.1.

**Lemma 4.1** (Regularization) For all $\kappa, \alpha > 0$ holds $4$

$$M[\kappa, \alpha]\{u\} = 0 \quad \forall u \in ST(\Sigma). \quad (4.12)$$

$^4$ See Remark 4.2 for notation policies.
The proof starts with the key observation that for arguments \( u \in ST(\Sigma) \) and identical scaling parameters the multi-potential satisfies the transmission conditions (1.2), because all jumps across interfaces cancel. Details of the proof can be found after [13, Lemma 4.1], see also [36, Lemma 6.3.11]. In the end of Section 4.4, we will see that the choice of \( \kappa > 0 \) does not actually matter.

Using Lemma 4.1, we can add zero to \((4.11)\) in the form of

\[
M[\kappa](u) = \left( \gamma_{el}^{\Sigma} M[\kappa, \kappa](u) \right) - \left( \gamma_{m}^{\Sigma} M[\kappa, 1](u) \right) = 0, \quad u \in ST(\Sigma), \kappa > 0.
\]

This gives us a regularized formulation.

**Formulation 4.2 (Regularized Second-Kind Formulation in Energy Trace Spaces)**

For some fixed \( \kappa > 0 \) seek \( e \in ST(\Sigma) \) such that

\[
\langle \langle \text{Id}_\kappa - (M - M[\kappa]) e, v \rangle \rangle = \langle \langle \text{Id}_\kappa e_{\text{inc}}, v \rangle \rangle, \quad \forall v \in MT(\Sigma).
\]

We now examine the difference operator of Formulation 4.2 more closely in order to justify the attribute “regularized”. For \( u = (u_0, \ldots, u_L) \in ST(\Sigma), u_i = (u_i, \nu_i) \in T(\partial \Omega) \), mainly using definitions, we find

\[
(M - M[\kappa])(u)
= \left( \gamma_{el}^{\Sigma} \sum_{i=0}^{L} (\kappa_i^2 G_i[\kappa_i] - \kappa_i^2 G_i[\kappa])(u_i) \right) - \left( \gamma_{m}^{\Sigma} \sum_{i=0}^{L} (G_i[\kappa_i] - G_i[\kappa])(u_i) \right)
= \left( \gamma_{el}^{j} \sum_{i=0}^{L} \left( -\kappa_i^2 D_i[\kappa_i] + \kappa_i^2 D_i[\kappa] \right)(u_i) + (\kappa_i^2 V_i[\kappa_i] - \kappa_i^2 V_i[\kappa])(\nu_i) + \text{grad}(V_i[\kappa_i] - V_i[\kappa])(\text{div}\nu_i) \right)_{j=0}^{L}
= \left( \gamma_{m}^{j} \sum_{i=0}^{L} \left( D_i[\kappa_i] + D_i[\kappa] \right)(u_i) + (V_i[\kappa_i] - V_i[\kappa])(\nu_i) \right)_{j=0}^{L}.
\]
\[
\begin{align*}
\sum_{i=0}^{L} \left\{ \gamma_i \left(-\kappa^2 D_i[\kappa] + \kappa^2 D_i[\kappa]\right)(\mathbf{u}_i) \\
+ \gamma_i \left(\kappa^2 V_i[\kappa] - \kappa^2 V_i[\kappa]\right)(\mathbf{v}_i) \\
+ \gamma_i \left(\text{grad}(V_i[\kappa] - V_i[\kappa])(\text{div}_T \mathbf{v}_i)\right) \right\}
\end{align*}
\]
(4.13)

where we used the following abbreviations for “difference potentials”:

\[
\begin{align*}
\delta D_i(\mathbf{u}_i) &:= (-\kappa^2 D_i[\kappa] + \kappa^2 D_i[\kappa])(\mathbf{u}_i), \\
\delta V_i(\mathbf{v}_i) &:= (\kappa^2 V_i[\kappa] - \kappa^2 V_i[\kappa])(\mathbf{v}_i), \\
\delta W_i(\mathbf{v}_i) &:= \text{grad}(V_i[\kappa] - V_i[\kappa])(\text{div}_T \mathbf{v}_i), \\
\delta C_i(\mathbf{v}_i) &:= \text{curl}(V_i[\kappa] - V_i[\kappa])(\mathbf{v}_i).
\end{align*}
\]

The color code has been borrowed from (4.2) to underscore similarities of (4.2) and (4.13). The reader is encouraged to verify that Formulation 4.2 boils down to (4.2) for \( L = 1 \) and \( \kappa = \kappa_0 \). Again, in (4.13), we have ended up with differences of boundary integral operators which reward us with the benefits of a cancellation of some leading singularities of their kernels as in Section 4.1. In particular, Formulation 4.2 remains meaningful in \( L^2 \); it is genuinely 2nd-kind.

**Theorem 4.3 (Continuity of Differenced BIOs in \( \mathcal{MC}_L(\Sigma) \))** For any real valued \( \kappa > 0 \), the linear operator \( \text{id}_L - (M - M[\kappa]) \) can be extended to a continuous mapping \( \mathcal{MC}_L(\Sigma) \to \mathcal{MC}_L(\Sigma) \).

The proof of Theorem 4.3 can be found in [36, Proof of Thm. 6.3.13]. The assertion can also be concluded from considerations in Section 4.4 below.

**Remark 4.4 (Fredholmness of Index Zero)** When trying to generalize the result stated in Remark 4.1 to the multi-domain setting, we face problems at triple points, see [36, Remark 6.3.14]. Hitherto it remains open, whether the boundary integral operator of Formulation 4.2 is Fredholm with index 0. Furthermore, the question of existence and uniqueness of solutions of Formulation 4.2 has no answer yet.
4.3 Second-Kind Single-Trace Formulation in $\mathcal{S}\mathcal{L}_t(\Sigma)$

As a consequence of Theorem 4.3, the bilinear form on the left-hand side of Formulation 4.2 remains continuous on $\mathcal{M}\mathcal{L}_t(\Sigma) \times \mathcal{M}\mathcal{L}_t(\Sigma)$, since the pairing $\langle \langle \cdot, \cdot \rangle \rangle$ extends naturally to $L^2_t$. This leads to the lifted version of Formulation 4.2.

Formulation 4.4 Fix $\kappa > 0$ and find $e \in \mathcal{S}\mathcal{L}_t(\Sigma)$ such that

$$\langle \langle (\text{Id}_\kappa - (M - M[\kappa]^-)) e, v \rangle \rangle = \langle \langle \text{Id}_\kappa e_{\text{inc}}, v \rangle \rangle, \quad \forall v \in \mathcal{M}\mathcal{L}_t(\Sigma).$$

Remark 4.5 Electric or magnetic traces of the solution of (1.1) may fail to belong to $L^2_t(\Sigma)$, but they will certainly be contained in some Sobolev space $H^{-\delta}_t(\Sigma)$ of low-regularity tangential vectorfields from some $0 < \delta < \frac{1}{2}$ [20]. This suggests that we consider Formulation 4.4 in $H^{-\delta}_t(\partial\Omega_i) \times H^2_\Sigma(\partial\Omega_i)$. Since this does not make any difference algorithmically, we are not going to elaborate on this. In any case, Formulation 4.4 will provide an outer approximation of $\gamma_{\text{tot}}^\Sigma E$.

There is a glaring mismatch of trial and test space in Formulation 4.4 and Formulation 4.2, respectively. A remedy is suggested by the following consequence of Lemma 3.2 (similar to [12, Proposition 5.1] in the acoustic case).

Theorem 4.5

$$\langle \langle (\text{Id}_\kappa - (M - M[\kappa]^-)) u, v \rangle \rangle = 0, \quad \forall u \in \mathcal{S}\mathcal{L}_t(\Sigma), \forall v \in \mathcal{S}\mathcal{L}_t(\Sigma).$$

The proof of the this theorem runs parallel to the proof of the corresponding result for acoustic scattering, see [36, Thm. 6.3.19] for details.

Theorem 4.5 implies, together with the polarity property from Lemma 3.2, that the variational equation from Formulation 4.4 is trivially satisfied for all test functions $v \in \mathcal{S}\mathcal{L}_t(\Sigma)$ since $\mathcal{S}\mathcal{L}_t(\Sigma)$ is a closed subspace of $\mathcal{M}\mathcal{L}_t(\Sigma)$. It is sufficient to test with elements in any complement space $\mathcal{S}\mathcal{L}_t^c(\Sigma)$ of $\mathcal{S}\mathcal{L}_t(\Sigma)$ in $\mathcal{M}\mathcal{L}_t(\Sigma)$. For the sake of easy implementation we choose $\mathcal{S}\mathcal{L}_t^c(\Sigma) := \mathcal{S}\mathcal{L}_t^\perp(\Sigma)$, the $L^2_t$-orthogonal complement space. It has the following simple characterization:

Corollary 4.2 (Orthogonal Complement of $L^2_t$-Single-Trace Space)

The orthogonal complement of the skeleton $L^2_t$-single-trace space is given by

$$\mathcal{S}\mathcal{L}_t^\perp(\Sigma) := \left\{ \left( \begin{array}{c} u_0 \\ \vdots \\ u_L \end{array} \right), \left( \begin{array}{c} \nu_0 \\ \vdots \\ \nu_L \end{array} \right) \right| \begin{array}{c} u_j|_{\Gamma_i} = -u_i|_{\Gamma_j}, \quad \nu_j|_{\Gamma_i} = \nu_i|_{\Gamma_j}, \forall j < i \in \{0, \ldots, L\} \end{array} \right\}.$$

Remark 4.6 (Polarity Complement of Energy Single-Trace Space) Since Theorem 4.5 also holds in $\mathcal{M}\mathcal{T}(\Sigma)$, in Formulation 4.2 it would be possible to test with elements in any complement space $\mathcal{S}\mathcal{T}^\perp(\Sigma)$ of $\mathcal{S}\mathcal{T}(\Sigma) \subset \mathcal{M}\mathcal{T}(\Sigma)$. 
A polarity-complement using the polarity property from Lemma 3.1 seems to be the best choice. But there is no simple representation like the one provided by Corollary 4.2 for the orthogonal complement space $\mathcal{S}\mathcal{L}_t^+(\Sigma)$ of $\mathcal{S}\mathcal{L}_t(\Sigma)$.

The reason for the lack of such an explicit representation is that restriction to interfaces is in general not well-defined in $\mathcal{M}\mathcal{T}(\Sigma)$. Thus Corollary 4.2 reveals a crucial advantage of the $L^2$-setting.

Replacing the test space from Formulation 4.4 by the $L^2$-orthogonal complement space, we obtain the following formulation.

**Formulation 4.6** For $\kappa > 0$ find $\vec{c} \in \mathcal{S}\mathcal{L}_t(\Sigma)$ such that

$$\langle (\text{Id}_\kappa - (\mathbf{M} - \mathbf{M}[\kappa])) \vec{c}, \vec{v} \rangle = \langle \text{Id}_\kappa \vec{c}_{\text{inc}}, \vec{v} \rangle \quad \forall \vec{v} \in \mathcal{S}\mathcal{L}_t^+(\Sigma).$$

### 4.3.1 Second-Kind Formulation in $L^2_t(\Sigma)$

The definitions of the skeleton spaces $\mathcal{S}\mathcal{L}_t(\Sigma)$ and $\mathcal{S}\mathcal{L}_t^+(\Sigma)$ suggest an identification of the spaces with the so-called *interface based* $L^2_t$-skeleton space

$$\mathcal{L}^2_t(\Sigma) := \bigtimes_{j<i} \left( L^2_t(\Gamma_{ij}) \times L^2_t(\Gamma_{ji}) \right) \cong L^2_t(\Sigma) \times L^2_t(\Sigma). \quad (4.15)$$

The identification $\cong$ is provided by the restriction of tangential vectorfields to interfaces, which is well defined in $L^2_t(\Sigma)$, and an additional reordering. Definition 3.4 immediately provides the isomorphism $J : \mathcal{L}^2_t(\Sigma) \rightarrow \mathcal{S}\mathcal{L}_t(\Sigma)$,

$$J \left( (\vec{u}_{ij})_{j<i} \right) := \left( \begin{array}{c} \vec{u}_0 \\ \nu_0 \\ \vdots \\ \vec{u}_L \\ \nu_L \end{array} \right), \quad \vec{u}_{ij} := \vec{u}_{ij}, \quad \nu_{ij} := \nu_{ij}.$$

for $i, j \in \{0, \ldots, L\}$. We also introduce an isomorphism $J_L : \mathcal{L}^2_t(\Sigma) \rightarrow \mathcal{S}\mathcal{L}_t(\Sigma)$ that identifies elements in $\mathcal{S}\mathcal{L}_t(\Sigma)$ with elements in $\mathcal{L}^2_t(\Sigma)$ by

$$J_L \left( (\vec{u}_{ij})_{j<i} \right) := \left( \begin{array}{c} \vec{u}_0 \\ \nu_0 \\ \vdots \\ \vec{u}_L \\ \nu_L \end{array} \right), \quad \vec{u}_{ij} := \vec{u}_{ij}, \quad \nu_{ij} := \nu_{ij}.$$

Moreover, it is immediate that

$$\langle J \vec{u}, J_L \vec{v} \rangle = 2 \langle [\vec{u}, \vec{v}]_{\mathcal{L}^2_t(\Sigma)}, \vec{u}, \vec{v} \rangle \in \mathcal{L}^2_t(\Sigma), \quad (4.17)$$

where $[\vec{u}, \vec{v}]_{\mathcal{L}^2_t(\Sigma)}$ refers to the inner product of $\mathcal{L}^2_t(\Sigma)$ which for $\vec{u} = (\vec{u}_{ij})_{j<i}, \nu_{ij})_{j<i} \in \mathcal{L}^2_t(\Sigma)$ is given by

$$[\vec{u}, \vec{v}]_{\mathcal{L}^2_t(\Sigma)} := \sum_{j<i} [\vec{u}_{ij}, \nu_{ij}]_{\Gamma_{ij}} + [\nu_{ij}, \varphi_{ij}]_{\Gamma_{ij}}.$$
Formulation 4.7 (Final Second-Kind Formulation)

Seek \( e \in L^2_t(\Sigma) \) such that

\[
\langle \langle (\text{Id}_\kappa - (\mathbf{M} - \mathbf{M}[\kappa])) J e, J_\perp v \rangle \rangle = \langle \langle \text{Id}_{\kappa,\text{inc}} e, J_\perp v \rangle \rangle \quad \forall v \in L^2_t(\Sigma).
\]

Finally we have found a formulation whose Galerkin discretization is straightforward, see Section 5 below.

Remark 4.7 Theorem 4.3 implies only that the operator \( \text{Id}_\kappa - (\mathbf{M} - \mathbf{M}[\kappa]) \) is continuous. We pointed out that a proof for Fredholmness of index zero is known in the case of \( L = 1 \), while it is not clear how to generalize the proof to the multi-domain setting \( L > 1 \). This means that apart from the question of uniqueness of solutions of Formulation 4.7, we cannot guarantee that the operator on the left hand side of Formulation 4.7 is Fredholm of index zero. Thus, well-posedness of the above formulation remains an open problem. Nevertheless, the numerical results provided in Section 6 are promising.

4.4 Interface Based Boundary Integral Equations

Assembly of boundary element Galerkin matrices relies on explicit integral representations of the boundary integral operators occurring in (4.13). Thus, we study the difference potentials from (4.14) for arguments in \( L^2_t(\partial\Omega_i) \).

– By simple calculations we find [9, Eq. (35)] for \( x \notin \partial\Omega_i \)

\[
\delta D_i(u_i)(x) = \int_{\partial\Omega_i} (-\kappa^2_i K_{D,i}[\kappa_i](x, y) + \kappa^2 K_{D,i}[\kappa_i](x, y)) u_i(y) \, dS(y),
\]

with Cauchy-singular kernel matrices \( (x \notin \partial\Omega_i, \, y \in \partial\Omega_i) \)

\[
K_{D,i}[\kappa_i](x, y) = n_i(y) \text{grad}_x (\Phi_\kappa(\|x - y\|)) + (\text{grad}_y (\Phi_\kappa(\|x - y\|)) \cdot n_i(y)) I_3 \in C^{3,3},
\]

behaving like \( \|K_{D,i}[\kappa_i](x, y)\| \sim \|x - y\|^{-2} \) for \( y \to x \).

– Immediately we get the weakly singular integral expression \( (x \in \mathbb{R}^3) \)

\[
\delta V_i(\nu_i)(x) = \int_{\partial\Omega_i} (\kappa^2_i \Phi_\kappa(\|x - y\|) - \kappa^2 \Phi_\kappa(\|x - y\|)) \nu_i(y) \, dS(y).
\]

– To manipulate \( \delta W_i \), we introduce the entire function \( \delta \Phi_\kappa(\xi) := (\Phi_{\kappa_i} - \Phi_{\kappa})(\xi), \, \xi \in \mathbb{R} \), and apply integration by parts on \( \partial\Omega_i \) and differentiation under the integral \( (x \in \mathbb{R}^3) \):}

\[
\delta W_i(\nu_i)(x)
\]
\[
\begin{align*}
= \text{grad}_x \int_{\partial \Omega_i} \delta \Phi_i(\|x - y\|)(\text{div}_\Gamma \nu_i)(y) \, dS(y) \\
= -\text{grad}_x \int_{\partial \Omega_i} (\text{grad}_y \delta \Phi_i(\|x - y\|)) \cdot \nu_i(y) \, dS(y) \\
= \int_{\partial \Omega_i} \left( \frac{\partial^2 \delta \Phi_i(\|x - y\|)}{\partial \xi^2} \frac{(x - y)(x - y)^T}{\|x - y\|^2} \\
+ \frac{\partial \delta \Phi_i(\|x - y\|)}{\partial \xi} \left( \frac{I_3}{\|x - y\|} - \frac{(x - y)(x - y)^T}{\|x - y\|^3} \right) \right) \nu_i(y) \, dS(y) \\
= \int_{\partial \Omega_i} (K_W[\kappa](x - y) - K_W[\kappa](x - y)) \nu_i(y) \, dS(y),
\end{align*}
\]

where \(K_W[\kappa](x) \in (C^\infty(\mathbb{R}^3 \setminus \{0\}))^{3,3}\) is a weakly singular integral kernel.

Even more straightforwardly, we obtain the representation (\(x \in \mathbb{R}^3\))

\[
\begin{align*}
\delta C_i(\nu_i)(x) &= \int_{\partial \Omega_i} -\frac{\partial \delta \Phi_i(\|x - y\|)}{\partial \xi} \frac{y - x}{\|x - y\|} \times \nu_i(y) \, dS(y) \\
&\quad + \int_{\partial \Omega_i} (K_C[\kappa_i](x - y) - K_C[\kappa_i](x - y)) \nu_i(y) \, dS(y),
\end{align*}
\]

with a bounded kernel matrix \(K_C[\kappa](x) \in (C^\infty(\mathbb{R}^3 \setminus \{0\}))^{3,3} \cap (L^\infty(\mathbb{R}^3))^{3,3}\).

A more detailed presentation of these kernels including series expansions for \(x \approx y\) can be found in [36, Appendix A]. In fact, the above manipulations confirm Theorem 4.5, because no hypersingular kernels occur so that boundary integral operators spawned by applying tangential traces to difference potentials can take arguments in \(L^2(\partial \Omega_i)\).

**Remark 4.8** In Formula (4.18) all derivatives have been shifted off the argument function \(\nu_i\). This was made possible by our regularization, which introduced more regular differences of singular kernels that allow differentiation. Hence, the regularity of \(\text{div}_\Gamma (n_i \times u)\) and \(\text{div}_\Gamma \nu\) becomes irrelevant. As regards boundary element Galerkin discretization, \(\text{div}_\Gamma\)-conformity of the trial and test spaces is no longer required!

Thanks to the regularity of their kernels the difference potentials can be considered on parts of a subdomain boundary, in particular on interfaces \(\Gamma_{ij}\). We pick \(u = (u_{ij})_{i < j} \in \mathcal{L}_i^2(\Sigma), \ u_{ij} = (\nu_{ij}) \in L^2(\Gamma_{ij}) \times L^2(\Gamma_{ij}), \ i, j \in \{0, \ldots, L\}\) and, by switching to summation over interfaces, convert (4.13) into

\[
(M - M[\kappa]) \mathbf{u} =
\]

\[
\begin{align*}
\delta \Phi_i(\|x - y\|)(\text{div}_\Gamma \nu_i)(y) \, dS(y) \\
-\text{grad}_x \int_{\partial \Omega_i} (\text{grad}_y \delta \Phi_i(\|x - y\|)) \cdot \nu_i(y) \, dS(y) \\
\int_{\partial \Omega_i} \left( \frac{\partial^2 \delta \Phi_i(\|x - y\|)}{\partial \xi^2} \frac{(x - y)(x - y)^T}{\|x - y\|^2} \\
+ \frac{\partial \delta \Phi_i(\|x - y\|)}{\partial \xi} \left( \frac{I_3}{\|x - y\|} - \frac{(x - y)(x - y)^T}{\|x - y\|^3} \right) \right) \nu_i(y) \, dS(y) \\
\int_{\partial \Omega_i} (K_W[\kappa](x - y) - K_W[\kappa](x - y)) \nu_i(y) \, dS(y),
\end{align*}
\]
The signs of the difference potential contributions are derived from (4.16). We point out that an extension by zero of \( \mathbf{u}_{ij} \) and \( \mathbf{\nu}_{ij} \) to both \( \partial \Omega_{l} \) and \( \partial \Omega_{j} \) has been tacitly assumed here.

Next, we exploit cancellation due to opposite orientations of the normals \( \mathbf{n}_{i} = -\mathbf{n}_{j} \) on \( \Gamma_{ij} \). We make the surprising discovery that all contributions due to the regularizing operator \( \mathbf{M}[\kappa] \) can be dropped: Eventually, the choice of \( \kappa > 0 \) does not matter!

\[
\begin{align*}
\gamma_{n}^{r} \sum_{j<i} \left( \frac{1}{2} \mathbf{D}[\mathbf{u}_{ij}] + \frac{1}{2} \mathbf{W}[\mathbf{\nu}_{ij}] - \mathbf{W}[\mathbf{\nu}_{ij}] \right) + \frac{1}{2} \mathbf{W}[\mathbf{\nu}_{ij}] + \frac{1}{2} \mathbf{W}[\mathbf{\nu}_{ij}] \\
\gamma_{n}^{r} \sum_{j>i} \left( -\mathbf{D}[\mathbf{u}_{ij} \times \mathbf{n}_{i}] + \mathbf{W}[\mathbf{\nu}_{ij} \times \mathbf{n}_{i}] + \mathbf{C}[\mathbf{\nu}_{ij}] \right) - \mathbf{D}[\mathbf{u}_{ij} \times \mathbf{n}_{i}] + \mathbf{W}[\mathbf{\nu}_{ij} \times \mathbf{n}_{i}] - \mathbf{C}[\mathbf{\nu}_{ij}] \\
\end{align*}
\]

Note that among these difference potentials \( \delta \mathbb{D}_{ij}(\mathbf{u}_{ij}) \) is the only one that may be discontinuous across \( \Gamma_{ij} \), because from the jump relations (2.9) we conclude (\( i, j, m \in \{0, \ldots, L\}, j < i \))

\[
\begin{align*}
\gamma_{el}^{L} \delta \mathbb{D}_{ij}(\mathbf{u}_{ij}) &= \frac{1}{2} (\kappa_{i}^{2} - \kappa_{j}^{2}) \mathbf{u}_{ij} + \delta \mathbb{D}_{ij}(\mathbf{u}_{ij}) \quad \text{on } \partial \Omega_{l} , \\
\gamma_{el}^{L} \delta \mathbb{D}_{ij}(\mathbf{u}_{ij}) &= \frac{1}{2} (\kappa_{i}^{2} - \kappa_{j}^{2}) \mathbf{u}_{ij} + \delta \mathbb{D}_{ij}(\mathbf{u}_{ij}) \quad \text{on } \partial \Omega_{j} , \\
\gamma_{el}^{m} \delta \mathbb{D}_{ij}(\mathbf{u}_{ij}) &= \delta \mathbb{D}_{ij}^{m}(\mathbf{u}_{ij}) \quad \text{on } \partial \Omega_{m} , m \notin \{i, j\} .
\end{align*}
\]
where the boundary integral operators $\delta D_{ij}^m : L^2(\Gamma_{ij}) \to L^2(\partial \Omega_m)$ are defined as

$$
(\delta D_{ij}^m u_{ij})(x) := \int_{\Gamma_{ij}} \pi_{t,m}(x) \delta K_{D,ij}(x, y) u_{ij}(y) \, dS(y), \quad x \in \partial \Omega_m, \quad (4.22)
$$

with $\pi_{t,m}(x)$ designating the orthogonal projection onto the tangent plane of $\partial \Omega_m$ in $x$. In a similar fashion we introduce the boundary integral operators listed in the following tables along with their kernels

<table>
<thead>
<tr>
<th>BI-Op.</th>
<th>$\delta D_{ij}^m$</th>
<th>$\delta V_{ij}^m$</th>
<th>$\delta W_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>kernel</td>
<td>$\pi_{t,m}(x) \delta K_{D,ij}(x, y)$</td>
<td>$\pi_{t,m}(x) \delta K_{V,ij}(x, y)$</td>
<td>$\pi_{t,m}(x) \delta K_{C,ij}(x, y)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>BI-Op.</th>
<th>$\delta V_{ij}^m$</th>
<th>$\delta W_{ij}$</th>
<th>$\delta C_{ij}^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>kernel</td>
<td>$\pi_{x,m}(x) \delta K_{V,ij}(x, y)$</td>
<td>$\pi_{x,m}(x) \delta K_{C,ij}(x, y)$</td>
<td>$\pi_{x,m}(x) \delta K_{C,ij}(x, y)$</td>
</tr>
</tbody>
</table>

Table 4.1: Boundary integral operators and their kernels

where $\pi_{x,m}(x)$ stands for the action of $-n_m(x) \times \pi_{t,m}$. Using these boundary integral operators and the jump relations (4.20) we can recast

$$(M - M[\kappa]) J u$

$$
= \left( \gamma_t^m \sum_{j \leq i} \left( \delta \mathcal{W}_{ij}(u_{ij}) + \delta \mathcal{V}_{ij}(\nu_{ij}) + \delta \mathcal{W}_{ij}(\nu_{ij}) \right) \\
+ \gamma_x^m \sum_{j \leq i} \left( -\delta \mathcal{W}_{ij}(u_{ij} \times n_i) + \delta \mathcal{V}_{ij}(u_{ij} \times n_i) + \delta \mathcal{C}_{ij}^{\nu}(\nu_{ij}) \right) \right)_{m=0}^{L}
$$

$$
= \left( \begin{array}{c}
\frac{m-1}{2}(\kappa_m^2 - \kappa_j^2)u_{mj} + \sum_{i=m+1}^{L} \frac{1}{2}(\kappa_m^2 - \kappa_j^2)u_{im} \\
+ \sum_{j \leq i} \delta D_{ij}(u_{ij}) + \delta V_{ij}^{\nu}(\nu_{ij}) + \delta W_{ij}^{\nu}(\nu_{ij}) \\
- \delta W_{ij}^{\nu}(u_{ij} \times n_i) + \delta V_{ij}^{\nu}(u_{ij} \times n_i) + \delta C_{ij}^{\nu}(\nu_{ij})
\end{array} \right)_{m=0}^{L} \quad (4.23)
$$

This formula can be inserted into the variational equation of Formulation 4.7 using

$$
\langle \langle w, J u, v \rangle \rangle = \sum_{l<k} \int_{\Gamma_{lk}} w_k \cdot v_{kl} + \omega_k \cdot \phi_{kl} + w_l \cdot v_{lk} - \omega_l \cdot \phi_{kl} \, dS,
$$

for $w = \left( w_0, \ldots, w_{m} \right) \in \mathcal{ML}(\Sigma)$ and $v = \left( v_{kl} \right)_{l<k} \in \mathcal{L}_{m}^{2}(\Sigma)$, which follows from (3.3). This implies an interface-oriented way to express the left-hand side of Formulation 4.7:

$$
\langle \langle (I_m - (M - M[\kappa])) J u, J u, v \rangle \rangle
$$
\[
\sum_{l<k} \int_{\Gamma_{kl}} (\kappa^2_l + \kappa^2_l) u_{kl} \cdot v_{kl} \, dS
\]
\[+ \sum_{l<k} \sum_{j<i} \int_{\Gamma_{kl}} \left( (\delta D^k_{ij} + \delta D^l_{ij})(\mathbf{u}_{ij}) + (\delta V^k_{ij} + \delta V^l_{ij})(\mathbf{v}_{ij}) + (\delta W^k_{ij} + \delta W^l_{ij})(\mathbf{\nu}_{ij}) \right) \cdot v_{kl} \, dS
\]
\[+ \int_{\Gamma_{kl}} \left( (-\delta \tilde{W}^k_{ij} + \delta \tilde{W}^l_{ij})(\mathbf{u}_{ij} \times \mathbf{n}) + (\delta \tilde{V}^k_{ij} - \delta \tilde{V}^l_{ij})(\mathbf{u}_{ij} \times \mathbf{n}) + (\delta C^k_{ij} - \delta C^l_{ij})(\mathbf{\nu}_{ij}) \right) \cdot v_{kl} \, dS
\]
\[= \sum_{l<k} \int_{\Gamma_{kl}} (\kappa^2_l + \kappa^2_l) u_{kl} \cdot v_{kl} \, dS
\]
\[+ 2 \sum_{l<k} \sum_{j<i} \int_{\Gamma_{kl}} \left( \delta D^k_{ij}(\mathbf{u}_{ij}) + \delta V^k_{ij}(\mathbf{v}_{ij}) + \delta W^k_{ij}(\mathbf{\nu}_{ij}) \right) \cdot v_{kl} \, dS
\]
\[+ \int_{\Gamma_{kl}} \left( -\delta \tilde{W}^k_{ij}(\mathbf{u}_{ij} \times \mathbf{n}) + \delta \tilde{V}^k_{ij}(\mathbf{u}_{ij} \times \mathbf{n}) + \delta C^k_{ij}(\mathbf{\nu}_{ij}) \right) \cdot v_{kl} \, dS.
\]

For Galerkin discretization this is the most appropriate form of Formulation 4.7.

5 Galerkin Discretization

For the Galerkin discretization of the variational problem from Formulation 4.7 it remains to specify finite dimensional boundary element spaces \( V_M \subset L^2(\Sigma) \) and sets of basis functions. This is done in the next Section. In Section 5.2, we explicitly state the Galerkin system corresponding to Formulation 4.7 for a simple geometric arrangement with \( L = 2 \).

5.1 Boundary Element Spaces

We rely on a mesh partition \( T_M \) of \( \Sigma \), compatible with the interfaces in the sense that the closure of each interface \( \Gamma_{ij} \) is the union of some (closed) cells of \( T_M \). For each cell \( \tau \) of \( T_M \) we assume the existence of a polygonal reference cell \( \hat{\tau} \subset \mathbb{R}^2 \) and a \( C^1 \)-diffeomorphism \( \Phi_{\tau} : \hat{\tau} \rightarrow \tau \).

Definition 5.1 (Piecewise polynomial Tangential Vector Fields) For \( k \in \mathbb{N}_0 \), we define the space of \( T_M \)-piecewise polynomial tangential vector fields of degree \( \leq k \) by
\[
\mathcal{V}_k(T_M)(\Sigma) = \left\{ \mathbf{v} \in L^\infty(\Sigma)^3 \mid \mathbf{v} \big|_\tau \in \mathbf{F}_{k,\tau} \left( \left( P_k(\hat{\tau}) \right)^2 \right) , \forall \tau \in T_M \right\}
\]
where $\mathbf{F}_{\tau,\tau}$ denotes the covariant transformation

$$
\mathbf{F}_{\tau,\tau}(\mathbf{p})(\mathbf{x}) := D\Phi_{\tau}(\mathbf{x})(\mathbf{G}_{\tau}(\mathbf{x}))^{-1}\mathbf{p}(\mathbf{x}),
$$

$$
\mathbf{G}_{\tau}(\mathbf{x}) := (D\Phi_{\tau}(\mathbf{x}))(\mathbf{x})\mathbf{D}\Phi_{\tau}(\mathbf{x}), \quad \mathbf{x} \in \tau,
$$

and $\mathcal{P}_k(\tau)$ is the space of polynomials of degree $\leq k$ on $\tau$.

Then, as trial and test space for Formulation 4.7 we may choose

$$
\mathcal{V}_M := \mathcal{V}_{\mathcal{T}_M}(\Sigma) \times \mathcal{V}_{\mathcal{T}_M}(\Sigma) \subset \mathcal{L}_V^2(\Sigma).
$$

(5.1)

Now we restrict ourselves to approximation by piecewise constant vector fields, that is, $k = 0$. In this case, writing $M$ for the number of cells of $\mathcal{T}_M$, we can use the basis $\{\chi^i_M\}_{i=1,\ldots,2M}$ of $\mathcal{V}_M(\Sigma)$, whose elements are given by

$$
\chi^i_M(\mathbf{x}) = \begin{cases} 
\mathbf{F}_{\tau,\tau}(\mathbf{p})(\mathbf{x}) & \mathbf{x} \in \tau_i, \\
0 & \mathbf{x} \notin \tau_i,
\end{cases}
\quad \tau_i \in \mathcal{T}_M := \{\tau_1, \ldots, \tau_M\},
$$

(5.2)

Thus we obtain a basis for $\mathcal{V}_M = \mathcal{V}_{\mathcal{T}_M}(\Sigma) \times \mathcal{V}_{\mathcal{T}_M}(\Sigma)$, namely

$$
\mathcal{B}_M := \{b^q_M\}_{q=1}^{\dim \mathcal{V}_M} = \left\{ \left(\chi^1_M, 0\right), \ldots, \left(\chi^{2M}_M, 0\right), \left(0, \chi^1_M\right), \ldots, \left(0, \chi^{2M}_M\right) \right\}.
$$

(5.3)

### 5.2 Galerkin Matrices

For the sake of lucidity, we will discuss the linear system of equations arising from the Galerkin discretization of Formulation 4.7 with $\mathcal{T}_M$-piecewise constant vector fields ($\mathcal{V}_M = \mathcal{V}_{\mathcal{T}_M}(\Sigma) \times \mathcal{V}_{\mathcal{T}_M}(\Sigma)$) only for special situation of two hemispherical subdomains, see Figure 5.1 for a cross-section. This situation is sufficiently general to convey all key considerations.

We order the cells of $\mathcal{T}_M$ according to their occurrence in the interfaces $I_{21}, I_{20}$, and $I_{10}$ and use the basis (5.3), designating the basis functions by $b^q_M$, $q = 1, \ldots, 2M$. We can represent an element $\mathbf{e}_M \in \mathcal{V}_M \subset \mathcal{L}_V^2(\Sigma)$ as $\mathbf{e}_M = \sum_{q=1}^{\dim \mathcal{V}_M} e_q M_M b^q_M$ with a coefficient vector $\mathbf{e}_M \in \mathbb{C}^{2M}$. The interface-aware ordering of mesh cells and the corresponding ordering of basis functions, induces a partitioning of the coefficient vector according to

$$
\mathbf{e}_M = \left( \mathbf{e}_{21,M}, \mathbf{e}_{20,M}, \mathbf{e}_{10,M}, \mathbf{e}_{21,M}, \mathbf{e}_{20,M}, \mathbf{e}_{10,M} \right),
$$

(5.4)

where $\mathbf{e}_{ij,M} \in \mathbb{C}^{2M_{ij}}$ is the $2M_{ij}$-vector of expansion coefficients, $M_{ij} \in \mathbb{N}$ the number of mesh cells $\subset I_{ij}$, belonging to basis functions supported on $I_{ij}$. We point out that the colors indicate the interface to which sub-vectors are associated. They match the colors in Figure 5.1.
Fig. 5.1: Cross-section of the model geometry \((L = 2)\) for studying the implementation of the second-kind formulation.

Inserting the ansatz into Formulation 4.7 and using the the interface-wise representation of its left hand side from (4.24) yields the following linear system of equations:

\[
G_M e_M = f_M, 
\]

with \(f_M \in \mathbb{C}^{2M}\) and the Galerkin matrix \(G_M \in \mathbb{C}^{2M,2M}\),

\[
f_M := \begin{pmatrix}
2\kappa_0^2 M_{21}^{21} & 0 & 0 & 0 & 0 & 0 \\
0 & 2\kappa_0^2 M_{20}^{20} & 0 & 0 & 0 & 0 \\
0 & 0 & 2\kappa_0^2 M_{10}^{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 2M_{21}^{21} & 0 & 0 \\
0 & 0 & 0 & 0 & 2M_{20}^{20} & 0 \\
0 & 0 & 0 & 0 & 0 & 2M_{10}^{10}
\end{pmatrix}
\begin{pmatrix}
e_{inc21,M} \\
e_{inc20,M} \\
e_{inc10,M} \\
e_{inc21,M} \\
e_{inc20,M} \\
e_{inc10,M}
\end{pmatrix},
\]

where \(\{e_{inc21,M}, e_{inc20,M}, e_{inc10,M}, \varphi_{inc21,M}, \varphi_{inc20,M}, \varphi_{inc10,M}\} \in \mathbb{C}^{2M}\) represents the coefficient vector of the interpolant of \(J^{-1} e_{inc} \in J^{-1} \Sigma_{tot} E_{inc} \in V_M\) and

\[
G_M = \begin{pmatrix}
(k_0^2 + \kappa_0^2) M_{21}^{21} & 0 & 0 & 0 & 0 & 0 \\
0 & (k_0^2 + \kappa_0^2) M_{20}^{20} & 0 & 0 & 0 & 0 \\
0 & 0 & (k_0^2 + \kappa_0^2) M_{10}^{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 2M_{21}^{21} & 0 & 0 \\
0 & 0 & 0 & 0 & 2M_{20}^{20} & 0 \\
0 & 0 & 0 & 0 & 0 & 2M_{10}^{10}
\end{pmatrix}
\]

\[
2 \begin{pmatrix}
\delta D_{21}^{21} & \delta D_{21}^{20} & \delta D_{10}^{20} & \delta V_{21}^{21} + \delta W_{21}^{21} & \delta V_{21}^{20} + \delta W_{21}^{20} & \delta V_{21}^{10} + \delta W_{21}^{10} \\
\delta D_{21}^{20} & \delta D_{20}^{20} & \delta D_{10}^{20} & \delta V_{21}^{20} + \delta W_{21}^{20} & \delta V_{21}^{20} + \delta W_{21}^{20} & \delta V_{21}^{10} + \delta W_{21}^{10} \\
\delta D_{10}^{20} & \delta D_{10}^{20} & \delta V_{21}^{10} + \delta W_{21}^{10} & \delta V_{21}^{10} + \delta W_{21}^{10} & \delta V_{21}^{10} + \delta W_{21}^{10}
\end{pmatrix}
\]

The Galerkin interface mass matrix for \(\Gamma_{ij}\) is given by

\[
M_{ij}^{13} := \left( \int_{\Gamma_{ij}} \chi_{ij,M}^q \chi_{ij,M}^s dS \right)_{s,q \in \{1,...,2M_{ij}\}},
\]
and the block matrices $\delta D^{k,l}_{ij}, \delta V^{k,l}_{ij}, \delta W^{k,l}_{ij}, \delta \tilde{V}^{k,l}_{ij}, \delta \tilde{W}^{k,l}_{ij}, \delta C^{k,l}_{ij} \in \mathbb{C}^{2M_{kl}, 2M_{ij}}$ have the entries

$$(\delta D^{k,l}_{ij})_{s,q} := \int_{\Gamma_{kl}} \delta D^{k}_{ij}(x^{q}_{ij,M}) \cdot \chi^{s}_{kl,M} dS,$$

$$(\delta V^{k,l}_{ij})_{s,q} := \int_{\Gamma_{kl}} \delta V^{k}_{ij}(x^{q}_{ij,M}) \cdot \chi^{s}_{kl,M} dS,$$

$$(\delta W^{k,l}_{ij})_{s,q} := \int_{\Gamma_{kl}} \delta W^{k}_{ij}(x^{q}_{ij,M}) \cdot \chi^{s}_{kl,M} dS,$$

$$(\delta \tilde{V}^{k,l}_{ij})_{s,q} := \int_{\Gamma_{kl}} \delta \tilde{V}^{k}_{ij}(x^{q}_{ij,M} \times n_i) \cdot \chi^{s}_{kl,M} dS,$$

$$(\delta \tilde{W}^{k,l}_{ij})_{s,q} := \int_{\Gamma_{kl}} \delta \tilde{W}^{k}_{ij}(x^{q}_{ij,M} \times n_i) \cdot \chi^{s}_{kl,M} dS.$$
Experiment II. The exciting field $E_{\text{inc}}$ is an incident plane wave with direction of propagation $d = \frac{1}{\sqrt{2}} (1, 0, 1)^\top$ and the polarization vector $p = (0, 1, 0)^\top$. As shown in Figure 6.2, the scatterer consists of two hemispheres $\Omega_2, \Omega_3$ with radius $\frac{1}{2}$ plus another medium occupying $\Omega_1 := Q \setminus \overline{E_{\text{inc}}(0)}$, where $Q := \{(x, y, z)^\top \in \mathbb{R}^3 : -0.7 < x < 0.7, -0.7 < y < 0.7, 0 < z < 0.7\}$. We use the wave number vector $(\kappa_0, \kappa_1, \kappa_2, \kappa_3) = (2, 3, 1, 4)$.
6.2 Convergence of Discretization Error

By uniform refinement we create sequences of nested meshes \( \{\mathcal{T}_M\}_{M} \) of \( \Sigma \), \( \mathcal{T}_M \) comprising \( M \) flat triangular cells. The results of [35, Chapter 8] suggest that the error introduced by our polyhedral approximation of \( \Sigma \) will not dominate the Galerkin discretization error for trial/test spaces of piecewise constant vector fields. The global mesh width \( h_M \) of \( \mathcal{T}_M \) is the maximal diameter of its cells. We monitor the behavior of the error in \( L^2_t(\Sigma) \)- and \( H^{-\frac{1}{2}}_t(\Sigma) \)-norms as a function of \( h_M \). The \( H^{-\frac{1}{2}}_t(\Sigma) \)-norm is computed via the inner product \( (\phi, \nu) \mapsto \langle \gamma^e_{i \text{el}} \mathbf{V}_i[0] \phi, \nu \rangle_{\partial \Omega} \), where \( \mathbf{V}_i[0] \) was defined in (2.5). The equivalence of norms follows from continuity properties and the ellipticity of \( \gamma^e_{i \text{el}} \mathbf{V}_i[0] \) in \( H^{-\frac{1}{2}}_t(\Sigma) \), shown for instance in [7, Proposition 2]. As a reference solution we use the discrete solution calculated with the second-kind formulation on a mesh obtained by one more step of refinement. In order to calculate the error \( H^{-\frac{1}{2}}_t(\Sigma) \)-norm, we project the first-kind approximation onto the space of piecewise constant vector fields.

We use \( M \in \{44, 176, 704, 2816, 11264, 45056\} \) for Experiment I, and \( M \in \{140, 560, 2240, 8960, 35840\} \) for Experiment II, respectively. The various estimated errors are plotted in Fig. 6.3 and indicate algebraic convergence.

The meshes were generated with Gmsh [21] and for visualization of the computed data (see Figures 6.1 and 6.2) we used ParaView [1].
Fig. 6.4: Convergence history of GMRES applied to diagonally scaled first and second-kind Galerkin systems. The reddish colors correspond to the results for the second-kind system.

of the Galerkin solutions for both traces in all norms. Rates are difficult to predict due to the presence of strong edge singularities of the fields. In both experiments we observe that the accuracy of the second-kind results matches that of the first-kind results.

6.3 Convergence of GMRES and Conditioning

Figure 6.4 displays the convergence history of the GMRES iterative solver when applied to the linear systems arising from the Galerkin discretization of first-kind and second-kind BIE for the two experimental setups. As initial guess the vector $x_0 = (1, \ldots, 1)^T$ is used. We observe that GMRES convergence for the second-kind system

- does not deteriorate on fine meshes,
- and is consistently significantly faster than for the first-kind system.

In order to study the dependence of the convergence of GMRES on wave numbers we fix two of them in the setting of Experiment I and vary the third, tracking the number of GMRES iterations required for a reduction of the residual $r_0$ by six orders of magnitude in the process. Concretely, for $\kappa_1 = 10$, $\kappa_2 = 1$, we consider $\kappa_0 \in \{2.5, 5, 7.5, 10, 12.5, 15, 17.5, 20, 22.5, 25\}$, and for $\kappa_0 = 10$, $\kappa_2 = 1$, we choose $\kappa_1 \in \{2.5, 5, 7.5, 10, 12.5, 15, 17.5, 20, 22.5, 25\}$. Again, throughout, the number of required iterations is substantially smaller for the second-kind Galerkin system compared to that for the first-kind approach. In both cases we observe a very similar pronounced dependence of the number of iterations on the wave number, which hints that it may be caused by an underlying physical instability.

Though the convergence of GMRES is not directly determined by the conditioning of a linear system [3], we also plot the Euclidean condition numbers
Fig. 6.5: Convergence of GMRES for varying wave numbers for the first and second-kind Galerkin systems for Experiment I.

Fig. 6.6: Euclidean condition numbers of diagonally rescaled Galerkin matrices for a sequence of meshes.

We observe that the conditioning of the second-kind matrices is independent of the mesh width, whereas or the first-kind Galerkin matrices the condition numbers grow like roughly like $O(h^{-2})$. Admittedly, the condition numbers of the second-kind matrices can be rather large, nevertheless. They will also strongly depend on the wave numbers and the geometry.

The spectrum of the system matrix, in particular its separation from 0, is more relevant for predicting GMRES convergence. Therefore, in Figure 6.7 we plot the spectra of the (diagonally scaled) Galerkin matrices in the complex plane. We see that the spectra of the second-kind Galerkin matrices are clus-
We derived a new second-kind boundary integral equation (BIE) formulation for electromagnetic scattering at penetrable composite objects. Though its rigorous theoretical analysis is still incomplete, its Galerkin boundary element discretization (BEM) performs well in numerical tests. Firstly, it delivers about the same accuracy as a popular first-kind BIE approach. Secondly, it gives rise to linear systems of equations for which GMRES enjoys fast and mesh-independent convergence, on fine meshes considerably faster than for first-kind Galerkin BEM. Dependence on wave numbers persists, however.
References


