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Elastic Scattering Coefficients and Enhancement of Nearly Elastic Cloaking *

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Abstract

The concept of scattering coefficients has played a pivotal role in a broad range of inverse scattering and imaging problems in acoustic and electromagnetic media. In view of their promising applications, we introduce the notion of scattering coefficients of an elastic inclusion in this article. First, we define elastic scattering coefficients and substantiate that they naturally appear in the expansions of elastic scattered field and far field scattering amplitudes corresponding to a plane wave incidence. Then an algorithm is developed and analyzed for extracting the elastic scattering coefficients from multi-static response measurements of the scattered field. Moreover, the estimate of the maximal resolving order is provided in terms of the signal-to-noise ratio. The decay rate and symmetry of the elastic scattering coefficients are also discussed. Finally, we design scattering-coefficients-vanishing structures and elucidate their utility for enhancement of nearly elastic cloaking.

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Key words. Elastic scattering, Scattering coefficients, Elastic cloaking, Inverse scattering.

1 Introduction

The notion of *scattering coefficients* of acoustic and electromagnetic inclusions emerged in an effort to design enhanced near invisibility cloaks [9, 11, 12]. These frequency-dependent geometric objects contain rich information about the contrast of material parameters, high order shape oscillations, frequency profile, and the maximal resolving power of the imaging setup. They have been effectively used for inverse medium scattering [7], echolocation and shape description [13], mathematical understanding of super-resolution phenomena in imaging [6] and phase-less reconstruction of domains [5]. In electromagnetic or acoustic media, scattering coefficients provide a natural extension to the concept of *polarization tensors* [8] with respect

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to frequency dependence. They are defined in terms of Fourier-Bessel coefficients (in 2D) or spherical harmonic coefficients (in 3D) of far-field scattering amplitude and can be retrieved with high accuracy from the multi-static response (MSR) measurements of the scattered field by solving a least-squares optimization problem. Multistatic imaging involves two steps. The first step consists of recording the waves generated by point sources on an array of receivers. The second step consists of processing the recorded matrix data in order to estimate some features of the medium [3, 8]. The interested readers are referred to [11, 12, 13] for further details.

The invisibility cloaking, proved to be scientifically realizable in many investigations, for instance, by Pendry et al. [37], Greenleaf, Lassas and Uhlmann [26, 27], Greenleaf et al. [24, 25], Leonhardt [29] and Milton, Briane and Willis[31], is an exciting area of interest nowadays. Significant progress has been made recently on the control of conductivity equations [10, 26, 27], acoustic [11, 16, 19], electromagnetic [12, 17] and elastic waves [28, 21, 22, 23] using curvilinear transformations of coordinates. In fact, a meta-material is perceived as an invisibility cloak that maps a concealment region into a surrounding shell by virtue of transformation and thereby making the material parameters strongly heterogeneous and anisotropic, however fulfilling impedance matching with the surrounding vacuum. The cloak, thus neither does scatter waves nor does it induce a shadow in the transmitted field. Reduction in the backscattering [10, 11, 12] and the anomalous localized resonances [4, 14, 20] are also used to design and enhance cloaking devices.

The purpose of this article is to introduce the notion of elastic scattering coefficients (ESC) of an inclusion embedded in a homogeneous medium. The impetus behind this study is the enhancement of nearly elastic cloaking and the promising applications of ESC in mathematical imaging and inverse scattering. We first define ESC of the inclusion using the eigen-functions of the Lamé equation and the integral representation of scattered elastic field in terms of hyper-singular boundary integral operators. Then, a least-squares optimization algorithm is designed for the reconstruction of significant ESC from the full aperture MSR data collected using a circular acquisition system. The stability, truncation error and maximal resolving order of the reconstruction procedure are analytically quantified. Finally, we design mathematical structures with vanishing scattering coefficients (S-vanishing structures) and elaborate a framework for the enhancement of nearly elastic cloaking. The results contained in this paper can cater to many inverse scattering problems, especially for shape identification and classification in elastic media. The interested readers are referred to [30] and articles cited therein for comprehensive details on shape identification in elastic media.

The contents of this article are organized in the following manner. Some notation and a few preliminary results on layer potential theory of elastic scattering are collected in Section 2. In Section 3, ESC are defined and their important features are discussed. Section 4 is dedicated to the reconstruction framework for ESC. The enhancement procedure for elastic cloaking is elaborated in Section 5. Finally, in Section 6, we sum up the important contributions of this investigation and discuss about interesting applications of ESC in mathematical imaging.

2 Elements of Layer Potential Theory

Since this article is concerned with elastic scattering and the integral formulation of the scattered field is the key component to define ESC, we feel it best to pause and introduce some background material from layer potential theory for elasticity. For details beyond those we provide in this section, please refer to the monograph [2].

2.1 Preliminaries and Notation

To simplify matters, throughout this article, we confine ourselves to the two-dimensional case. However, it is precisely that all the results and definitions in this section are valid in three-dimensions with obvious modifications.

For any sufficiently smooth, open and bounded domain $\Omega \subset \mathbb{R}^2$ with $\mathcal{C}^2-$ boundary $\partial\Omega$, we define $L^2(\Omega)$ in the usual way with norm

$$\|u\|_{L^2(\Omega)} := \left(\int_\Omega |u|^2 d\mathbf{x}\right)^{1/2},$$

and the Hilbert space $H^1(\Omega)$ by

$$H^1(\Omega) := \big\{u \in L^2(\Omega) \,|\, \nabla u \in L^2(\Omega)\big\},$$

with norm

$$\|u\|_{H^1(\Omega)} := \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{1/2}.$$

We define $H^2(\Omega)$ as the space of functions $u \in H^1(\Omega)$ such that $\partial_{ij} u \in L^2(\Omega)$ for all $i, j = 1, 2$, and $H^{3/2}(\Omega)$ as the interpolation space $[H^1(\Omega), H^2(\Omega)]_{1/2}$. Let $\mathbf{t}$ be the tangent vector to $\partial\Omega$ at point $\mathbf{x}$ and let $\partial/\partial\mathbf{t}$ denote the tangential derivative. Then, we say that $u \in H^1(\partial\Omega)$ if $u \in L^2(\partial\Omega)$ and $\partial u/\partial\mathbf{t} \in L^2(\partial\Omega)$. Refer to the monograph by Bergh and Löfström [18] for further details.

Consider a homogeneous isotropic elastic material, occupying a bounded domain $D \subset \mathbb{R}^2$ with connected $\mathcal{C}^2-$boundary ∂D, compressional and shear moduli $\lambda_1 \in \mathbb{R}_+$ and $\mu_1 \in \mathbb{R}_+$ respectively, and density $\rho_1 \in \mathbb{R}_+$. Let the exterior domain $\mathbb{R}^2 \setminus \overline{D}$ be loaded with different elastic material having parameters $\rho_0, \lambda_0, \mu_0 \in \mathbb{R}_+$ such that

$$(\lambda_0 - \lambda_1)(\mu_0 - \mu_1) > 0. \tag{2.1}$$

To facilitate latter analysis, we introduce piecewise defined parameters

$$\lambda(\mathbf{x}) := \lambda_0 \chi_{(\mathbb{R}^2\setminus D)}(\mathbf{x}) + \lambda_1 \chi_{(D)}(\mathbf{x}) \tag{2.2}$$

$$\mu(\mathbf{x}) := \mu_0 \chi_{(\mathbb{R}^2\setminus D)}(\mathbf{x}) + \mu_1 \chi_{(D)}(\mathbf{x}) \tag{2.3}$$

$$\rho(\mathbf{x}) := \rho_0 \chi_{(\mathbb{R}^2\setminus D)}(\mathbf{x}) + \rho_1 \chi_{(D)}(\mathbf{x}), \tag{2.4}$$

where χ_Ω represents the characteristic function of a domain Ω. We also define the linear elasticity operator $\mathcal{L}_{\lambda_0,\mu_0}$ and the surface traction operator (or conormal derivativative) $\partial/\partial\nu$, associated with parameters (λ_0, μ_0) by

$$\mathcal{L}_{\lambda_0,\mu_0}[\mathbf{w}] := [\mu_0 \Delta\mathbf{w} + (\lambda_0 + \mu_0)\nabla\nabla\cdot\mathbf{w}] \tag{2.5}$$

and

$$\frac{\partial\mathbf{w}}{\partial\nu} := \lambda_0(\nabla\cdot\mathbf{w})\mathbf{n} + 2\mu_0\left(\nabla^s\mathbf{w}\right)\mathbf{n} \tag{2.6}$$

for all sufficiently smooth vector fields $\mathbf{w} : \mathbb{R}^2 \to \mathbb{R}^2$, where $\mathbf{n} \in \mathbb{R}^2$ represents the outward unit normal to ∂D, $\nabla^s\mathbf{w} = (\nabla\mathbf{w} + (\nabla\mathbf{w})^\top)/2$ is the linear elastic strain and the superscript $\top$ reflects a transpose operation.

Let $\omega > 0$ be the angular frequency of the mechanical oscillations. We denote the outgoing fundamental solution of the time-harmonic elasticity equation in $\mathbb{R}^2$ with parameters $(\lambda_0, \mu_0, \rho_0)$ by $\mathbf{\Gamma}^\omega$, that is, for all $\mathbf{x} \in \mathbb{R}^2$

$$(\mathcal{L}_{\lambda_0,\mu_0} + \rho_0\omega^2\mathcal{I})\mathbf{\Gamma}^\omega(\mathbf{x}) = -\delta_0(\mathbf{x})\mathbf{I}_2, \qquad \forall\mathbf{x} \in \mathbb{R}^2, \tag{2.7}$$

subject to the *Kupradze's* outgoing radiation conditions where $\delta_{\mathbf{y}}$ is the Dirac mass at $\mathbf{y}$, $\mathcal{I}$ is the identity operator and $\mathbf{I}_2 \in \mathbb{R}^{2\times 2}$ is the identity matrix. Let $\kappa_\alpha := \omega/c_\alpha$ for $\alpha = P, S$, where the constants $c_S = \sqrt{\mu_0/\rho_0}$ and $c_P = \sqrt{\lambda_0 + 2\mu_0/\rho_0}$ refer to background shear and pressure wave speeds respectively. It is well known, see, for instance, [32], that

$$\mathbf{\Gamma}^\omega(\mathbf{x}) = \frac{1}{\mu_0}\left[\left(\mathbf{I}_2 + \frac{1}{\kappa_S^2}\nabla\nabla^\top\right) g(\mathbf{x}, \kappa_S) - \frac{1}{\kappa_S^2}\nabla\nabla^\top g(\mathbf{x}, \kappa_P)\right], \quad \mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}. \tag{2.8}$$

The function $g(\cdot, \kappa)$ is the fundamental solution to the Helmholtz operator $-(\Delta + \kappa^2\mathcal{I})$ in $\mathbb{R}^2$ with wave-number $\kappa \in \mathbb{R}_+$, that is,

$$(\Delta + \kappa^2\mathcal{I})g(\mathbf{x}, \kappa) = -\delta_{\mathbf{0}}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2,$$

subject to the *Sommerfeld's* outgoing radiation condition

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|^{1/2}\left[\frac{\partial g(\mathbf{x}, \kappa)}{\partial \mathbf{n}} - i\kappa g(\mathbf{x}, \kappa)\right] = 0, \qquad \mathbf{x} \in \mathbb{R}^2,$$

where $\partial/\partial\mathbf{n}$ represents the normal derivative. In two dimensions,

$$g(\mathbf{x}, \kappa) = \frac{i}{4} H_0^{(1)}(\kappa|\mathbf{x}|), \quad \forall \mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \tag{2.9}$$

where $H_0^{(1)}$ is the Hankel function of first kind of order zero (see, for instance, [33]). Throughout this article, we use the convention $\mathbf{\Gamma}^\omega(\mathbf{x}, \mathbf{y}) = \mathbf{\Gamma}^\omega(\mathbf{x} - \mathbf{y})$ and reserve the notation α and β to represent pressure (P) and shear (S) wave-modes, that is, $\alpha, \beta \in \{P, S\}$.

2.2 Scattered Field and Integral Representation

Let us begin this subsection by introducing the elastic single layer potential

$$\mathcal{S}_D^\omega[\boldsymbol{\varphi}](\mathbf{x}) := \int_{\partial D} \mathbf{\Gamma}^\omega(\mathbf{x}, \mathbf{y})\boldsymbol{\varphi}(\mathbf{y})d\sigma(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^2 \setminus \partial D \tag{2.10}$$

for all densities $\boldsymbol{\varphi} \in L^2(\partial D)^2$. We also need the boundary integral operator

$$(\mathcal{K}_D^\omega)^*[\boldsymbol{\varphi}](\mathbf{x}) = \text{p.v.} \int_{\partial D} \frac{\partial}{\partial \nu_{\mathbf{x}}} \mathbf{\Gamma}^\omega(\mathbf{x}, \mathbf{y})\boldsymbol{\varphi}(\mathbf{y})d\sigma(\mathbf{y}), \quad \text{a.e.} \quad \mathbf{x} \in \partial D$$

for all $\boldsymbol{\varphi} \in L^2(\partial D)^2$, where p.v. stands for Cauchy principle value of the integral and the surface traction of matrix $\mathbf{\Gamma}^\omega$ is defined columnwise, that is, for all constant vectors $\mathbf{p} \in \mathbb{R}^2$

$$\left[\frac{\partial \mathbf{\Gamma}^\omega}{\partial \nu}\right]\mathbf{p} = \frac{\partial\left[\mathbf{\Gamma}^\omega \mathbf{p}\right]}{\partial \nu}.$$

We recall that the traces $\mathcal{S}_D^\omega[\boldsymbol{\varphi}]\big|_\pm$ and $\partial(\mathcal{S}_D^\omega[\boldsymbol{\varphi}])/\partial\nu\big|_\pm$ are well-defined and satisfy the jump conditions (see, for instance, [1, Section 3.4.3])

$$\begin{cases} \mathcal{S}_D^\omega[\boldsymbol{\varphi}]\big|_+ = \mathcal{S}_D^\omega[\boldsymbol{\varphi}]\big|_- \\ \dfrac{\partial(\mathcal{S}_D^\omega[\boldsymbol{\varphi}])}{\partial \nu}\Big|_\pm(\mathbf{x}) = \left(\pm\dfrac{1}{2}I + (\mathcal{K}_D^\omega)^*\right)\boldsymbol{\varphi}(\mathbf{x}), \qquad \text{a.e.} \quad \mathbf{x} \in \partial D. \end{cases} \tag{2.11}$$

Here and throughout this investigation subscripts + and − indicate the limiting values across the interface ∂D from outside and from inside domain D respectively, that is, for any function ψ

$$(\psi(\mathbf{x}))\big|_{\pm} = \lim_{\epsilon\to 0^+} \psi(\mathbf{x} \pm \epsilon\mathbf{n}), \quad \mathbf{x} \in \partial D.$$

Consider a time harmonic incident elastic field $\mathbf{U}$ satisfying

$$(\mathcal{L}_{\lambda_0,\mu_0} + \rho_0\omega^2\mathcal{I})\mathbf{U}(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \mathbb{R}^2. \tag{2.12}$$

Then the total displacement field in the presence of inclusion D, represented by $\mathbf{u}$, satisfies the transmission problem

$$\begin{cases} (\mathcal{L}_{\lambda,\mu} + \rho\omega^2\mathcal{I})\mathbf{u}(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \mathbb{R}^2, \\ (\mathbf{u} - \mathbf{U})(\mathbf{x}) \text{ satisfies Kupradze's radiation condition when } |\mathbf{x}| \to \infty\ . \end{cases} \tag{2.13}$$

We recall from [2, Theorem 1.8] that the total field $\mathbf{u}$ admits the integral representation

$$\mathbf{u}(\mathbf{x}, \omega) = \begin{cases} \mathbf{U}(\mathbf{x}, \omega) + \mathcal{S}_D^\omega[\boldsymbol{\psi}](\mathbf{x}, \omega), & \mathbf{x} \in \mathbb{R}^2 \setminus \overline{D}, \\ \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}](\mathbf{x}, \omega), & \mathbf{x} \in D, \end{cases} \tag{2.14}$$

in terms of single layer potentials $\mathcal{S}_D^\omega$ and $\widetilde{\mathcal{S}}_D^\omega$, where unknown densities $\boldsymbol{\varphi}, \boldsymbol{\psi} \in L^2(\partial D)^2$ satisfy the system of integral equations

$$\begin{pmatrix} \widetilde{\mathcal{S}}_D^\omega & -\mathcal{S}_D^\omega \\ \dfrac{\partial}{\partial\widetilde{\nu}}\widetilde{\mathcal{S}}_D^\omega\Big|_- & -\dfrac{\partial}{\partial\nu}\mathcal{S}_D^\omega\Big|_+ \end{pmatrix} \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} \mathbf{U} \\ \dfrac{\partial \mathbf{U}}{\partial\nu} \end{pmatrix}\Bigg|_{\partial D}. \tag{2.15}$$

Here a superposed $\sim$ is used to distinguish the single layer potential and the surface traction defined using interior Lamé parameters $(\lambda_1, \mu_1, \rho_1)$. To simply matters the dependence of $\mathbf{u}$, $\mathbf{U}$, $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ on frequency ω is suppressed, whenever no confusion may arise.

The following result from [2, Theorem 1.7] guarantees the unique solvability of the system (2.15) and consequently that of problems (2.13) and (2.14).

Theorem 2.1. *Let D be a Lipschitz bounded domain in $\mathbb{R}^2$ with parameters $0 < \lambda_1, \mu_1, \rho_1 < \infty$ satisfying condition (2.1) and let $\omega^2\rho_1$ be different from Dirichlet eigenvalues of the operator $-\mathcal{L}_{\lambda_1,\mu_1}$ on D. Then for any function $\mathbf{U} \in H^1(\partial D)^2$ there exists a unique solution $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in L^2(\partial D)^2 \times L^2(\partial D)^2$ to the integral system (2.15). Moreover, there exists a constant $C > 0$ such that*

$$\|\boldsymbol{\varphi}\|_{L^2(\partial D)^2} + \|\boldsymbol{\psi}\|_{L^2(\partial D)^2} \leq C\left(\|\mathbf{U}\|_{H^1(\partial D)^2} + \left\|\frac{\partial \mathbf{U}}{\partial\nu}\right\|_{L^2(\partial D)^2}\right). \tag{2.16}$$

3 Elastic Scattering Coefficients

This section is dedicated to defining ESC in two dimensions. To facilitate the definition of ESC, we first recall some background material on cylindrical eigen-functions of the Lamé equation and present the multipolar expansions of the exterior scattered elastic field and the Kupradze fundamental solution $\mathbf{\Gamma}^\omega$ in the next subsection.

3.1 Cylindrical Elastic Waves and Multipolar Expansions

We define $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ and write $\mathbb{S} := \{\mathbf{x} \in \mathbb{R}^2 \,|\, \mathbf{x}\cdot\mathbf{x} = 1\}$. The position vector $\mathbf{x} \in \mathbb{R}^2$ can be equivalently expressed as $\mathbf{x} = (|\mathbf{x}|\cos\varphi_{\mathbf{x}}, |\mathbf{x}|\sin\varphi_{\mathbf{x}})$ where $\varphi_{\mathbf{x}} \in [0, 2\pi)$ denotes the polar angle of $\mathbf{x}$. Denote by $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta\}$ the orthonormal basis vectors for the polar coordinate system in two dimensions, that is,

$$\hat{\mathbf{e}}_r = (\cos\varphi_{\mathbf{x}}, \sin\varphi_{\mathbf{x}}), \quad \hat{\mathbf{e}}_\theta = -(\sin\varphi_{\mathbf{x}}, \cos\varphi_{\mathbf{x}}).$$

Consider the surface vector harmonics in two-dimensions

$$\mathbf{P}_m(\hat{\mathbf{x}}) = e^{im\varphi_{\mathbf{x}}}\hat{\mathbf{e}}_r \quad \text{and} \quad \mathbf{S}_m(\hat{\mathbf{x}}) = e^{im\varphi_{\mathbf{x}}}\hat{\mathbf{e}}_\theta \quad \text{for all } \ m \in \mathbb{Z}. \tag{3.1}$$

It is known, see [32] for instance, that these cylindrical surface vector potentials enjoy the orthogonality properties

$$\int_{\mathbb{S}} \mathbf{P}_n(\hat{\mathbf{x}}) \cdot \overline{\mathbf{P}_m(\hat{\mathbf{x}})} d\sigma(\hat{\mathbf{x}}) = 2\pi\delta_{nm}, \tag{3.2}$$

$$\int_{\mathbb{S}} \mathbf{S}_n(\hat{\mathbf{x}}) \cdot \overline{\mathbf{S}_m(\hat{\mathbf{x}})} d\sigma(\hat{\mathbf{x}}) = 2\pi\delta_{nm}, \tag{3.3}$$

$$\int_{\mathbb{S}} \mathbf{P}_m(\hat{\mathbf{x}}) \cdot \overline{\mathbf{S}_m(\hat{\mathbf{x}})} d\sigma(\hat{\mathbf{x}}) = 0, \tag{3.4}$$

for all $n, m \in \mathbb{Z}$, where δ_{nm} is the Kronecker's delta function and $d\sigma$ is the infinitesimal differential element on $\mathbb{S}$.

Let $H_m^{(1)}$ and J_m be cylindrical Hankel and Bessel functions of first kind of order $m \in \mathbb{Z}$, respectively. Then, for each $\kappa > 0$, we construct the functions $v_m(\cdot, \kappa)$ and $w_m(\cdot, \kappa)$ by

$$v_m(\mathbf{x}, \kappa) := H_m^{(1)}(\kappa|\mathbf{x}|)e^{im\varphi_{\mathbf{x}}} \quad \text{and} \quad w_m(\mathbf{x}, \kappa) := J_m(\kappa|\mathbf{x}|)e^{im\varphi_{\mathbf{x}}}. \tag{3.5}$$

It is easy to verify that v_m are outgoing radiating solutions to the Helmholtz equation $\Delta v + \kappa^2 v = 0$ in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ and that w_m are entire functions to $\Delta v + \kappa^2 v = 0$ in $\mathbb{R}^2$ respectively.

Using surface vector harmonics $\mathbf{P}_m$, $\mathbf{S}_m$ and functions v_m, w_m, we define

$$\begin{aligned}\mathbf{H}_m^P(\mathbf{x}, \kappa_P) :=& \nabla v_m(\mathbf{x}, \kappa_P) \\ =& \kappa_P \left(H_m^{(1)}(\kappa_P|\mathbf{x}|)\right)' \mathbf{P}_m(\hat{\mathbf{x}}) + \frac{im}{|\mathbf{x}|} H_m^{(1)}(\kappa_P|\mathbf{x}|)\mathbf{S}_m(\hat{\mathbf{x}}),\end{aligned} \tag{3.6}$$

$$\begin{aligned}\mathbf{H}_m^S(\mathbf{x}, \kappa_S) :=& \nabla \times (\hat{\mathbf{e}}_{\mathbf{z}} v_m(\mathbf{x}, \kappa_S)) \\ =& \frac{im}{|\mathbf{x}|} H_m^{(1)}(\kappa_S|\mathbf{x}|)\mathbf{P}_m(\hat{\mathbf{x}}) - \kappa_S \left(H_m^{(1)}(\kappa_S|\mathbf{x}|)\right)' \mathbf{S}_m(\hat{\mathbf{x}}),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\mathbf{J}_m^P(\mathbf{x}, \kappa_P) :=& \nabla w_m(\mathbf{x}, \kappa_P) \\ =& \kappa_P \left(J_m(\kappa_P|\mathbf{x}|)\right)' \mathbf{P}_m(\hat{\mathbf{x}}) + \frac{im}{|\mathbf{x}|} J_m(\kappa_P|\mathbf{x}|)\mathbf{S}_m(\hat{\mathbf{x}}),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\mathbf{J}_m^S(\mathbf{x}, \kappa_S) :=& \nabla \times (\hat{\mathbf{e}}_{\mathbf{z}} w_m(\mathbf{x}, \kappa_S)) \\ =& \frac{im}{|\mathbf{x}|} J_m(\kappa_S|\mathbf{x}|)\mathbf{P}_m(\hat{\mathbf{x}}) - \kappa_S \left(J_m(\kappa_S|\mathbf{x}|)\right)' \mathbf{S}_m(\hat{\mathbf{x}}),\end{aligned} \tag{3.9}$$

for all $\kappa_\alpha > 0$ and $m \in \mathbb{Z}$, where $\hat{\mathbf{e}}_\mathbf{z} = (0,0,1)$ is a unit normal vector to the $(x_1, x_2)-$plane and

$$\left(H_m^{(1)}\right)'(t) := \frac{d}{dt}\left[H_m^{(1)}(t)\right] \quad \text{and} \quad (J_m)'(t) := \frac{d}{dt}\left[J_m(t)\right]. \tag{3.10}$$

For simplicity, we suppress the dependence of $\mathbf{J}_m^\alpha$ and $\mathbf{H}_m^\alpha$ on wave-numbers κ_α henceforth.

The functions $\mathbf{J}_m^P$ and $\mathbf{J}_m^S$ are the interior longitudinal and transverse eigen-vectors of the Lamé system in $\mathbb{R}^2$. Similarly, $\mathbf{H}_m^P$ and $\mathbf{H}_m^S$ are the exterior eigen-vectors of the Lamé system in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ [15]. Following result on the completeness and linear independence of the interior eigen-vectors $\left(\mathbf{J}_m^P, \mathbf{J}_m^S\right)$ and exterior eigen-vectors $\left(\mathbf{H}_m^P, \mathbf{H}_m^S\right)$ with respect to $L^2(\partial D)^2-$norm holds. The interested readers are referred to [38, Lemmas 1-3] for further details.

Lemma 3.1. *Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain containing origin and ∂D be a closed Lyapunov curve. Then the set $\{\mathbf{H}_m^P, \mathbf{H}_m^S : m \in \mathbb{Z}\}$ is complete and linearly independent in $L^2(\partial D)^2$. Moreover, if $\rho_1\omega^2$ is not a Dirichlet eigenvalue of the Lamé equation on D, then the set $\{\mathbf{J}_m^P, \mathbf{J}_m^S : m \in \mathbb{Z}\}$ is also complete and linearly independent in $L^2(\partial D)^2$.*

As a direct consequence of Lemma 3.1, corresponding to every incident field $\mathbf{U}$ satisfying (2.12), there exist constants $a_m^P, a_m^S \in \mathbb{C}$ for all $m \in \mathbb{Z}$ such that

$$\mathbf{U}(\mathbf{x}) = \sum_{m\in\mathbb{Z}} \left(a_m^S \mathbf{J}_m^S(\mathbf{x}) + a_m^P \mathbf{J}_m^P(\mathbf{x})\right), \quad \mathbf{x} \in \mathbb{R}^2. \tag{3.11}$$

In particular, a general plane incident wave of the form

$$\begin{aligned}\mathbf{U}(\mathbf{x}) = &\ \frac{1}{\rho_0 c_S^2} e^{i\kappa_S \mathbf{x}\cdot\mathbf{d}}\, \mathbf{d}^\perp + \frac{1}{\rho_0 c_P^2} e^{i\kappa_P \mathbf{x}\cdot\mathbf{d}}\, \mathbf{d} \\ = & -\left(\frac{i}{\rho_0 c_S^2 \kappa_S} \nabla \times \left[\hat{\mathbf{e}}_\mathbf{z}\, e^{i\kappa_S \mathbf{x}\cdot\mathbf{d}}\right] + \frac{i}{\rho_0 c_P^2 \kappa_P}\left[\nabla e^{i\kappa_P \mathbf{x}\cdot\mathbf{d}}\right]\right)\end{aligned} \tag{3.12}$$

can be written in the form (3.11) with

$$a_m^\beta := a_m^\beta(\mathbf{U}) = -\frac{i}{\rho_0 c_\beta^2 \kappa_\beta} e^{im(\pi/2-\theta)}, \quad \beta \in \{P, S\}, \tag{3.13}$$

where $\mathbf{d} = (\cos\theta, \sin\theta) \in \mathbb{S}$ is the direction of incidence and $\mathbf{d}^\perp$ is a vector perpendicular to $\mathbf{d}$. In fact, this decomposition is a simple consequence of Jacobi-Anger decomposition of the scalar plane wave

$$e^{i\kappa \mathbf{x}\cdot\mathbf{d}} = \sum_{m\in\mathbb{Z}} e^{im(\pi/2-\varphi_\mathbf{x})} J_m(\kappa|\mathbf{x}|) e^{im\theta_\mathbf{x}}.$$

Moreover, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ such that $|\mathbf{x}| > |\mathbf{y}|$ and for any vector $\mathbf{p} \in \mathbb{R}^2$ independent of $\mathbf{x}$

$$\mathbf{\Gamma}^\omega(\mathbf{x}, \mathbf{y})\mathbf{p} = -\frac{i}{4\rho_0 c_S^2} \sum_{n\in\mathbb{Z}} \mathbf{H}_n^S(\mathbf{x}) \left[\overline{\mathbf{J}_n^S(\mathbf{y})} \cdot \mathbf{p}\right] - \frac{i}{4\rho_0 c_P^2} \sum_{n\in\mathbb{Z}} \mathbf{H}_n^P(\mathbf{x}) \left[\overline{\mathbf{J}_n^P(\mathbf{y})} \cdot \mathbf{p}\right]. \tag{3.14}$$

Refer, for instance, to [39], for the derivation of this expansion.

3.2 Scattering Coefficients of Elastic Inclusions

Note that the multipolar expansion (3.14) of the fundamental solution $\mathbf{\Gamma}^\omega$ enables us to derive the expansion

$$\mathcal{S}_D^\omega[\boldsymbol{\psi}](\mathbf{x}) = -\frac{i}{4\rho_0 c_P^2}\sum_{n\in\mathbb{Z}} \mathbf{H}_n^P(\mathbf{x})\int_{\partial D}\left[\overline{\mathbf{J}_n^P(\mathbf{y})}\cdot\boldsymbol{\psi}(\mathbf{y})\right]d\sigma(\mathbf{y}) - \frac{i}{4\rho_0 c_S^2}\sum_{n\in\mathbb{Z}} \mathbf{H}_n^S(\mathbf{x})\int_{\partial D}\left[\overline{\mathbf{J}_n^S(\mathbf{y})}\cdot\boldsymbol{\psi}(\mathbf{y})\right]d\sigma(\mathbf{y}) \tag{3.15}$$

of the single layer potential $\mathcal{S}_D^\omega[\boldsymbol{\psi}]$ for all $\mathbf{x}\in\mathbb{R}^2\setminus\overline{D}$ sufficiently far from the boundary ∂D. Consequently, by virtue of expansion (3.15) and the integral representation (2.14), the scattered field can be expanded as

$$\mathbf{u}(\mathbf{x}) - \mathbf{U}(\mathbf{x}) = -\frac{i}{4\rho_0}\sum_{n\in\mathbb{Z}}\left[\frac{b_n^S}{c_S^2}\mathbf{H}_n^S(\mathbf{x}) + \frac{b_n^P}{c_P^2}\mathbf{H}_n^P(\mathbf{x})\right], \tag{3.16}$$

where

$$b_n^\alpha = \int_{\partial D}\left[\overline{\mathbf{J}_n^\alpha(\mathbf{y})}\cdot\boldsymbol{\psi}(\mathbf{y})\right]d\sigma(\mathbf{y}), \quad \alpha\in\{S,P\}, \quad \forall\, n\in\mathbb{Z}. \tag{3.17}$$

Definition 3.2. *Let $(\boldsymbol{\varphi}_m^\beta, \boldsymbol{\psi}_m^\beta)$, $m\in\mathbb{Z}$, be the solution of (2.15) corresponding to $\mathbf{U} = \mathbf{J}_m^\beta$. Then the elastic scattering coefficients $W_{m,n}^{\alpha,\beta}$ of $D \Subset \mathbb{R}^2$ are defined by*

$$W_{m,n}^{\alpha,\beta} = W_{m,n}^{\alpha,\beta}[D,\lambda_0,\lambda_1,\mu_0,\mu_1,\rho_0,\rho_1,\omega] := \int_{\partial D}\left[\overline{\mathbf{J}_n^\alpha(\mathbf{y})}\cdot\boldsymbol{\psi}_m^\beta(\mathbf{y})\right]d\sigma(\mathbf{y}), \quad m,n\in\mathbb{Z}, \tag{3.18}$$

where α and β indicate wave-modes P or S.

Following result on the decay rate of the ESC holds.

Lemma 3.3. *There exist constants $C_{\alpha,\beta}>0$ for each wave-mode $\alpha,\beta = P,S$ such that*

$$\left|W_{m,n}^{\alpha,\beta}[D,\lambda_0,\lambda_1,\mu_0,\mu_1,\rho_0,\rho_1,\omega]\right| \le \frac{C_{\alpha,\beta}^{|n|+|m|-2}}{|n|^{|n|-1}|m|^{|m|-1}} \tag{3.19}$$

for all $m,n\in\mathbb{Z}$ and $m,n\to\infty$.

Proof. The proof of the estimate (3.19) is very similar to [11, Lemma 2.1] for the acoustic scattering coefficients. For the sake of completeness, we fix the ideas of the proof in the sequel. Recall the asymptotic behavior

$$J_m(t) \simeq \sqrt{\frac{1}{2\pi|m|}}\left(\frac{et}{2|m|}\right)^{|m|} \tag{3.20}$$

of Bessel functions of first kind with respect to the order $|m|\to\infty$ and for a fixed argument t [35, Formula 10.19.1]. Then, by using the recurrence formulae [35, Formula 10.6.2],

$$J_m'(t) = -J_{m+1}(t) + \frac{m}{t}J_m(t) \quad \text{and} \quad J_m'(t) = J_{m-1}(t) - \frac{m}{t}J_m(t) \tag{3.21}$$

one obtains

$$|J'_m(t)| \leq \sqrt{\frac{1}{2\pi(|m|+1)}} \left(\frac{et}{2(|m|+1)}\right)^{|m|+1} + \frac{|m|}{t}\sqrt{\frac{1}{2\pi|m|}} \left(\frac{et}{2|m|}\right)^{|m|}. \tag{3.22}$$

Consequently, by the definition (3.8)-(3.9) of $\mathbf{J}_n^\alpha(\mathbf{x})$, and Theorem 2.1, we have

$$\|\mathbf{J}_n^\alpha\|_{L^2(\partial D)^2} \leq \left(\frac{C_1^\alpha}{|n|}\right)^{|n|-1} \tag{3.23}$$

and

$$\left\|\boldsymbol{\psi}_m^\beta\right\|_{L^2(\partial D)^2} \leq C\left(\left\|\mathbf{J}_m^\beta\right\|_{L^2(\partial D)^2} + \left\|\frac{\partial \mathbf{J}_m^\beta}{\partial \nu}\right\|_{L^2(\partial D)^2}\right) \leq \left(\frac{C_2^\beta}{|m|}\right)^{|m|-1}, \tag{3.24}$$

for some constants C_1^α and C_2^β. Finally, the result follows by substituting the estimates for the norms of $\boldsymbol{\psi}_m^\beta$ and $\mathbf{J}_n^\alpha$ in the definition of the scattering coefficients and choosing $C_{\alpha,\beta}$ appropriately in terms of C_1^α and C_2^β. □

3.3 Connections with Scattered Field and Far Field Amplitudes

Consider a general plane incident field $\mathbf{U}$ of the form (3.12) admitting decomposition (3.11)-(3.13). By superposition principle the solution $(\boldsymbol{\varphi}, \boldsymbol{\psi})$ of (2.15) is given by

$$\boldsymbol{\psi}(\mathbf{x}) = \sum_{m\in\mathbb{Z}} \left[a_m^P \boldsymbol{\psi}_m^P + a_m^S \boldsymbol{\psi}_m^S\right] \quad \text{and} \quad \boldsymbol{\varphi}(\mathbf{x}) = \sum_{m\in\mathbb{Z}} \left[a_m^P \boldsymbol{\varphi}_m^P + a_m^S \boldsymbol{\varphi}_m^S\right]. \tag{3.25}$$

This, together with Definition 3.2 of the scattering coefficients and the expansion (3.16), renders the asymptotic expansion

$$\mathbf{u}(\mathbf{x}) - \mathbf{U}(\mathbf{x}) = \sum_{n\in\mathbb{Z}} \left[\gamma_n^P \mathbf{H}_n^P(\mathbf{x}) + \gamma_n^S \mathbf{H}_n^S(\mathbf{x})\right] \tag{3.26}$$

of the scattered field for all $\mathbf{x} \in \mathbb{R}^2 \setminus D$ sufficiently far from ∂D, where

$$\gamma_n^\beta = \sum_{m\in\mathbb{Z}} \left(d_m^P W_{m,n}^{\beta,P} + d_m^S W_{m,n}^{\beta,S}\right), \quad \beta \in \{P, S\}, \tag{3.27}$$

with

$$d_m^\beta := -\frac{i}{4\rho_0 c_\beta^2} a_m^\beta = -\frac{1}{4\rho_0^2 c_\beta^4 \kappa_\beta} e^{im(\pi/2-\theta)}. \tag{3.28}$$

In order to substantiate the connection between ESC and far field scattering amplitudes we recall that the cylindrical Hankel functions $H_n^{(1)}$ admit the far field behavior

$$H_n^{(1)}(\kappa|\mathbf{x}|) \sim \frac{e^{i\kappa|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} \sqrt{\frac{2}{\pi\kappa}} e^{-i\pi(n/2+1/4)} + O\left(|x|^{-3/2}\right) \tag{3.29}$$

$$\left(H_n^{(1)}(\kappa|\mathbf{x}|)\right)' \sim i\kappa \frac{e^{i\kappa|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} \sqrt{\frac{2}{\pi\kappa}} e^{-i\pi(n/2+1/4)} + O\left(|x|^{-3/2}\right) \tag{3.30}$$

as $|\mathbf{x}| \to \infty$ (see, for instance, [35, Formulae 10.2.5 and 10.17.11]). Here, the notation $\sim$ indicates that only leading order terms are retained. Consequently, the far field behavior of the functions $\mathbf{H}_n^P$ and $\mathbf{H}_n^S$ can be predicted as

$$\mathbf{H}_n^P(\mathbf{x}) \sim \frac{e^{i\kappa_P|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} A_n^{\infty,P}\mathbf{P}_n(\hat{\mathbf{x}}) \quad \text{and} \quad \mathbf{H}_n^S(\mathbf{x}) \sim \frac{e^{i\kappa_S|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} A_n^{\infty,S}\mathbf{S}_n(\hat{\mathbf{x}}), \quad \text{as } |\mathbf{x}| \to \infty, \tag{3.31}$$

where

$$A_n^{\infty,P} := (i+1)\kappa_P e^{-in\pi/2}\sqrt{\frac{\kappa_P}{\pi}} \quad \text{and} \quad A_n^{\infty,S} := -(i+1)\kappa_S e^{-in\pi/2}\sqrt{\frac{\kappa_S}{\pi}}. \tag{3.32}$$

Thus, for all $\mathbf{x} \in \mathbb{R}^2 \setminus D$ such that $|\mathbf{x}| \to \infty$ the scattered field $(\mathbf{u} - \mathbf{U})$ in (3.26) admits the asymptotic expansion

$$\mathbf{u}(\mathbf{x}) - \mathbf{U}(\mathbf{x}) = \frac{e^{i\kappa_P|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} \sum_{n\in\mathbb{Z}} \left[\gamma_n^P A_n^{\infty,P}\mathbf{P}_n(\hat{\mathbf{x}})\right] + \frac{e^{i\kappa_S|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} \sum_{n\in\mathbb{Z}} \left[\gamma_n^S A_n^{\infty,S}\mathbf{S}_n(\hat{\mathbf{x}})\right]. \tag{3.33}$$

On the other hand, the Kupradze radiation condition guarantees the existence of two analytic functions $\mathbf{u}_P^\infty : \mathbb{S} \to \mathbb{C}^2$ and $\mathbf{u}_S^\infty : \mathbb{S} \to \mathbb{C}^2$ such that

$$\mathbf{u}(\mathbf{x}) - \mathbf{U}(\mathbf{x}) = \frac{e^{i\kappa_S|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}}\mathbf{u}_P^\infty(\hat{\mathbf{x}}) + \frac{e^{i\kappa_P|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}}\mathbf{u}_S^\infty(\hat{\mathbf{x}}) + O\left(\frac{1}{|\mathbf{x}|^{3/2}}\right), \quad \text{as } |\mathbf{x}| \to \infty. \tag{3.34}$$

The functions $\mathbf{u}_P^\infty$ and $\mathbf{u}_S^\infty$ are respectively known as the longitudinal and transverse far-field patterns or the scattering amplitudes. Comparing (3.33) and (3.34) the following result is readily proved, which substantiates that the far-field scattering amplitudes admit natural expansions in terms of scattering coefficients.

Theorem 3.4. *Let* $\mathbf{U}$ *be the incident plane field given by* (3.11)*. Then the corresponding longitudinal and transverse scattering amplitudes can be written as*

$$\mathbf{u}_P^\infty[D, \lambda_0, \lambda_1, \mu_0, \mu_1, \rho_0, \rho_1, \omega](\hat{\mathbf{x}}) = \sum_{n\in\mathbb{Z}} \gamma_n^P A_n^{\infty,P}\mathbf{P}_n(\hat{\mathbf{x}}) \tag{3.35}$$

$$\mathbf{u}_S^\infty[D, \lambda_0, \lambda_1, \mu_0, \mu_1, \rho_0, \rho_1, \omega](\hat{\mathbf{x}}) = \sum_{n\in\mathbb{Z}} \gamma_n^S A_n^{\infty,S}\mathbf{S}_n(\hat{\mathbf{x}}), \tag{3.36}$$

where the coefficients γ_n^P *and* γ_n^S *are defined in* (3.27)*.*

3.4 Symmetry of Scattering Coefficients

We have the following results on the symmetry of ESC with respect to indices and wave-modes.

Lemma 3.5. *For all* $n, m \in \mathbb{Z}$ *and* $\alpha, \beta \in \{P, S\}$

$$W_{m,n}^{\alpha,\beta}[D, \lambda_0, \lambda_1, \mu_0, \mu_1, \rho_0, \rho_1, \omega] = \overline{W_{n,m}^{\beta,\alpha}[D, \lambda_0, \lambda_1, \mu_0, \mu_1, \rho_0, \rho_1, \omega]}. \tag{3.37}$$

Proof. Let us first fix some notation. For any $\mathbf{v}, \mathbf{w} \in H^{3/2}(D)^2$ and $a, b \in \mathbb{R}_+$, we define the quadratic form

$$\langle \mathbf{v}, \mathbf{w}\rangle_D^{a,b} := \int_D \left[a(\nabla\cdot\mathbf{v})(\nabla\cdot\mathbf{w}) + \frac{b}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^\top\right) : \left(\nabla\mathbf{w} + \nabla\mathbf{w}^\top\right)\right] d\mathbf{x}, \tag{3.38}$$

where : denotes the matrix contraction operator defined for two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ by $\mathbf{A} : \mathbf{B} := \sum_{i,j} a_{ij} b_{ij}$. It is easy to get from the definition of $\langle \cdot, \cdot \rangle_D^{a,b}$ that

$$\int_{\partial D} \mathbf{v} \cdot \frac{\partial \mathbf{w}}{\partial \nu} d\sigma(\mathbf{x}) = \int_D \mathbf{v} \cdot \mathcal{L}_{a,b}[\mathbf{w}] d\mathbf{x} + \langle \mathbf{v}, \mathbf{w} \rangle_D^{a,b}. \tag{3.39}$$

Note that if $\mathbf{w}$ is a solution of the Lamé equation $\mathcal{L}_{a,b}[\mathbf{w}] + c\omega^2 \mathbf{w} = \mathbf{0}$ then

$$\int_{\partial D} \mathbf{v} \cdot \frac{\partial \mathbf{w}}{\partial \nu} d\sigma(\mathbf{x}) = -c\omega^2 \int_D \mathbf{v} \cdot \mathbf{w} d\mathbf{x} + \langle \mathbf{v}, \mathbf{w} \rangle_D^{a,b} \tag{3.40}$$

and consequently from (3.39)

$$\int_{\partial D} \mathbf{v} \cdot \frac{\partial \mathbf{w}}{\partial \nu} d\sigma(\mathbf{x}) = \int_{\partial D} \frac{\partial \mathbf{v}}{\partial \nu} \cdot \mathbf{w} d\sigma(\mathbf{x}) - c\omega^2 \int_D \mathbf{v} \cdot \mathbf{w} d\mathbf{x} - \int_D \mathcal{L}_{a,b}[\mathbf{v}] \cdot \mathbf{w} d\mathbf{x}. \tag{3.41}$$

Moreover, if $\mathbf{v}$ solves $\mathcal{L}_{a,b}[\mathbf{v}] + c\omega^2 \mathbf{v} = \mathbf{0}$ then

$$\int_{\partial D} \mathbf{v} \cdot \frac{\partial \mathbf{w}}{\partial \nu} d\sigma(\mathbf{x}) = \int_{\partial D} \frac{\partial \mathbf{v}}{\partial \nu} \cdot \mathbf{w} d\sigma(\mathbf{x}). \tag{3.42}$$

We will also require the constants

$$\begin{aligned}
\eta_P :=& \frac{\mu_0}{\mu_1 - \mu_0}, \\
\widetilde{\eta}_P :=& \frac{\mu_1}{\mu_1 - \mu_0}, \\
\eta_S :=& \frac{\lambda_0 + \mu_0}{(\lambda_1 - \lambda_0) + (\mu_1 - \mu_0)}, \\
\widetilde{\eta}_S :=& \frac{\lambda_1 + \mu_1}{(\lambda_1 - \lambda_0) + (\mu_1 - \mu_0)}.
\end{aligned}$$

Let $(\boldsymbol{\varphi}_n^\alpha, \boldsymbol{\psi}_n^\alpha)$ and $(\boldsymbol{\varphi}_n^\beta, \boldsymbol{\psi}_m^\beta)$ be the solutions of (2.15) with $\mathbf{U} = \mathbf{J}^\alpha$ and $\mathbf{U} = \mathbf{J}^\beta$ respectively, that is,

$$\widetilde{\mathcal{S}}_D^\omega \boldsymbol{\varphi}_n^\alpha - \mathcal{S}_D^\omega \boldsymbol{\psi}_n^\alpha = \mathbf{J}_n^\alpha \big|_{\partial D} \tag{3.43}$$

$$\frac{\partial}{\partial \widetilde{\nu}} \widetilde{\mathcal{S}}_D^\omega \boldsymbol{\varphi}_n^\alpha \Big|_- - \frac{\partial}{\partial \nu} \mathcal{S}_D^\omega \boldsymbol{\psi}_n^\alpha \Big|_+ = \frac{\partial \mathbf{J}_n^\alpha}{\partial \nu} \Big|_{\partial D}, \tag{3.44}$$

and

$$\widetilde{\mathcal{S}}_D^\omega \boldsymbol{\varphi}_m^\beta - \mathcal{S}_D^\omega \boldsymbol{\psi}_m^\beta = \mathbf{J}_m^\beta \big|_{\partial D} \tag{3.45}$$

$$\frac{\partial}{\partial \widetilde{\nu}} \widetilde{\mathcal{S}}_D^\omega \boldsymbol{\varphi}_m^\beta \Big|_- - \frac{\partial}{\partial \nu} \mathcal{S}_D^\omega \boldsymbol{\psi}_m^\beta \Big|_+ = \frac{\partial \mathbf{J}_m^\beta}{\partial \nu} \Big|_{\partial D}. \tag{3.46}$$

Then, by making use of the jump conditions (2.11), $W_{m,n}^{\alpha,\beta}$ can be expressed as

$$W_{m,n}^{\alpha,\beta} = \int_{\partial D} \overline{\mathbf{J}_n^\alpha} \cdot \boldsymbol{\psi}_m^\beta d\sigma(\mathbf{x}) = \int_{\partial D} \overline{\mathbf{J}_n^\alpha} \cdot \left[\frac{\partial}{\partial \nu} \mathcal{S}_D^\omega [\boldsymbol{\psi}_m^\beta] \Big|_+ - \frac{\partial}{\partial \nu} \mathcal{S}_D^\omega [\boldsymbol{\psi}_m^\beta] \Big|_- \right] d\sigma(\mathbf{x}).$$

Further, by invoking (3.46) and subsequently using (3.41) and (3.42), one gets the expression

$$W_{m,n}^{\alpha,\beta} = -\int_{\partial D} \overline{\mathbf{J}_n^\alpha} \cdot \frac{\partial \mathbf{J}_m^\beta}{\partial \nu} d\sigma(\mathbf{x}) + \int_{\partial D} \overline{\mathbf{J}_n^\alpha} \cdot \left[\frac{\partial}{\partial \widetilde{\nu}} \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta]\Big|_- - \frac{\partial}{\partial \nu} \mathcal{S}_D^\omega[\boldsymbol{\psi}_m^\beta]\Big|_- \right] d\sigma(\mathbf{x})$$

$$= -\int_{\partial D} \overline{\mathbf{J}_n^\alpha} \cdot \frac{\partial \mathbf{J}_m^\beta}{\partial \nu} d\sigma(\mathbf{x}) + \int_{\partial D} \left[\frac{\partial \overline{\mathbf{J}_n^\alpha}}{\partial \widetilde{\nu}} \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] - \frac{\partial \overline{\mathbf{J}_n^\alpha}}{\partial \nu} \cdot \mathcal{S}_D^\omega[\boldsymbol{\psi}_m^\beta] \right] d\sigma(\mathbf{x})$$

$$- \rho_1 \omega^2 \int_D \overline{\mathbf{J}_n^\alpha} \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\mathbf{x} - \int_D \mathcal{L}_{\lambda_1,\mu_1}[\overline{\mathbf{J}_n^\alpha}] \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\mathbf{x}.$$

This, together with (3.45), leads to

$$W_{m,n}^{\alpha,\beta} = -\int_{\partial D} \overline{\mathbf{J}_n^\alpha} \cdot \frac{\partial \mathbf{J}_m^\beta}{\partial \nu} d\sigma(\mathbf{x}) + \int_{\partial D} \frac{\partial \overline{\mathbf{J}_n^\alpha}}{\partial \widetilde{\nu}} \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\sigma(\mathbf{x}) - \int_{\partial D} \frac{\partial \overline{\mathbf{J}_n^\alpha}}{\partial \nu} \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\sigma(\mathbf{x})$$
$$+ \int_{\partial D} \frac{\partial \overline{\mathbf{J}}_n^\alpha}{\partial \nu} \cdot \mathbf{J}_m^\beta \cdot d\sigma(\mathbf{x}) - \rho_1 \omega^2 \int_D \overline{\mathbf{J}_n^\alpha} \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\mathbf{x} - \int_D \mathcal{L}_{\lambda_1,\mu_1}[\overline{\mathbf{J}_n^\alpha}] \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\mathbf{x}.$$

It is easy to see that the first and the fourth terms cancel out each other thanks to (3.42). Therefore,

$$W_{m,n}^{\alpha,\beta} = \int_{\partial D} \left[\frac{\partial \overline{\mathbf{J}_n^\alpha}}{\partial \widetilde{\nu}} - \frac{\partial \overline{\mathbf{J}_n^\alpha}}{\partial \nu} \right] \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\sigma(\mathbf{x}) - \rho_1 \omega^2 \int_D \overline{\mathbf{J}_n^\alpha} \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\mathbf{x}$$
$$- \int_D \mathcal{L}_{\lambda_1,\mu_1}[\overline{\mathbf{J}_n^\alpha}] \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\mathbf{x}. \tag{3.47}$$

Now remark that $\nabla \cdot \mathbf{J}_n^S = 0 = \nabla \times \mathbf{J}_n^P$. Therefore, it is easy to verify by definition of the surface traction operator that

$$\frac{\partial \overline{\mathbf{J}_n^\alpha}}{\partial \widetilde{\nu}} - \frac{\partial \overline{\mathbf{J}_n^\alpha}}{\partial \nu} = \frac{1}{\eta_\alpha} \frac{\partial \overline{\mathbf{J}_n^\alpha}}{\partial \nu} = \frac{1}{\widetilde{\eta}_\alpha} \frac{\partial \overline{\mathbf{J}_n^\alpha}}{\partial \widetilde{\nu}}. \tag{3.48}$$

Thus, using right most quantity of (3.48) in (3.47) and subsequently invoking identity (3.41), one gets

$$\widetilde{\eta}_\alpha W_{m,n}^{\alpha,\beta} = \int_{\partial D} \frac{\partial \overline{\mathbf{J}_n^\alpha}}{\partial \widetilde{\nu}} \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\sigma(\mathbf{x}) - \widetilde{\eta}_\alpha \rho_1 \omega^2 \int_D \overline{\mathbf{J}_n^\alpha} \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\mathbf{x}$$
$$- \widetilde{\eta}_\alpha \int_D \mathcal{L}_{\lambda_1,\mu_1}[\overline{\mathbf{J}_n^\alpha}] \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\mathbf{x}$$
$$= \int_{\partial D} \overline{\mathbf{J}_n^\alpha} \cdot \frac{\partial}{\partial \widetilde{\nu}} \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta]\Big|_- d\sigma(\mathbf{x}) + (1-\widetilde{\eta}_\alpha) \rho_1 \omega^2 \int_D \overline{\mathbf{J}_n^\alpha} \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\mathbf{x}$$
$$+ (1-\widetilde{\eta}_\alpha) \int_D \mathcal{L}_{\lambda_1,\mu_1}[\overline{\mathbf{J}_n^\alpha}] \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\mathbf{x}.$$

This, together with (3.43) and (3.46), provides

$$\widetilde{\eta}_\alpha W_{m,n}^{\alpha,\beta} = \int_{\partial D} \overline{\widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_n^\alpha]} \cdot \frac{\partial}{\partial \widetilde{\nu}} \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta]\Big|_- d\sigma(\mathbf{x}) - \int_{\partial D} \overline{\mathcal{S}_D^\omega[\boldsymbol{\psi}_n^\alpha]} \cdot \frac{\partial}{\partial \nu} \mathcal{S}_D^\omega[\boldsymbol{\psi}_m^\beta]\Big|_+ d\sigma(\mathbf{x})$$
$$- \int_{\partial D} \overline{\mathcal{S}_D^\omega[\boldsymbol{\psi}_n^\alpha]} \cdot \frac{\partial \mathbf{J}_m^\beta}{\partial \nu} d\sigma(\mathbf{x}) + (1-\widetilde{\eta}_\alpha) \rho_1 \omega^2 \int_D \overline{\mathbf{J}_n^\alpha} \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\mathbf{x}$$
$$+ (1-\widetilde{\eta}_\alpha) \int_D \mathcal{L}_{\lambda_1,\mu_1}[\overline{\mathbf{J}_n^\alpha}] \cdot \widetilde{\mathcal{S}}_D^\omega[\boldsymbol{\varphi}_m^\beta] d\mathbf{x}. \tag{3.49}$$

Similarly, substituting the first relation of (3.48) back in (3.47) and invoking (3.45), one obtains

$$
\begin{aligned}
\eta_\alpha W^{\alpha,\beta}_{m,n} =& \int_{\partial D} \frac{\partial \overline{\mathbf{J}^\alpha_n}}{\partial \nu} \cdot \widetilde{\mathcal{S}}^\omega_D[\boldsymbol{\varphi}^\beta_m] d\sigma(\mathbf{x}) - \eta_\alpha \rho_1 \omega^2 \int_D \overline{\mathbf{J}^\alpha_n} \cdot \widetilde{\mathcal{S}}^\omega_D[\boldsymbol{\varphi}^\beta_m] d\mathbf{x} \\
&- \eta_\alpha \int_D \mathcal{L}_{\lambda_1,\mu_1}[\overline{\mathbf{J}^\alpha_n}] \cdot \widetilde{\mathcal{S}}^\omega_D[\boldsymbol{\varphi}^\beta_m] d\mathbf{x} \\
=& \int_{\partial D} \frac{\partial \overline{\mathbf{J}^\alpha_n}}{\partial \nu} \cdot \mathcal{S}^\omega_D[\boldsymbol{\psi}^\beta_m] d\sigma(\mathbf{x}) + \int_{\partial D} \frac{\partial \overline{\mathbf{J}^\alpha_n}}{\partial \nu} \cdot \mathbf{J}^\beta_m d\sigma(\mathbf{x}) \\
&- \eta_\alpha \rho_1 \omega^2 \int_D \overline{\mathbf{J}^\alpha_n} \cdot \widetilde{\mathcal{S}}^\omega_D[\boldsymbol{\varphi}^\beta_m] d\mathbf{x} - \eta_\alpha \int_D \mathcal{L}_{\lambda_1,\mu_1}[\overline{\mathbf{J}^\alpha_n}] \cdot \widetilde{\mathcal{S}}^\omega_D[\boldsymbol{\varphi}^\beta_m] d\mathbf{x}. \qquad (3.50)
\end{aligned}
$$

Finally, subtracting (3.50) from (3.49) and noting that $\widetilde{\eta}_\alpha - \eta_\alpha = 1$, one finds out that

$$
\begin{aligned}
W^{\alpha,\beta}_{m,n} =& \int_{\partial D} \overline{\widetilde{\mathcal{S}}^\omega[\boldsymbol{\varphi}^\alpha_n]} \cdot \frac{\partial}{\partial \widetilde{\nu}} \widetilde{\mathcal{S}}^\omega_D[\boldsymbol{\varphi}^\beta_m]\Big|_- d\sigma(\mathbf{x}) - \int_{\partial D} \overline{\mathcal{S}^\omega_D[\boldsymbol{\psi}^\alpha_n]} \cdot \frac{\partial}{\partial \nu} \mathcal{S}^\omega_D[\boldsymbol{\psi}^\beta_m]\Big|_+ d\sigma(\mathbf{x}) \\
&- \int_{\partial D} \overline{\mathcal{S}^\omega_D[\boldsymbol{\psi}^\alpha_n]} \cdot \frac{\partial \mathbf{J}^\beta_m}{\partial \nu} d\sigma(\mathbf{x}) - \int_{\partial D} \frac{\partial \overline{\mathbf{J}^\alpha_n}}{\partial \nu} \cdot \mathcal{S}^\omega_D[\boldsymbol{\psi}^\beta_m] d\sigma(\mathbf{x}) - \int_{\partial D} \frac{\partial \overline{\mathbf{J}^\alpha_n}}{\partial \nu} \cdot \mathbf{J}^\beta_m d\sigma(\mathbf{x}). \qquad (3.51)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
W^{\beta,\alpha}_{n,m} =& \int_{\partial D} \overline{\widetilde{\mathcal{S}}^\omega[\boldsymbol{\varphi}^\beta_m]} \cdot \frac{\partial}{\partial \widetilde{\nu}} \widetilde{\mathcal{S}}^\omega_D[\boldsymbol{\varphi}^\alpha_n]\Big|_- d\sigma(\mathbf{x}) - \int_{\partial D} \overline{\mathcal{S}^\omega_D[\boldsymbol{\psi}^\beta_m]} \cdot \frac{\partial}{\partial \nu} \mathcal{S}^\omega_D[\boldsymbol{\psi}^\alpha_n]\Big|_+ d\sigma(\mathbf{x}) \\
&- \int_{\partial D} \overline{\mathcal{S}^\omega_D[\boldsymbol{\psi}^\beta_m]} \cdot \frac{\partial \mathbf{J}^\alpha_n}{\partial \nu} d\sigma(\mathbf{x}) - \int_{\partial D} \frac{\partial \overline{\mathbf{J}^\beta_m}}{\partial \nu} \cdot \mathcal{S}^\omega_D[\boldsymbol{\psi}^\alpha_n] d\sigma(\mathbf{x}) - \int_{\partial D} \frac{\partial \overline{\mathbf{J}^\beta_m}}{\partial \nu} \cdot \mathbf{J}^\alpha_n d\sigma(\mathbf{x}) \\
=& \int_{\partial D} \frac{\partial}{\partial \widetilde{\nu}} \overline{\widetilde{\mathcal{S}}^\omega[\boldsymbol{\varphi}^\beta_m]}\Big|_- \cdot \widetilde{\mathcal{S}}^\omega_D[\boldsymbol{\varphi}^\alpha_n] d\sigma(\mathbf{x}) - \int_{\partial D} \frac{\partial}{\partial \nu} \overline{\mathcal{S}^\omega_D[\boldsymbol{\psi}^\beta_m]}\Big|_+ \cdot \mathcal{S}^\omega_D[\boldsymbol{\psi}^\alpha_n] d\sigma(\mathbf{x}) \\
&- \int_{\partial D} \overline{\mathcal{S}^\omega_D[\boldsymbol{\psi}^\beta_m]} \cdot \frac{\partial \mathbf{J}^\alpha_n}{\partial \nu} d\sigma(\mathbf{x}) - \int_{\partial D} \frac{\partial \overline{\mathbf{J}^\beta_m}}{\partial \nu} \cdot \mathcal{S}^\omega_D[\boldsymbol{\psi}^\alpha_n] d\sigma(\mathbf{x}) - \int_{\partial D} \overline{\mathbf{J}^\beta_m} \cdot \frac{\partial \mathbf{J}^\alpha_n}{\partial \nu} d\sigma(\mathbf{x}). \qquad (3.52)
\end{aligned}
$$

The proof is completed by taking complex conjugate of expression (3.52) and comparing the result with equation (3.51). □

Lemma 3.6. *For all $m, n \in \mathbb{Z}$ and $\alpha, \beta \in \{P, S\}$,*

$$
W^{\alpha,\beta}_{-m,-n}[D, \lambda_0, \lambda_1, \mu_0, \mu_1, \rho_0, \rho_1, \omega] = (-1)^{m+n} \overline{W^{\alpha,\beta}_{m,n}[D, \lambda_0, \lambda_1, \mu_0, \mu_1, \rho_0, \rho_1, \omega]}. \qquad (3.53)
$$

Proof. Let $\boldsymbol{\psi}^\beta_{-m}$ be the unique solution of the integral system (2.15) with $\mathbf{U}(\mathbf{x}) := \mathbf{J}^\beta_{-m}(\mathbf{x})$. Then by definition

$$
W^{\alpha,\beta}_{-m,-n}[D] := \int_{\partial D} \left[\overline{\mathbf{J}^\alpha_{-n}(\mathbf{y})} \cdot \boldsymbol{\psi}^\beta_{-m}(\mathbf{y}) \right] d\sigma(\mathbf{y}). \qquad (3.54)
$$

On the other hand, recall that the cylindrical Bessel functions possess the connection property [35, Formula 10.4.1],

$$
J_{-m}(\mathbf{x}) = (-1)^m J_m(\mathbf{x}). \qquad (3.55)
$$

Therefore, for all $m \in \mathbb{Z}$ and $\mathbf{x} \in \mathbb{R}^2$

$$
\mathbf{J}^\beta_{-m}(\mathbf{x}) := (-1)^m \overline{\mathbf{J}^\beta_m(\mathbf{x})}. \qquad (3.56)
$$

Consequently,

$$W^{\alpha,\beta}_{-m,-n}[D] =(-1)^n \int_{\partial D} \left[\mathbf{J}^{\alpha}_n(\mathbf{y}) \cdot \boldsymbol{\psi}^{\beta}_{-m}(\mathbf{y})\right] d\sigma(\mathbf{y}).$$

Using similar arguments as in Lemma 3.5 it can be proved after fairly easy manipulations that

$$\int_{\partial D} \left[\mathbf{J}^{\alpha}_n(\mathbf{y}) \cdot \boldsymbol{\psi}^{\beta}_{-m}(\mathbf{y})\right] d\sigma(\mathbf{y}) = \int_{\partial D} \left[\mathbf{J}^{\beta}_{-m}(\mathbf{y}) \cdot \boldsymbol{\psi}^{\alpha}_n(\mathbf{y})\right] d\sigma(\mathbf{y}). \tag{3.57}$$

Thus

$$\begin{aligned} W^{\alpha,\beta}_{-m,-n}[D] =&(-1)^n \int_{\partial D} \left[\mathbf{J}^{\beta}_{-m}(\mathbf{y}) \cdot \boldsymbol{\psi}^{\alpha}_n(\mathbf{y})\right] d\sigma(\mathbf{y}) \\ =&(-1)^{n+m} \int_{\partial D} \left[\overline{\mathbf{J}^{\beta}_{m}(\mathbf{y})} \cdot \boldsymbol{\psi}^{\alpha}_n(\mathbf{y})\right] d\sigma(\mathbf{y}) \\ =&(-1)^{m+n} W^{\beta,\alpha}_{n,m}[D], \end{aligned}$$

which gives the required result by virtue of Lemma 3.5. □

4 Reconstruction of Scattering Coefficients

In the previous section we have defined and discussed interesting properties of ESC of an inclusion. The aim of this section is to substantiate that these frequency dependent geometric quantities can be recovered from the measurements of the scattered field. Towards this end, we design a procedure based on least-squares minimization using MSR data. To simplify the matters we consider the full aperture case with the circular acquisition system.

4.1 MSR Data Acquisition

Let $\{\mathbf{x}_s\}_{s=1,\cdots,N_s}$ and $\{\mathbf{x}_r\}_{r=1,\cdots,N_r}$ be the sets of locations of the point sources and point receivers, and $\{\mathbf{d}_s, \mathbf{d}_s^{\perp}\}_{s=1,\cdots,N_s}$ and $\{\mathbf{d}_r, \mathbf{d}_r^{\perp}\}_{r=1,\cdots,N_r}$ (such that $\mathbf{d}_s \cdot \mathbf{d}_s^{\perp} = 0 = \mathbf{d}_r \cdot \mathbf{d}_r^{\perp}$) be the corresponding unit directions of incidences and receptions respectively for some $N_r, N_s \in \mathbb{N}$. Let the points$\{\mathbf{x}_s\}$ and $\{\mathbf{x}_r\}$ be uniformly distributed over the circle $\partial B_R(\mathbf{0})$ with radius R centered at origin such that $|\mathbf{x}_r| = R = |\mathbf{x}_s|$ and $\theta_r = \theta_{\mathbf{x}_r} = 2\pi r/N_r$ and $\theta_s = \theta_{\mathbf{x}_s} = 2\pi s/N_s$. We consider a regime in which R is large enough so that the terms of order $O(R^{-3/2})$ are negligible. For simplicity, we assume that D contains the origin which is reasonable since we are in sufficiently far field regime and the inclusion D can be envisioned as *sufficiently centered* in $B_R(\mathbf{0})$.

Let $\mathbf{F}_s$ and $\mathbf{G}_s$, for all $s = 1, \cdots, N_s$, be the pressure and shear type incident waves emitted from point $\mathbf{x}_s$ with direction of incidence $\mathbf{d}_s$, that is,

$$\mathbf{F}_s(\mathbf{x}) := \frac{1}{\rho_0 c_P^2} \mathbf{d}_s e^{i\kappa_P \mathbf{x}\cdot\mathbf{d}_s} \quad \text{and} \quad \mathbf{G}_s(\mathbf{x}) := \frac{1}{\rho_0 c_S^2} \mathbf{d}_s^{\perp} e^{i\kappa_S \mathbf{x}\cdot\mathbf{d}_s}. \tag{4.1}$$

Let $\mathbf{u}_{\mathbf{F}_s}(\mathbf{x})$ and $\mathbf{u}_{\mathbf{G}_s}(\mathbf{x})$ be the corresponding total fields. For all incident fields $\mathbf{F}_s$ and $\mathbf{G}_s$ we record the scattered fields at all points $\mathbf{x}_r$ in directions $\mathbf{d}_r$ and $\mathbf{d}_r^{\perp}$ so that we can form four

MSR matrices $\mathbf{A}^{\ell,\ell'} = \left(A_{sr}^{\ell,\ell'}\right)_{s=1,\cdots N_s, r=1,\cdots,N_r}$ for $\ell,\ell' \in \{\|,\perp\}$ with elements

$$A_{sr}^{\|,\|} = \Big([\mathbf{u}_{\mathbf{F}_s}(\mathbf{x}_r) - \mathbf{F}_s(\mathbf{x}_r)] \cdot \mathbf{d}_r \Big)_{sr} \tag{4.2}$$

$$A_{sr}^{\|,\perp} = \Big([\mathbf{u}_{\mathbf{F}_s}(\mathbf{x}_r) - \mathbf{F}_s(\mathbf{x}_r)] \cdot \mathbf{d}_r^{\perp} \Big)_{sr} \tag{4.3}$$

$$A_{sr}^{\perp,\|} = \Big([\mathbf{u}_{\mathbf{G}_s}(\mathbf{x}_r) - \mathbf{G}_s(\mathbf{x}_r)] \cdot \mathbf{d}_r \Big)_{sr} \tag{4.4}$$

$$A_{sr}^{\perp,\perp} = \Big([\mathbf{u}_{\mathbf{G}_s}(\mathbf{x}_r) - \mathbf{G}_s(\mathbf{x}_r)] \cdot \mathbf{d}_r^{\perp} \Big)_{sr} \tag{4.5}$$

at a given fixed frequency ω. Note that by virtue of expansions (3.11), (3.26) and (3.27), the elements of the MSR matrices admit the expansions

$$A_{sr}^{\|,\|} = \sum_{n,m\in\mathbb{Z}} d_m^P(s) \left(W_{m,n}^{P,P} \Big[\mathbf{H}_n^P(\mathbf{x}_r) \cdot \mathbf{d}_r \Big] + W_{m,n}^{S,P} \Big[\mathbf{H}_n^S(\mathbf{x}_r) \cdot \mathbf{d}_r \Big] \right) \tag{4.6}$$

$$A_{sr}^{\|,\perp} = \sum_{n,m\in\mathbb{Z}} d_m^P(s) \left(W_{m,n}^{P,P} \Big[\mathbf{H}_n^P(\mathbf{x}_r) \cdot \mathbf{d}_r^{\perp} \Big] + W_{m,n}^{S,P} \Big[\mathbf{H}_n^S(\mathbf{x}_r) \cdot \mathbf{d}_r^{\perp} \Big] \right) \tag{4.7}$$

$$A_{sr}^{\perp,\|} = \sum_{n,m\in\mathbb{Z}} d_m^S(s) \left(W_{m,n}^{P,S} \Big[\mathbf{H}_n^P(\mathbf{x}_r) \cdot \mathbf{d}_r \Big] + W_{m,n}^{S,S} \Big[\mathbf{H}_n^S(\mathbf{x}_r) \cdot \mathbf{d}_r \Big] \right) \tag{4.8}$$

$$A_{sr}^{\perp,\perp} = \sum_{n,m\in\mathbb{Z}} d_m^S(s) \left(W_{m,n}^{P,S} \Big[\mathbf{H}_n^P(\mathbf{x}_r) \cdot \mathbf{d}_r^{\perp} \Big] + W_{m,n}^{S,S} \Big[\mathbf{H}_n^S(\mathbf{x}_r) \cdot \mathbf{d}_r^{\perp} \Big] \right), \tag{4.9}$$

where $d_m^P(s) = d_m^P(\mathbf{F}_s)$ and $d_m^S(s) = d_m^S(\mathbf{G}_s)$ are the coefficients given by (3.28) corresponding to incident fields $\mathbf{F}_s$ and $\mathbf{G}_s$ respectively. Here the parameter s in the argument of d_m^{β} reflects its connection with s-th incident field.

Let us now introduce a cut-off parameter K such that the terms with $|n| > K$ or $|m| > K$ are truncated in the expansions (4.6)-(4.9) and let $\mathbf{E}^{\ell,\ell'} = (E_{sr}^{\ell,\ell'}) \in \mathbb{C}^{N_s\times N_r}$ for $\ell,\ell' \in \{\|,\perp\}$ be the corresponding matrices of truncation errors thus induced. Let us also introduce the matrices $\mathbf{W}^{\alpha,\beta} \in \mathbb{C}^{(2K+1)\times(2K+1)}$, $\mathbf{X}^{\alpha} \in \mathbb{C}^{N_s\times(2K+1)}$, $\mathbf{Y}_{\|}^{\alpha} \in \mathbb{C}^{N_r\times(2K+1)}$ and $\mathbf{Y}_{\perp}^{\alpha} \in \mathbb{C}^{N_r\times(2K+1)}$ by

$$\left(\mathbf{W}^{\alpha,\beta}\right)_{mn} := W_{m,n}^{\alpha,\beta}, \qquad \left(\mathbf{X}^{\beta}\right)_{sm} := d_m^P(s),$$
$$\left(\mathbf{Y}_{\|}^{\alpha}\right)_{rn} := \overline{\mathbf{H}_n^{\alpha}(\mathbf{x}_r)} \cdot \mathbf{d}_r, \qquad (\mathbf{Y}_{\perp}^{\alpha})_{rn} := \overline{\mathbf{H}_n^{\alpha}(\mathbf{x}_r)} \cdot \mathbf{d}_r^{\perp}. \tag{4.10}$$

and the block matrices

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^{\parallel,\parallel} & \mathbf{A}^{\parallel,\perp} \\ \mathbf{A}^{\perp,\parallel} & \mathbf{A}^{\perp,\perp} \end{pmatrix} \in \mathbb{C}^{2N_s \times 2N_r}$$

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}^{P,P} & \mathbf{W}^{S,P} \\ \mathbf{W}^{P,S} & \mathbf{W}^{S,S} \end{pmatrix} \in \mathbb{C}^{(4K+2)\times(4K+2)}$$

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}^{PP} & \mathbf{E}^{SP} \\ \mathbf{E}^{PS} & \mathbf{E}^{SS} \end{pmatrix} \in \mathbb{C}^{2N_s \times 2N_r}$$

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}^{P} & \mathbf{O}_{2K+1} \\ \mathbf{O}_{2K+1} & \mathbf{X}^{S} \end{pmatrix} \in \mathbb{C}^{2N_s \times (4K+2)}$$

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}^{P}_{\parallel} & \mathbf{Y}^{S}_{\parallel} \\ \mathbf{Y}^{P}_{\perp} & \mathbf{Y}^{S}_{\perp} \end{pmatrix} \in \mathbb{C}^{2N_r \times (4K+2)}, \tag{4.11}$$

where $\mathbf{O}_{2K+1} \in \mathbb{R}^{N_s \times (2K+1)}$ is the zero matrix. It can be seen after fairly easy manipulations that the global MSR matrix can be expressed as

$$\mathbf{A} = \mathbf{X}\mathbf{W}\mathbf{Y}^* + \mathbf{E}, \tag{4.12}$$

where $*$ reflects the Hermitian transpose of a matrix, i.e. $\mathbf{A}^* = \overline{\mathbf{A}}^\top$.

The following result is readily proved thanks to Lemma 3.5.

Lemma 4.1. *The global block matrix* $\mathbf{W}$ *is Hermitian, that is,* $\mathbf{W} = \mathbf{W}^*$.

4.2 Least-squares Minimization Algorithm

Let us define the linear transformation $\mathbf{L} : \mathbb{C}^{(4K+2)\times(4K+2)} \to \mathbb{C}^{2N_s \times 2N_r}$ by

$$\mathbf{L}(\mathbf{M}) := \mathbf{X}\mathbf{M}\mathbf{Y}^* \tag{4.13}$$

and let $\mathbf{N}_{\text{noise}} \in \mathbb{C}^{2N_s \times 2N_r}$ denote the measurement noise. For simplicity, we assume that each entry $(\mathbf{N}_{\text{noise}})_{sr}$ is an independent and identically distributed complex random noise with mean zero and variance σ^2_{noise} such that

$$\mathbf{N}_{\text{noise}} = \sigma_{\text{noise}}\mathbf{N}_0 \quad \text{with} \quad (\mathbf{N}_0)_{sr} \sim \mathcal{N}(0,1). \tag{4.14}$$

In this subsection we consider the noisy measurements

$$\mathbf{A} = \mathbf{X}\mathbf{W}\mathbf{Y}^* + \mathbf{E} + \mathbf{N}_{\text{noise}} = \mathbf{L}(\mathbf{W}) + \mathbf{E} + \mathbf{N}_{\text{noise}} \tag{4.15}$$

and design a procedure to retrieve the solution $\mathbf{W}$. Let us reconstruct a least-squares minimization solution for the linear system (4.15) in $\ker \mathbf{L}^\perp$ by

$$\widehat{\mathbf{W}} := \underset{\mathbf{M}\in\ker\mathbf{L}^\perp}{\arg\min} \|\mathbf{L}(\mathbf{M}) - \mathbf{A}\|_F, \tag{4.16}$$

where $\|\cdot\|_F$ denotes the Frobenius norm of matrices and $\ker \mathbf{L}$ denotes the kernel of the linear operator $\mathbf{L}$. Note that if the cut-off parameter K is such that $(2K+1) < N_r, N_s$ and both matrices $\mathbf{X}$ and $\mathbf{Y}$ are full rank then $\mathbf{L}$ is rank preserving and $\ker \mathbf{L}$ is trivial. Consequently, the admissible set for the least-squares minimization turns out to be $\mathbb{R}^{(4K+2)\times(4K+2)}$ and $\widehat{\mathbf{W}}$ can be explicitly calculated in the absence of measurement noise. In this case $\mathbf{X}^\alpha$ is a Fourier matrix by virtue of (3.6), (3.7) and (3.13) and

$$(\mathbf{X}^\alpha)^* \mathbf{X}^\alpha = \frac{N_s}{|b_\alpha|^2}\mathbf{I}_{2K+1} \quad \text{with} \quad b_\alpha = -\frac{1}{4\rho_0^2 c_\beta^4 \kappa_\beta}, \tag{4.17}$$

where $\mathbf{I}_{2K+1} \in \mathbb{R}^{(2K+1)\times(2K+1)}$ is the identity matrix. Consequently,

$$\mathbf{X}^*\mathbf{X} = \begin{pmatrix} \left(\mathbf{X}^P\right)^* \mathbf{X}^P & \mathbf{O}_{2K+1} \\ \mathbf{O}_{2K+1} & \left(\mathbf{X}^S\right)^* \mathbf{X}^S \end{pmatrix} = N_s \mathbf{Z}_{\mathbf{X}} \tag{4.18}$$

with

$$\mathbf{Z}_{\mathbf{X}} := \begin{pmatrix} |b_P|^{-2}\mathbf{I}_{2K+1} & \mathbf{O}_{2K+1} \\ \mathbf{O}_{2K+1} & |b_S|^{-2}\mathbf{I}_{2K+1} \end{pmatrix}. \tag{4.19}$$

Note also that

$$\left(\mathbf{Y}_\parallel^\alpha\right)^* \mathbf{Y}_\parallel^\beta = N_r \mathbf{C}^{\alpha,\beta} \quad \text{and} \quad \left(\mathbf{Y}_\perp^\alpha\right)^* \mathbf{Y}_\perp^\beta = N_r \mathbf{D}^{\alpha,\beta}, \tag{4.20}$$

where $\mathbf{C}^{\alpha,\beta}, \mathbf{D}^{\alpha,\beta} \in \mathbb{R}^{(2K+1)\times(2K+1)}$ are diagonal matrices

$$\mathbf{C}^{\alpha,\beta} := \begin{pmatrix} g_{-K}^\alpha \overline{g_{-K}^\beta} & & & \\ & g_{1-K}^\alpha \overline{g_{1-K}^\beta} & & \\ & & \ddots & \\ & & & g_K^\alpha \overline{g_K^\beta} \end{pmatrix}, \tag{4.21}$$

$$\mathbf{D}^{\alpha,\beta} := \begin{pmatrix} h_{-K}^\alpha \overline{h_{-K}^\beta} & & & \\ & h_{1-K}^\alpha \overline{h_{1-K}^\beta} & & \\ & & \ddots & \\ & & & h_K^\alpha \overline{h_K^\beta} \end{pmatrix}, \tag{4.22}$$

with

$$g_m^P := \kappa_P \left(H_m^{(1)}(\kappa_P R)\right)' \quad \text{and} \quad g_m^S := \frac{im}{R} H_m^{(1)}(\kappa_S R), \tag{4.23}$$

$$h_m^P := \frac{im}{R} H_m^{(1)}(\kappa_P R) \quad \text{and} \quad h_m^S := -\kappa_S \left(H_m^{(1)}(\kappa_S R)\right)'. \tag{4.24}$$

Therefore,

$$\begin{aligned} (\mathbf{Y})^* \mathbf{Y} &= \begin{pmatrix} \left(\mathbf{Y}_\parallel^P\right)^* \mathbf{Y}_\parallel^P + \left(\mathbf{Y}_\perp^P\right)^* \mathbf{Y}_\perp^P & \left(\mathbf{Y}_\parallel^P\right)^* \mathbf{Y}_\parallel^S + \left(\mathbf{Y}_\perp^P\right)^* \mathbf{Y}_\perp^S \\ \left(\mathbf{Y}_\parallel^S\right)^* \mathbf{Y}_\parallel^P + \left(\mathbf{Y}_\perp^S\right)^* \mathbf{Y}_\perp^P & \left(\mathbf{Y}_\parallel^S\right)^* \mathbf{Y}_\parallel^S + \left(\mathbf{Y}_\perp^S\right)^* \mathbf{Y}_\perp^S \end{pmatrix} \\ &= N_r \begin{pmatrix} \mathbf{C}^{P,P} + \mathbf{D}^{P,P} & \mathbf{C}^{P,S} + \mathbf{D}^{P,S} \\ \mathbf{C}^{S,P} + \mathbf{D}^{S,P} & \mathbf{C}^{S,S} + \mathbf{D}^{S,S} \end{pmatrix}. \end{aligned} \tag{4.25}$$

It can be easily proved that $\mathbf{Y}^*\mathbf{Y}$ becomes diagonal when the radius R of the imaging domain $\partial B_R(\mathbf{0})$ is sufficiently large. Precisely, the following result holds.

Lemma 4.2. *For the radius R of the ball $B_R(\mathbf{0})$ approaching to infinity the matrix $\mathbf{Y}^*\mathbf{Y}$ admits a decomposition*

$$\mathbf{Y}^*\mathbf{Y} = N_r\mathbf{Z}_{\mathbf{Y}} + \mathbf{Q} \quad \textit{with} \quad \mathbf{Z}_{\mathbf{Y}} := \begin{pmatrix} \mathbf{C}^{P,P} & \mathbf{O}_{2K+1} \\ \mathbf{O}_{2K+1} & \mathbf{D}^{S,S} \end{pmatrix}, \tag{4.26}$$

where $\mathbf{Q} = (q_{\ell\ell'})_{\ell,\ell'=1\cdots,4K+2}$ is such that $|q_{\ell\ell'}| \leq CR^{-2}$ for some constant $C \in \mathbb{R}_+$ independent of R.

Proof. In the sequel C denotes a generic constant and varies at each step. Note that the matrix $\mathbf{Y}^*\mathbf{Y}$ can be decomposed as

$$\mathbf{Y}^*\mathbf{Y} := N_r \begin{pmatrix} \mathbf{C}^{P,P} & \mathbf{O}_{2K+1} \\ \mathbf{O}_{2K+1} & \mathbf{D}^{S,S} \end{pmatrix} + N_r \begin{pmatrix} \mathbf{D}^{P,P} & \mathbf{C}^{P,S} + \mathbf{D}^{P,S} \\ \mathbf{C}^{S,P} + \mathbf{D}^{S,P} & \mathbf{C}^{S,S} \end{pmatrix}. \tag{4.27}$$

Recall that $\mathbf{C}^{\alpha,\beta}$ and $\mathbf{D}^{\alpha,\beta}$ are diagonal matrices and in particular

$$\left(\mathbf{C}^{S,S}\right)_{mm} = \frac{m^2}{R^2}|H_m^{(1)}(\kappa_S R)|^2 \quad \text{and} \quad \left(\mathbf{D}^{P,P}\right)_{mm} = \frac{m^2}{R^2}|H_m^{(1)}(\kappa_P R)|^2.$$

Thus, by virtue of the decay property (3.29) of $H_m^{(1)}$, as $R \to \infty$ we have

$$|\left(\mathbf{C}^{S,S}\right)_{mn}| \leq \frac{C}{R^3} \quad \text{and} \quad |\left(\mathbf{D}^{P,P}\right)_{mn}| \leq \frac{C}{R^3}.$$

Similarly, the decay properties (3.29)-(3.30) furnish

$$|\left(\mathbf{C}^{P,S}\right)_{mn}| \leq \frac{C}{R^2} \quad \text{and} \quad |\left(\mathbf{C}^{S,P}\right)_{mn}| \leq \frac{C}{R^2},$$

$$|\left(\mathbf{D}^{P,S}\right)_{mn}| \leq \frac{C}{R^2} \quad \text{and} \quad |\left(\mathbf{D}^{S,P}\right)_{mn}| \leq \frac{C}{R^2}$$

as $R \to \infty$. This shows the decay of the elements of second matrix on right hand side (RHS) of (4.27), which leads to the required form of $\mathbf{Y}^*\mathbf{Y}$ for $R \to \infty$. □

An important consequence of Lamma 4.2 and the orthogonality relation (4.18) is the following result substantiating that the linear operator $\mathbf{L}$ possesses a left pseudo-inverse when $R \to \infty$.

Theorem 4.3. *Let $(2K+1) \leq N_r, N_s$ and matrices $\mathbf{X}$ and $\mathbf{Y}$ be full-rank. Then, the linear operator $\mathbf{L} : \mathbb{C}^{(4K+2)\times(4K+2)} \to \mathbb{C}^{2N_s\times 2N_r}$ possesses a left pseudo-inverse*

$$\mathbf{L}^\dagger(\mathbf{A}) := \frac{1}{N_s N_r}\mathbf{Z}_{\mathbf{X}}^{-1}\mathbf{X}^*\mathbf{A}\mathbf{Y}\mathbf{Z}_{\mathbf{Y}}^{-1} \tag{4.28}$$

when $R \to \infty$.

Proof. Since $(2K+1) \le N_s, N_r$, $\mathbf{X}$ and $\mathbf{Y}$ are full-rank, and $R \to \infty$, it is easy to see that both $\mathbf{X}$ and $\mathbf{Y}$ possess left pseudo-inverses, denoted by $\mathbf{X}^\dagger$ and $\mathbf{Y}^\dagger$ respectively, thanks to the orthogonality property (4.18) and Lemma 4.2. Precisely,

$$\mathbf{X}^\dagger := \left(\mathbf{X}^*\mathbf{X}\right)^{-1}\mathbf{X}^* = \frac{1}{N_s}\mathbf{Z}_{\mathbf{X}}^{-1}\mathbf{X}^* \quad \text{and} \quad \mathbf{Y}^\dagger := \left(\mathbf{Y}^*\mathbf{Y}\right)^{-1}\mathbf{Y}^* = \frac{1}{N_r}\mathbf{Z}_{\mathbf{Y}}^{-1}\mathbf{Y}^*$$

as $R \to \infty$. Consequently, we have

$$\begin{aligned}
\frac{1}{N_s N_r}\mathbf{Z}_{\mathbf{X}}^{-1}\mathbf{X}^*\mathbf{A}\mathbf{Y}\mathbf{Z}_{\mathbf{Y}}^{-1} =& \frac{1}{N_r}\left(\mathbf{X}^*\mathbf{X}\right)^{-1}\mathbf{X}^*\left(\mathbf{X}\widehat{\mathbf{W}}\mathbf{Y}^*\right)\mathbf{Y}\mathbf{Z}_{\mathbf{Y}}^{-1} \\
=& \frac{1}{N_r}\widehat{\mathbf{W}}\left(\mathbf{Y}^*\mathbf{Y}\right)\mathbf{Z}_{\mathbf{Y}}^{-1} \\
=& \widehat{\mathbf{W}}.
\end{aligned}$$

This completes the proof. □

4.3 Stability Analysis

In this section we perform a stability analysis for the linear operator $\mathbf{L}$. We substantiate that the operator $\mathbf{L}$ is ill-conditioned for $K \to +\infty$. It simply means that only a certain number of lower order scattering coefficients can be recovered stably which in turn contain only lower order information of the shape oscillations of boundary ∂D. The limit on the information about the shape and morphology of the inclusion D that can be obtained stably is determined by the maximum resolving order and the stability estimate for the operator $\mathbf{L}$ thereby defining the resolution limit of the imaging paradigm. Towards this end, the following result characterizes the singular values and the singular vectors of the operator $\mathbf{L}$.

Theorem 4.4. *Let $N_s, N_r \ge 2K+1$ and $R \to \infty$. Then the right singular vectors of $\mathbf{L}$ are coincident with the canonical basis of $\mathbb{R}^{(4K+2)\times(4K+2)}$ and the $(p,q)-$th singular value of the operator $\mathbf{L}$ is given by*

$$\sigma_{pq} := 4\rho_0^2\omega^2\sqrt{N_s N_r}\begin{cases} c_P^2\left|\left(H^{(1)}_{q-1-K}(\kappa_P R)\right)'\right|, & 1 \le p, q \le 2K+1, \\ c_S^2\left|\left(H^{(1)}_{q-2-3K}(\kappa_S R)\right)'\right|, & 2K+2 \le p, q \le 4K+2. \end{cases} \tag{4.29}$$

Proof. Let us define the inner product of two complex matrices $\mathbf{N}$ and $\mathbf{M}$ by

$$\langle \mathbf{N}, \mathbf{M}\rangle := \sum_{\ell,\ell'}\left(\mathbf{N}^*\right)_{\ell\ell'}\left(\mathbf{M}\right)_{\ell\ell'}.$$

Let $\mathbf{V}_{pq} \in \mathbb{R}^{(4K+2)\times(4K+2)}$ for each $p, q = 1, 2, \cdots, 4K+2$ be such that

$$\left(\mathbf{V}_{pq}\right)_{\ell\ell'} = \delta_{p\ell}\delta_{q\ell'}, \quad \forall\, \ell, \ell' = 1, 2, \cdots, 4K+2.$$

It is easy to verify that for $R \to \infty$, thanks to diagonality result (4.18) and Lemma 4.2,

$$\begin{aligned}
\langle \mathbf{L}\left(\mathbf{V}_{pq}\right), \mathbf{L}\left(\mathbf{V}_{p'q'}\right)\rangle =& \langle \mathbf{X}\mathbf{V}_{pq}\mathbf{Y}^*, \mathbf{X}\mathbf{V}_{p'q'}\mathbf{Y}^*\rangle \\
=& N_s N_r \langle \mathbf{V}_{pq}, \mathbf{Z}_{\mathbf{X}}\mathbf{V}_{p'q'}\mathbf{Z}_{\mathbf{Y}}\rangle \\
=& \delta_{pp'}\delta_{qq'}N_s N_r |f_q|^2,
\end{aligned} \tag{4.30}$$

where

$$
|f_q| := \begin{cases} \dfrac{|g^P_{q-1-K}|}{|b_P|}, & 1 \le p,q \le 2K+1, \\ \dfrac{|h^S_{q-2-3K}|}{|b_S|}, & 2K+2 \le p,q \le 4K+2. \end{cases} \tag{4.31}
$$

On substituting the values of g^P_{q-1-K}, h^S_{q-2-3K} and b_α from (4.23), (4.24) and (4.17), one arrives at

$$
|f_q| := 4\rho_0^2\omega^2 \begin{cases} c_P^2 \left| \left(H^{(1)}_{q-1-K}(\kappa_P R) \right)' \right|, & 1 \le p,q \le 2K+1, \\ c_S^2 \left| \left(H^{(1)}_{q-2-3K}(\kappa_S R) \right)' \right|, & 2K+2 \le p,q \le 4K+2. \end{cases} \tag{4.32}
$$

This shows that the canonical basis $\left\{ \mathbf{V}_{pq} \right\}_{p,q=1,\cdots,4K+2}$ forms the set of right singular vectors of $\mathbf{L}$ and the (p,q)-th singular value of the operator $\mathbf{L}$ is thus rendered by $\|\mathbf{L}\left(\mathbf{V}_{pq}\right)\|_F$ and is given by (4.29). Moreover, the left singular vectors of $\mathbf{L}$ are furnished by the relation $\widetilde{\mathbf{V}}_{pq} := \mathbf{L}\left(\mathbf{V}_{pq}\right)/\sigma_{pq}$. □

It should be observed that the quantities $|g^P_{2K+1}|$ and $|h^S_{2K+1}|$ diverge when $K \to \infty$. Consequently, the operator $\mathbf{L}$ is unbounded. Indeed, we have the following estimate for the condition number of $\mathbf{L}$ thanks to Theorem 4.4.

Corollary 4.5. *Under the assumptions of Theorem 4.4,*

$$
\operatorname{cond}(\mathbf{L}) \lesssim \left(C_R^P K \right)^{(K+1)} \quad \text{as} \quad K \to +\infty, \tag{4.33}
$$

where $C_R^\alpha := 2/e\kappa_\alpha R$.

Proof. Let $\sigma_{\max}$ and $\sigma_{\min}$ be the largest and the smallest singular values of the operator $\mathbf{L}$. Recall the asymptotic behavior of the Bessel functions of first and second kind

$$
J_m(t) \simeq \sqrt{\frac{1}{2\pi|m|}} \left(\frac{et}{2|m|} \right)^{|m|} \quad \text{and} \quad Y_m(t) \simeq -\sqrt{\frac{2}{\pi|m|}} \left(\frac{et}{2|m|} \right)^{-|m|} \tag{4.34}
$$

with respect to the order $|m| \to \infty$ at a fixed argument t [35, Formulae 10.19.1 and 10.19.2]. Consequently, an easy commutation shows that

$$
\left| H^{(1)}_m(\kappa_\alpha R) \right| \lesssim \left(C_R^\alpha |m| \right)^{|m|} + \left(C_R^\alpha |m| \right)^{-|m|} \quad \text{as } |m| \to \infty. \tag{4.35}
$$

Moreover, invoking the recurrence relation [35, Formula 10.6.2]

$$
\left(H^{(1)}_m(t) \right)' = H^{(1)}_{m-1}(t) - \frac{m}{t} H^{(1)}_m(t) \tag{4.36}
$$

it is easy to get that

$$
\begin{aligned}
\left| \left(H^{(1)}_m(\kappa_\alpha R) \right)' \right| &\lesssim \left(C_R^\alpha (m-1) \right)^{(m-1)} + \left(C_R^\alpha (m-1) \right)^{-(m-1)} + \frac{e}{2} C_R^\alpha m \left(\left(C_R^\alpha m \right)^m + \left(C_R^\alpha m \right)^{-m} \right) \\
&\lesssim \left(C_R^\alpha m \right)^{m+1}, \qquad (4.37)
\end{aligned}
$$

when $m \to +\infty$. Consequently,

$$\sigma_{(2K+1)(2K+1)} \lesssim \left(C_R^P K\right)^{K+1} \quad \text{and} \quad \sigma_{(4K+2)(4K+2)} \lesssim \left(C_R^S K\right)^{K+1}. \tag{4.38}$$

Finally, we note that the relation $\sigma_{\max} \simeq \sigma_{(2K+1)(2K+1)}$ holds when K is large enough, which follows from the fact that $C_R^P > C_R^S$ (this is due to the inequality $c_P > c_S$, since $\mu_0, \lambda_0 > 0$). Moreover, the smallest singular value $\sigma_{\min}$ is bounded. Therefore,

$$\text{cond}(\mathbf{L}) = \frac{\sigma_{\max}}{\sigma_{\min}} \lesssim \left(C_R^P K\right)^{K+1}. \tag{4.39}$$

□

4.4 Error Analysis

Let us now analyze the error committed by truncating the infinite series in the MSR data. But before further discussion we recall the following result from [13, Appendix A].

Lemma 4.6. *For $c > 0$ and $N \in \mathbb{N}$ such that $N > c/e$*

$$\sum_{n>N} \left(\frac{c}{n}\right)^n \le \left(\frac{c}{N}\right)^N \left(\frac{1}{1+\ln(N/c)}\right). \tag{4.40}$$

The following is the main result of this section.

Theorem 4.7. *Let C_R^α and $C_{\alpha,\beta} > 1$ be the constants defined in Corollary 4.5 and Lemma 3.3 respectively. Let the radius R of the measurement domain $B_R(\boldsymbol{0})$ be such that $h := \max_{\alpha,\beta}\{2C_{\alpha,\beta}^2 C_R^\alpha\} < 1$. Then there exists a sufficiently large truncation order K satisfying $K > \max\limits_{\alpha,\beta}\{C_{\alpha,\beta}/(C_R^\alpha e)\}$ such that*

$$|E_{sr}^{\alpha,\beta}| = O(h^{K-1}). \tag{4.41}$$

Proof. We prove the result for the truncation error $E_{sr}^{P,P}$ only. The rest of the estimates can be obtained following the same procedure. We first split the summations into three different contributions as

$$\begin{aligned} E_{sr}^{P,P} = &\left(\sum_{\substack{|m|\le K\\|n|>K}} + \sum_{\substack{|m|>K\\|n|\le K}} + \sum_{\substack{|m|>K\\|n|>K}}\right) \alpha_m^P(s)\left(W_{m,n}^{P,P}\Big[\mathbf{H}_n^P(\mathbf{x}_r)\cdot\mathbf{d}_r\Big] + W_{m,n}^{S,P}\Big[\mathbf{H}_n^S(\mathbf{x}_r)\cdot\mathbf{d}_r\Big]\right) \\ = &I_1 + I_2 + I_3. \end{aligned} \tag{4.42}$$

Then, thanks to Lemma 3.3 and invoking the definitions (3.28) and (3.6)-(3.7) of d_m^P and $\mathbf{H}_n^\alpha$ respectively, we have

$$\begin{aligned} |I_1| \le &\frac{1}{4\rho_0^2 c_P^4 \kappa_P}\Bigg(\sum_{|m|\le K} \frac{C_{P,P}^{|m|-1}}{|m|^{|m|-1}} \sum_{|n|>K} \frac{C_{P,P}^{|n|-1}}{|n|^{|n|-1}} \left|\kappa_P \left(H_n^{(1)}(\kappa_P R)\right)'\right| \\ &+ \sum_{|m|\le K} \frac{C_{S,P}^{|m|-1}}{|m|^{|m|-1}} \sum_{|n|>K} \frac{C_{S,P}^{|n|-1}}{|n|^{|n|-1}} \frac{|n|}{R} \left|H_n^{(1)}(\kappa_S R)\right|\Bigg). \end{aligned} \tag{4.43}$$

We recall again the estimates

$$\left|H_n^{(1)}(\kappa_\alpha R)\right| \lesssim (C_R^\alpha |n|)^{|n|} + (C_R^\alpha |n|)^{-|n|},$$
$$\left|\left(H_n^{(1)}(\kappa_\alpha R)\right)'\right| \lesssim \frac{e}{2}(C_R^\alpha |n|)^{|n|+1} + (C_R^\alpha(|n|-1))^{|n|-1} + \frac{e}{2}(C_R^\alpha |n|)^{1-|n|} + (C_R^\alpha(|n|-1))^{1-|n|},$$

as $|n| \to \infty$ and note that, up to some factors independent of K,

$$\sum_{|m|\le K} \frac{C_{P,P}^{|m|-1}}{|m|^{|m|-1}} \lesssim C_{P,P}^{K-1} \quad \text{and} \quad \sum_{|m|\le K} \frac{C_{S,P}^{|m|-1}}{|m|^{|m|-1}} \lesssim C_{S,P}^{K-1}.$$

Therefore

$$\begin{aligned}|I_1| \lesssim & \frac{C_{P,P}^{K-1}}{4\rho_0^2 c_P^4} \sum_{|n|>K} \left[\frac{e}{2}\left(C_R^P |n|\right)^2 \left(C_{P,P} C_R^P\right)^{|n|-1} + \left(1 - \frac{1}{|n|}\right)^{|n|-1} \left(C_{P,P} C_R^P\right)^{|n|-1} \right. \\ & \left. + \frac{e}{2}\left(\frac{C_{P,P}/C_R^P}{|n|^2}\right)^{|n|-1} + \left(\frac{C_{P,P}/C_R^P}{|n|(|n|-1)}\right)^{|n|-1}\right] \\ & + \frac{e\kappa_S C_{S,P}^{K-1}}{8\rho_0^2 c_P^4 \kappa_P} \sum_{|n|>K} \left[\left(C_R^S |n|\right)^2 \left(C_{S,P} C_R^S\right)^{|n|-1} + \left(\frac{C_{S,P}/C_R^S}{|n|^2}\right)^{|n|-1}\right]. \end{aligned} \tag{4.44}$$

Thanks to Lemma 4.6 the third, fourth and sixth terms on RHS of (4.44) are negligible for all $K > \max_{\alpha,\beta}\{C_{\alpha,\beta}/C_R^\alpha e\}$. Moreover, it can be easily verified that

$$\frac{n^2}{2^{n+2}} \le 1 \quad \text{and} \quad \frac{1}{2^{n-1}}\left(1 - \frac{1}{n}\right)^{n-1} \le 1, \quad \forall\, n \in \mathbb{N} \tag{4.45}$$

and

$$(C_R^\alpha)^2 < C_R^\alpha < C_{\alpha,\beta} C_R^\alpha < C_{\alpha,\beta}^2 C_R^\alpha < \max_{\alpha,\beta}\{C_{\alpha,\beta}^2 C_R^\alpha\} < \frac{1}{2}. \tag{4.46}$$

Therefore we have

$$\begin{aligned}\frac{eC_{P,P}^{K-1}}{8} \sum_{|n|>K} \left(C_R^P |n|\right)^2 \left(C_{P,P} C_R^P\right)^{|n|-1} &\lesssim eC_{P,P}^{K-1} \sum_{|n|>K} \frac{|n|^2}{2^{|n|+2}} \left(2C_{P,P} C_R^P\right)^{|n|-1} \\ &\lesssim eC_{P,P}^{K-1} \left(2C_{P,P} C_R^P\right)^{K-1} \\ &\le eh^{K-1}, \end{aligned} \tag{4.47}$$

$$\begin{aligned}&\frac{C_{P,P}^{K-1}}{4} \sum_{|n|>K} \left(1 - \frac{1}{|n|}\right)^{|n|-1} \left(C_{P,P} C_R^P\right)^{|n|-1} \\ &\lesssim C_{P,P}^{K-1} \sum_{|n|>K} \frac{1}{2^{|n|-1}} \left(1 - \frac{1}{|n|}\right)^{|n|-1} \left(2C_{P,P} C_R^P\right)^{|n|-1} \\ &\lesssim C_{P,P}^{K-1} \left(2C_{P,P} C_R^P\right)^{K-1} \\ &\le h^{K-1}, \end{aligned} \tag{4.48}$$

and

$$\frac{e\kappa_S C_{S,P}^{K-1}}{8\kappa_P}\sum_{|n|>K}\left(C_R^S|n|\right)^2\left(C_{S,P}C_R^S\right)^{|n|-1} \lesssim \frac{c_P C_{S,P}^{K-1}}{c_S}\sum_{|n|>K}\frac{|n|^2}{2^{|n|+2}}\left(2C_{S,P}C_R^S\right)^{|n|-1}$$
$$\lesssim \frac{c_P C_{S,P}^{K-1}}{c_S}\left(2C_{S,P}C_R^S\right)^{K-1}$$
$$\leq \frac{c_P}{c_S}h^{K-1}. \tag{4.49}$$

Substituting the estimates (4.47)-(4.49) in (4.44) one arrives at

$$|I_1| \lesssim \frac{1}{\rho_0 c_P^4}\left(e+1+\frac{c_P}{c_S}\right)h^{K-1}.$$

The estimate for $|I_2|$ follows by changing the role of m and n. Moreover, by proceeding in a similar fashion, it can be easily established that

$$|I_3| \lesssim \left(\frac{h}{K}\right)^{K-1}.$$

Combining the estimates for $|I_1|$, $|I_2|$ and $|I_3|$, one obtains $|E_{sr}^{P,P}| \lesssim h^{K-1}$. This completes the proof.

□

4.5 Maximal Resolving Order

In this subsection, we determine the maximal resolving order K for the reconstruction framework. In order to do so we first estimate the *strength* of the recorded signals in terms of the geometry of inclusion D and the radius of the recording circle. Then we define the signal-to-noise ratio (SNR) in terms of signal strength and noise standard deviation σ_{noise}. Towards this end, it is easy to see from the integral representation (2.14) that

$$A_{sr}^{\parallel,\parallel} = \mathcal{S}_D^\omega[\boldsymbol{\psi}_{\mathbf{F}_s}](\mathbf{x}_r)\cdot\mathbf{d}_r = \mathcal{S}_D^\omega[\boldsymbol{\psi}_{\mathbf{F}_s}](R\,\mathbf{d}_r)\cdot\mathbf{d}_r, \tag{4.50}$$

where $\boldsymbol{\psi}_{\mathbf{F}_s}$ is the solution of (2.15) corresponding to $\mathbf{U}=\mathbf{F}_s$. By virtue of the far field behavior

$$\boldsymbol{\Gamma}^\omega(\mathbf{x},\mathbf{y}) \simeq \frac{e^{i\kappa_P|\mathbf{x}|}}{\sqrt{|x|}}\left(\frac{i+1}{4\rho_0 c_P^2\sqrt{\pi\kappa_P}}\hat{\mathbf{x}}\otimes\hat{\mathbf{x}}e^{-i\kappa_P\hat{\mathbf{x}}\cdot\mathbf{y}}\right)$$
$$+\frac{e^{i\kappa_S|\mathbf{x}|}}{\sqrt{|x|}}\left(\frac{i+1}{4\rho_0 c_S^2\sqrt{\pi\kappa_S}}\left(\mathbf{I}_2-\hat{\mathbf{x}}\otimes\hat{\mathbf{x}}\right)e^{-i\kappa_S\hat{\mathbf{x}}\cdot\mathbf{y}}\right) \tag{4.51}$$

of the fundamental solution for a bounded $\mathbf{y} \in \mathbb{R}^2$ and $\mathbf{x} \in \mathbb{R}^2$ such that $|\mathbf{x}| \to \infty$ one has

$$\begin{aligned} A^{\|,\|} \simeq & \frac{1}{\sqrt{R}} \frac{(i+1)e^{i\kappa_P R}}{4\rho_0 c_P^2 \sqrt{\pi\kappa_P}} \int_{\partial D} [(\mathbf{d}_r \otimes \mathbf{d}_r)\boldsymbol{\psi}_{\mathbf{F}_s}(\mathbf{y})] \cdot \mathbf{d}_r \, e^{-i\kappa_P |\mathbf{y}| \cos(\theta_r - \theta_{\mathbf{y}})} \, d\sigma(\mathbf{y}) \\ & + \frac{1}{\sqrt{R}} \frac{(i+1)e^{i\kappa_S R}}{4\rho_0 c_S^2 \sqrt{\pi\kappa_S}} \int_{\partial D} [\mathbf{d}_r^\perp \otimes \mathbf{d}_r^\perp)\boldsymbol{\psi}_{\mathbf{F}_s}(\mathbf{y})] \cdot \mathbf{d}_r \, e^{-i\kappa_S |\mathbf{y}| \cos(\theta_r - \theta_{\mathbf{y}})} \, d\sigma(\mathbf{y}) \\ \simeq & \frac{1}{\sqrt{R}} \frac{(i+1)e^{i\kappa_P R}}{4\rho_0 c_P^2 \sqrt{\pi\kappa_P}} \int_{\partial D} [\boldsymbol{\psi}_{\mathbf{F}_s}(\mathbf{y}) \cdot \mathbf{d}_r] \, e^{-i\kappa_P |\mathbf{y}| \cos(\theta_r - \theta_{\mathbf{y}})} \, d\sigma(\mathbf{y}) \\ & + \frac{1}{\sqrt{R}} \frac{(i+1)e^{i\kappa_S R}}{4\rho_0 c_S^2 \sqrt{\pi\kappa_S}} \int_{\partial D} [\boldsymbol{\psi}_{\mathbf{F}_s}(\mathbf{y}) \cdot \mathbf{d}_r^\perp] \, e^{-i\kappa_S |\mathbf{y}| \cos(\theta_r - \theta_{\mathbf{y}})} \, d\sigma(\mathbf{y}). \end{aligned} \tag{4.52}$$

On the other hand, by (2.16)

$$\|\boldsymbol{\psi}_{\mathbf{F}_s}\|_{L^2(\partial D)^2} \leq \|\mathbf{F}_s\|_{H^1(\partial D)^2} + \left\| \frac{\partial \mathbf{F}_s}{\partial \nu} \right\|_{L^2(\partial D)^2} \leq C\sqrt{|\partial D|} \tag{4.53}$$

for some constant independent of R and $|\partial D|$. Thus, by taking the modulus on both sides of (4.52), substituting the above estimate for $\|\boldsymbol{\psi}_{\mathbf{F}_s}\|$ and using the Cauchy-Schwartz inequality, one obtains the estimate

$$|A^{\|,\|}| \leq C \frac{|\partial D|}{\sqrt{R}}. \tag{4.54}$$

The constant C above depends only on the material parameters of the background domain, inclusion D and the frequency ω of the incident field but is independent of R and ∂D. Similarly, the terms of other MSR matrices can be also bounded by $|\partial D|/\sqrt{R}$. This endorses that the measured signals are of order $|\partial D|/\sqrt{R}$. Therefore, we define SNR by

$$\mathrm{SNR} := \frac{|\partial D|/\sqrt{R}}{\sigma_{\mathrm{noise}}}. \tag{4.55}$$

Now we are ready to estimate the maximal resolving order K. In the sequel, $\mathbb{E}$ denotes the expectation with respect to the statistics of the noise $\mathbf{N}_{\mathrm{noise}}$. Moreover, we work in the regime when $R \to \infty$ (or $O(R^{-3/2})$ terms are negligible) and the truncation error is much smaller than the noise standard deviation, which in tern is much smaller than the order of the signal (or simply SNR is much larger than 1), that is,

$$h^{K-1} \ll \sigma_{\mathrm{noise}} \ll |\partial D|/\sqrt{R}. \tag{4.56}$$

From the injectivity of operator $\mathbf{L}$ for $R \to \infty$ and the relation (4.15) we have

$$\begin{aligned} \mathbb{E}\left(\left| \left(\widehat{\mathbf{W}} - \mathbf{W} \right)_{mn} \right|^2 \right)^{1/2} &= \mathbb{E}\left(\left| (\mathbf{L}^\dagger(\mathbf{E} + \mathbf{N}_{\mathrm{noise}})_{mn} \right|^2 \right)^{1/2} \\ &\leq \left| \mathbf{L}^\dagger(\mathbf{E})_{mn} \right| + \left| \mathbf{L}^\dagger(\mathbf{N}_{\mathrm{noise}})_{mn} \right| \\ &\lesssim \sigma_{mn}^{-1} \left(\|\mathbf{E}\|_F + \|\mathbf{N}_{\mathrm{noise}}\|_F \right) \\ &\lesssim \sigma_{mn}^{-1} \left(h^{K-1} + \sigma_{\mathrm{noise}} \sqrt{N_s N_r} \right), \end{aligned} \tag{4.57}$$

where Cauchy-Schwartz inequality has been invoked to arrive at the last identity. By assumption (4.56), the first term on RHS of (4.57) is negligible. Thus,

$$\mathbb{E}\left(\left|\left(\widehat{\mathbf{W}}-\mathbf{W}\right)_{mn}\right|^2\right)^{1/2} \lesssim \sigma_{mn}^{-1}\sigma_{\text{noise}}\sqrt{N_sN_r}. \tag{4.58}$$

This indicates that the discrepancy between the estimated and the measured scattering coefficients approaches zero very rapidly for all $m,n>K$ when $K\to\infty$ thanks to the estimation of the magnitude of σ_{mn}. It simply means that the scattering coefficients of an inclusion D can be approximated arbitrarily closely and up to *any order* by the elements of $\widehat{\mathbf{W}}$ in the sense of mean-squared error when the noise regime is characterized by (4.56). However, in view of the decay rate (3.19) of $W_{mn}^{\alpha,\beta}$, it is reasonable to determine an adoptive resolving order K by restricting the reconstruction error to be smaller than the signal level. In particular, for any threshold reconstruction error $\varepsilon>0$, one can see from (4.58) and (3.19) that

$$\mathbb{E}\left(\left|\left(\widehat{\mathbf{W}}-\mathbf{W}\right)_{mn}\right|^2\right)^{1/2} \lesssim \sigma_{mn}^{-1}\sigma_{\text{noise}}\sqrt{N_sN_r} \lesssim \varepsilon\left(\frac{C_{\max}}{K}\right)^{2K-2},$$

where $C_{\max}:=\max_{\alpha,\beta}\{C_{\alpha,\beta}\}$. After simple manipulations analogous to those in the proof of Corollary 4.5 and using the behavior of the $\left(H_n^{(1)}(\kappa_\alpha R)\right)'$ for large n, one can show that $\sigma_{mn}^{-1}=O\left((C_R^SK)^{1-K}\right)$ for all $m,n>K$. Therefore, under noise regime characterize by (4.56),

$$\left(C_R^SK\right)^{1-K}K^{2K-2}\lesssim\varepsilon\frac{C_{\max}^{2K-2}}{\sigma_{\text{noise}}}$$

or equivalently

$$K^{K-1}\lesssim\varepsilon\frac{(C_{\max}^2C_R^S)^{K-1}}{\sigma_{\text{noise}}}\lesssim\varepsilon\frac{h^{K-1}}{\sigma_{\text{noise}}}\le\varepsilon\text{SNR}$$

and the maximal resolving order is defined by

$$K=\max\left\{N\in\mathbb{N}\,|N^{N-1}\le\varepsilon\text{SNR}\right\}. \tag{4.59}$$

5 Nearly Elastic Cloaking

As an application of the elastic scattering coefficients we consider the elastic cloaking problem. The aim here is to construct an effective nearly cloaking structure at a fixed frequency for making the objects inside the unit disk *invisible*. We extend the approach of Ammari et al. [10, 11, 12] for conductivity, Helmholtz and Maxwell equations to the Lamé system. Towards this end, we first design S-vanishing structures by cancelling the first ESCs in the next subsection.

5.1 S-vanishing Structures

For positive numbers r_j $(j=1,2,\cdots,L+1)$ with $2=r_1>r_2>\cdots>r_{L+1}=1$ we construct a multi-layered structure by defining

$$\begin{aligned}
A_0 :=&\left\{\mathbf{x}\in\mathbb{R}^2\,\middle|\quad |\mathbf{x}|>2\right\}\\
A_j :=&\left\{\mathbf{x}\in\mathbb{R}^2\,\middle|\quad r_{j+1}\le|\mathbf{x}|<r_j\right\},\quad j=1,\cdots,L\\
A_{L+1} :=&\left\{\mathbf{x}\in\mathbb{R}^2\,\middle|\quad |\mathbf{x}|<1\right\}.
\end{aligned}$$

Let $(\lambda_j, \mu_j, \rho_j)$ be the Lamé parameters and densities of A_j for $j = 0, \cdots, L+1$. In particular, λ_0, μ_0 and ρ_0 are the parameters of the background medium. In the sequel, the piecewise constant parameters λ, μ and ρ are redefined as

$$\lambda(\mathbf{x}) = \sum_{j=0}^{L+1} \lambda_j\, \chi_{(A_j)}(\mathbf{x}), \quad \mu(\mathbf{x}) = \sum_{j=0}^{L+1} \mu_j\, \chi_{(A_j)}(\mathbf{x}) \quad \text{and} \quad \rho(\mathbf{x}) = \sum_{j=0}^{L+1} \rho_j\, \chi_{(A_j)}(\mathbf{x}). \tag{5.1}$$

in accordance with the aforementioned multi-layered structure. The scattering coefficients $W_{m,n}^{\alpha,\beta} = W_{m,n}^{\alpha,\beta}(\lambda, \mu, \rho, \omega)$ can be defined analogously to (3.18) and the total field $\mathbf{u} = (u_1, u_2)^\top$ solves the equation

$$\mathcal{L}_{\lambda,\mu}\mathbf{u} + \rho\omega^2\mathbf{u} = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{5.2}$$

Since the aforementioned multi-layered structure is circularly symmetric it is easy to check that

$$W_{m,n}^{\alpha,\beta} = 0 \qquad \text{for all} \quad \alpha, \beta \in \{P, S\} \quad \text{and} \quad n \neq m.$$

Therefore, we have the following definition of the S-vanishing structures.

Definition 5.1 (S-vanishing Structure). *The medium (λ, μ, ρ) defined by (5.1) is called an S-vanishing structure of order N at frequency ω if $W_{n,n}^{\alpha,\beta} = 0$ for all $|n| \le N$ and $\alpha, \beta \in \{P, S\}$. Analogously, it is called an S-vanishing structure for compressional (resp. shear) waves if $W_{n,n}^{\alpha,P} = 0$ (resp. $W_{n,n}^{\alpha,S} = 0$) for all $|n| \le N$ and $\alpha \in \{P, S\}$.*

In the rest of this subsection we aim to construct an S-vanishing structure for general elastic waves. To facilitate the later analysis we adopt the notation $T_{\lambda,\mu}$ for the surface traction operator $\partial/\partial\nu$ associated with elastic moduli λ and μ. In order to design envisioned structure it suffices to construct (λ, μ, ρ) such that $W_n^{\alpha,\beta} := W_{n,n}^{\alpha,\beta} = 0$ for all $0 \le n \le N$ and $\alpha, \beta \in \{P, S\}$ thanks to Lemma 3.6. We assume that D is a cavity, that is, the scattered field $\mathbf{u}$ satisfies the traction-free boundary condition $T_{\lambda_{L+1},\mu_{L+1}}\mathbf{u} := \partial\mathbf{u}/\partial\nu = 0$ on $|\mathbf{x}| = 1$. Note that the two-dimensional surface traction admits the expression

$$T_{\lambda,\mu}\mathbf{w} = 2\mu(\nu \cdot \nabla w_1, \nu \cdot \nabla w_2) + \lambda\mathbf{n}\,\mathrm{div}\,\mathbf{w} + \mu\,\mathbf{t}\,(\partial_2 w_1 - \partial_1 w_2), \quad \mathbf{w} = (w_1, w_2),$$

in terms of the normal and tangent vectors $\mathbf{n} = (n_1, n_2)$ and $\mathbf{t} = (-n_1, n_2)$ on the surface respectively. Here and in the sequel we use the notation $T_{\lambda,\mu}\mathbf{w}$ to indicate the dependance of $\partial\mathbf{w}/\partial\nu$ on the parameters λ and μ. We look for solutions $\mathbf{u}_n$ to (5.2) of the form

$$\mathbf{u}_n(\mathbf{x}) = \widehat{a}_j^{n,P}\mathbf{J}_n^P(\mathbf{x}) + \widehat{a}_j^{n,S}\mathbf{J}_n^S(\mathbf{x}) + a_j^{n,P}\mathbf{H}_n^P(\mathbf{x}) + a_j^{n,S}\mathbf{H}_n^S(\mathbf{x}), \quad \mathbf{x} \in A_j,\ j = 0, \cdots, L,$$

with the unknown coefficients $\widehat{a}_j^{n,\alpha}, a_j^{n,\alpha} \in \mathbb{C}$, to be determined later. Intuitively, one should look for solutions $\mathbf{u}_n$ whose coefficients fulfill the relations

$$\widehat{a}_0^{n,P}\ \widehat{a}_0^{n,S} \neq 0 \quad \text{and} \quad a_0^{n,P} = a_0^{n,S} = 0 \quad \text{for all} \quad n = 0, \cdots, N. \tag{5.3}$$

Note that by (3.16) and (3.17), the scattering coefficients in this case turn out to be

$$\begin{cases} W_n^{\alpha,P} = i4\rho_0 c_\alpha^2 a_0^{n,\alpha} = 0 & \text{when } \widehat{a}_0^{n,P} = 1 \text{ and } \widehat{a}_0^{n,S} = 0, \\ W_n^{\alpha,S} = i4\rho_0 c_\alpha^2 a_0^{n,\alpha} = 0 & \text{when } \widehat{a}_0^{n,P} = 0 \text{ and } \widehat{a}_0^{n,S} = 1, \end{cases} \tag{5.4}$$

where c_P and c_S are the pressure and shear wave speeds, respectively. The solution $\mathbf{u}_n$ satisfies the transmission conditions

$$\mathbf{u}_n|_+ = \mathbf{u}_n|_- \quad \text{and} \quad T_{\lambda_{j-1},\mu_{j-1}}\mathbf{u}_n|_+ = T_{\lambda_j,\mu_j}\mathbf{u}_n|_- \quad \text{on} \quad |\mathbf{x}| = r_j,\ \forall j = 1, \cdots, L. \tag{5.5}$$

Fairly easily calculations indicate that on $|\mathbf{x}| = r$

$$\begin{aligned}
\widehat{\mathbf{e}}_r \cdot [T_{\lambda,\mu}\mathbf{H}_n^P(\mathbf{x})] &= 2\mu\frac{\partial^2 v_n(\mathbf{x},\kappa_P)}{\partial r^2} + \lambda\Delta v_n(\mathbf{x},\kappa_P) \\
&= 2\mu\kappa_P^2(H_n^{(1)})''(r\kappa_P)\mathbf{e}^{in\varphi_{\mathbf{x}}} - \lambda\kappa_P^2 H_n^{(1)}(r\kappa_P)\mathbf{e}^{in\varphi_{\mathbf{x}}} \\
&= \frac{1}{r^2}\left(-2\mu r\kappa_p(H_n^{(1)})'(r\kappa_P) + (2\mu n^2 - (\lambda+2\mu)r^2\kappa_p^2)H_n^{(1)}(r\kappa_P)\right)\mathbf{e}^{in\varphi_{\mathbf{x}}}, \\
&=: \frac{1}{r^2}B_n^P(r\kappa_P,\lambda,\mu)\mathbf{e}^{in\varphi_{\mathbf{x}}},
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\mathbf{e}}_\theta \cdot [T_{\lambda,\mu}\mathbf{H}_n^P(\mathbf{x})] &= 2\mu\left(-\frac{1}{r^2}\frac{\partial v_n(\mathbf{x},\kappa_P)}{\partial\varphi_{\mathbf{x}}} + \frac{1}{r}\frac{\partial^2 v_n(\mathbf{x},\kappa_P)}{\partial r\partial\varphi_{\mathbf{x}}}\right) \\
&= \frac{1}{r^2}(2i\mu n)\left(-H_n^{(1)}(r\kappa_P) + r\kappa_P(H_n^{(1)})'(r\kappa_P)\right)\mathbf{e}^{in\varphi_{\mathbf{x}}} \\
&=: \frac{1}{r^2}C_n^P(r\kappa_P,\lambda,\mu)\mathbf{e}^{in\varphi_{\mathbf{x}}},
\end{aligned}$$

where

$$\begin{aligned}
B_n^P(t,\lambda,\mu) &:= -2\mu t(H_n^{(1)})'(t) + (2\mu n^2 - (\lambda+2\mu)t^2)H_n^{(1)}(t), \\
C_n^P(t,\lambda,\mu) &:= (2i\mu n)\left(-H_n^{(1)}(t) + t(H_n^{(1)})'(t)\right).
\end{aligned}$$

In the sequel, we use shorthand notation $B_{n,j}^P = B_n^P(r_j\kappa_P,\lambda_j,\mu_j)$ and $C_{n,j}^P = C_n^P(r_j\kappa_P,\lambda_j,\mu_j)$ for simplicity. It holds that

$$T_{\lambda_j,\mu_j}\mathbf{H}_n^P(\mathbf{x}) = \frac{1}{r_j^2}\left(B_{n,j}^P\,\mathbf{P}_n(\hat{\mathbf{x}}) + C_{n,j}^P\,\mathbf{S}_n(\hat{\mathbf{x}})\right) \quad \text{on } |\mathbf{x}| = r_j.$$

Analogously, we obtain

$$T_{\lambda_j,\mu_j}\mathbf{H}_n^S(\mathbf{x}) = \frac{1}{r_j^2}\left(B_{n,j}^S\,\mathbf{P}_n(\hat{\mathbf{x}}) + C_{n,j}^S\,\mathbf{S}_n(\hat{\mathbf{x}})\right) \quad \text{on } |\mathbf{x}| = r_j,$$

with

$$\begin{aligned}
B_{n,j}^S =& B_n^S(t)\big|_{t=r_j\kappa_S} := (2i\mu n)\left(H_n^{(1)}(r_j\kappa_S) + r_j\kappa_S\left(H_n^{(1)}\right)'(r_j\kappa_S)\right), \\
C_{n,j}^S =& C_n^S(t)\big|_{t=r_j\kappa_S} := 2\mu(r_j\kappa_S)\left(H_n^{(1)}\right)'(r_j\kappa_S) + \left(-2\mu n^2 + 3\mu(r_j\kappa_S)^2 H_n^{(1)}(r_j\kappa_S)\right),
\end{aligned}$$

and

$$T_{\lambda_j,\mu_j}\mathbf{J}_n^\alpha(\mathbf{x}) = \frac{1}{r_j^2}\left(\widehat{B}_{n,j}^\alpha\,\mathbf{P}_n(\hat{\mathbf{x}}) + \widehat{C}_{n,j}^\alpha\,\mathbf{S}_n(\hat{\mathbf{x}})\right), \quad \alpha = P, S,$$

where $\widehat{B}_{n,j}^\alpha$ and $\widehat{C}_{n,j}^\alpha$ are defined in the same way as $B_{n,j}^\alpha$ and $C_{n,j}^\alpha$ with $H_n^{(1)}$ replaced by J_n. Hence, the transmission conditions in (5.5) can be written as

$$\frac{1}{r_{j-1}^2}\mathbf{M}_{n,j-1}(\widehat{a}_{j-1}^{n,P},\widehat{a}_{j-1}^{n,S},a_{j-1}^{n,P},a_{j-1}^{n,S})^\top = \frac{1}{r_j^2}\mathbf{M}_{n,j}(\widehat{a}_j^{n,P},\widehat{a}_j^{n,S},a_j^{n,P},a_j^{n,S})^\top, \tag{5.6}$$

for $j=1,\cdots,L$, where $\mathbf{M}_{n,j}$, $j=0,\cdots,L$, $n=0,\cdots,N$, is the 4×4 matrix defined by

$$\mathbf{M}_{n,j} := \begin{pmatrix} t_{j,P}J_n^{'}(t_{j,P}) & inJ_n(t_{j,S}) & t_{j,P}(H_n^{(1)})'(t_{j,P}) & inH_n^{(1)}(t_{j,S}) \\ inJ_n(t_{j,P}) & -t_{j,S}J_n^{'}(t_{j,S}) & inH_n^{(1)}(t_{j,P}) & -t_{j,S}H_n^{(1)}(t_{j,S}) \\ \widehat{B}_{n,j}^P(t_{j,P}) & \widehat{B}_{n,j}^S(t_{j,S}) & B_{n,j}^P(t_{j,P}) & B_{n,j}^S(t_{j,S}) \\ \widehat{C}_{n,j}^P(t_{j,P}) & \widehat{C}_{n,j}^S(t_{j,P}) & C_{n,j}^S(t_{j,S}) & C_{n,j}^S(t_{j,S}) \end{pmatrix}, \quad t_{j,\alpha} := r_j\kappa_\alpha.$$

The traction-free boundary condition on $|\mathbf{x}| = r_{L+1} = 1$ amounts to

$$\mathbf{M}_{n,L+1}(\widehat{a}_L^{n,P}, \widehat{a}_L^{n,S}, a_L^{n,P}, a_L^{n,S})^\top = (0,0,0,0)^\top, \tag{5.7}$$

for $n=0,\cdots,N$ with

$$\mathbf{M}_{n,L+1} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \widehat{B}_{n,L}^P & \widehat{B}_{n,L}^S & B_{n,L}^P & B_{n,L}^S \\ \widehat{C}_{n,L}^P & \widehat{C}_{n,L}^S & C_{n,L}^P & C_{n,L}^S \end{pmatrix}.$$

Combining (5.6) and (5.7) we obtain

$$\begin{cases} \mathbf{Q}^{(n)}(\widehat{a}_0^{n,P}, \widehat{a}_0^{n,S}, a_0^{n,P}, a_0^{n,S})^\top = (0,0,0,0)^\top, \\ \mathbf{Q}^{(n)} = \mathbf{Q}^{(n)}(\lambda,\mu,\rho\omega^2) := \left(\dfrac{r_L}{r_0}\right)^2 \mathbf{M}_{n,L+1}\displaystyle\prod_{j=1}^{L}\mathbf{M}_{n,j}^{-1}\mathbf{M}_{n,j-1} = \begin{pmatrix} 0 & 0 \\ \mathbf{Q}_{21}^{(n)} & \mathbf{Q}_{22}^{(n)} \end{pmatrix}, \end{cases} \tag{5.8}$$

where $\mathbf{Q}_{21}^{(n)}, \mathbf{Q}_{22}^{(n)}$ are 2×2 matrix functions of λ, μ and $\rho\omega^2$.

Exactly like the acoustic case [11] one can show that the determinant of $\mathbf{Q}_{22}^{(n)}$ is non-vanishing. Therefore, it suffices to look for the parameters λ_j, μ_j, ρ_j ($j=1,2,\cdots,L$) from the nonlinear algebraic equations

$$(\mathbf{Q}_{21}^{(n)})_{i,k}(\lambda,\mu,\rho\omega^2) = 0, \qquad i,k=1,2, \quad n=1,2\cdots.$$

We are interested in a nearly S-vanishing structure of order N at low frequencies, that is, a structure (λ,μ,ρ) such that

$$W_n^{\alpha,\beta}(\lambda,\mu,\rho,\omega) = o(\omega^{2N}) \quad \text{for all} \quad \alpha,\beta\in\{P,S\},\ |n|\le N, \quad \text{as} \quad \omega\to 0.$$

Towards this end, we need to study the asymptotic behavior of $W_n^{\alpha,\beta}(\lambda,\mu,\rho,\omega)$ as ω tends to zero. In view of (5.4) and (5.8) we find out that

$$(W_n^{\alpha,P}, W_n^{\alpha,S})^\top = i4\rho_0c_\alpha^2\,(a_0^{n,P}, a_0^{n,S})^\top = -i4\rho_0c_\alpha^2\,(\mathbf{Q}_{22}^{(n)})^{-1}\,\mathbf{Q}_{21}^{(n)}\,(\widehat{a}_0^{n,P}, \widehat{a}_0^{n,S})^\top \tag{5.9}$$

where $\widehat{a}_0^{n,P}$ and $\widehat{a}_0^{n,S}$ are selected depending on (5.4).

Let $\mathbf{W}_n$ denote the 2×2 matrix

$$\mathbf{W}_n = \begin{pmatrix} \mathbf{W}_n^{P,P} & \mathbf{W}_n^{S,P} \\ \mathbf{W}_n^{P,S} & \mathbf{W}_n^{S,S} \end{pmatrix}.$$

Then, the following result based on relation (5.9) elucidates the low frequency asymptotic behavior of $\mathbf{W}_n$.

Theorem 5.2. *For all $n \in \mathbb{N}$, we have*

$$\mathbf{W}_n(\lambda,\mu,\rho,\omega) = \omega^{2n}\left(\mathbf{V}_{n,0}(\lambda,\mu,\rho) + \sum_{l=0}^{N-n}\sum_{j=0}^{(L+1)l} \omega^{2l}(\ln\omega)^j\,\mathbf{V}_{n,l,j}(\lambda,\mu,\rho)\right) + \mathbf{\Upsilon}_n \qquad (5.10)$$

as $\omega \to 0$, where matrices $\mathbf{V}_{n,0}$ and $\mathbf{V}_{n,l,j}$ are defined by

$$\mathbf{V}_{n,0} = \begin{pmatrix} \mathbf{V}_{n,0}^{P,P} & \mathbf{V}_{n,0}^{S,P} \\ \mathbf{V}_{n,0}^{P,S} & \mathbf{V}_{n,0}^{S,S} \end{pmatrix} \quad \text{and} \quad \mathbf{V}_{n,l,j} = \begin{pmatrix} \mathbf{V}_{n,l,j}^{P,P} & \mathbf{V}_{n,l,j}^{S,P} \\ \mathbf{V}_{n,l,j}^{P,S} & \mathbf{V}_{n,l,j}^{S,S}, \end{pmatrix}$$

in terms of some $V_{n,0}^{\alpha,\beta}$ and $V_{n,l,j}^{\alpha,\beta}$ dependent on λ, μ, ρ but independent of ω. The residual matrix $\mathbf{\Upsilon}_n = (\Upsilon_{ik}^n)_{i,k=1,2}$ is such that $|\Upsilon_{ik}^n| \le C\omega^{2N}$, for all $i,k = 1,2$, where constant $C \in \mathbb{R}_+$ is independent of ω.

The analytic expressions of the quantities $\mathbf{V}_{n,0}^{\alpha,\beta}$ and $\mathbf{V}_{n,l,j}^{\alpha,\beta}$ in terms of λ_j, μ_j and ρ_j are very complicated, but can be extracted by, for example, using the symbolic toolbox of MATLAB. Theorem 5.2 follows from (5.9) and the low-frequency asymptotics of $\mathbf{Q}_{22}^{(n)}(\lambda,\mu,\rho\omega^2)$ and $\mathbf{Q}_{21}^{(n)}(\lambda,\mu,\rho\omega^2)$ as $\omega \to 0$. The latter can be derived based on the definition given in (5.8) in combination with the expansion formula of Bessel and Neumann functions and their derivatives for small arguments. For the sake of completeness below we sketch the proof of Theorem 5.2.

Proof of Theorem 5.2. Recall that for $t \to 0$

$$\begin{aligned} J_n(t) &= \frac{t^n}{2^n\Gamma(n+1)} + O(t^{n+1}), \\ J_n'(t) &= \frac{nt^{n-1}}{2^n\Gamma(n+1)} + O(t^{n+1}), \\ Y_n(t) &= -\frac{2^n\Gamma(n)}{\pi t^n} + O(t^{-n+1}), \\ Y_n'(t) &= \frac{2^n\Gamma(n+1)}{\pi t^{n+1}} + O(t^{-n}). \end{aligned}$$

Hence, by the definition of $B_n^\alpha(t,\lambda,\mu)$, $C_n^\alpha(t,\lambda,\mu)$, $\widehat{B}_n^\alpha(t,\lambda,\mu)$ and $\widehat{C}_n^\alpha(t,\lambda,\mu)$, we have

$$\begin{aligned} B_n^P(t,\lambda,\mu) &= -C_n^S(t,\lambda,\mu) = -\frac{i\mu 2^{n+1}\Gamma(n+1)}{\pi t^n} + O(t^{-n+1}) \\ C_n^P(t,\lambda,\mu) &= -B_n^S(t,\lambda,\mu) = -\frac{\mu n 2^{n+1}\Gamma(n+1)}{\pi t^n} + O(t^{-n+1}) \\ \widehat{B}_n^P(t,\lambda,\mu) &= -\widehat{C}_n^S(t,\lambda,\mu) = -\frac{i\mu n t^n}{2^{n-1}\Gamma(n+1)} + O(t^{n+1}) \\ \widehat{C}_n^P(t,\lambda,\mu) &= -\widehat{B}_n^S(t,\lambda,\mu) = -\frac{\mu n^2 t^n}{2^{n-1}\Gamma(n+1)} + O(t^{n+1}), \end{aligned}$$

as $t \to 0$. Inserting the previous asymptotic behavior into the expression of $\mathbf{M}_{n,j}$ we get

$$\mathbf{M}_{n,j} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \qquad (5.11)$$

where

$$\mathbf{A}_{11} = \quad \frac{n}{2^n\Gamma(n+1)}\begin{pmatrix} t_{j,P}^{n-1} & it_{j,S}^{n} \\ it_{j,P}^{n} & -t_{j,S}^{n-1}\end{pmatrix} + O(\omega^n),$$

$$\mathbf{A}_{12} = \quad \frac{2^n\Gamma(n+1)}{\pi}\begin{pmatrix} it_{j,P}^{-n-1} & t_{j,S}^{-n} \\ t_{j,P}^{-n} & -it_{j,S}^{-n-1}\end{pmatrix} + O(\omega^{-n}),$$

$$\mathbf{A}_{21} = -\frac{\mu n}{2^{n-1}\Gamma(n+1)}\begin{pmatrix} it_{j,P}^{n} & nt_{j,S}^{n} \\ nt_{j,P}^{n} & -it_{j,S}^{n-1}\end{pmatrix} + O(\omega^{n+1}),$$

$$\mathbf{A}_{22} = -\frac{2^{n+1}\mu\Gamma(n+1)}{\pi}\begin{pmatrix} it_{j,P}^{-n} & t_{j,S}^{-n} \\ t_{j,P}^{-n} & -it_{j,S}^{-n-1}\end{pmatrix} + O(\omega^{-n+1}).$$

This implies that

$$\mathbf{M}_{n,j} = \begin{pmatrix} O(\omega^{n-1}) & O(\omega^{-n-1}) \\ O(\omega^{n}) & O(\omega^{-n})\end{pmatrix}, \quad j = 1, \cdots, L, \tag{5.12}$$

$$\mathbf{M}_{n,L} = \begin{pmatrix} 0 & 0 \\ O(\omega^{n}) & O(\omega^{-n})\end{pmatrix} \quad \text{as } \omega \to 0. \tag{5.13}$$

Moreover, the inverse of $\mathbf{M}_{n,j}$ can be expressed as

$$\mathbf{M}_{n,j}^{-1} = \begin{pmatrix} \mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{B}^{-1} \\ -\mathbf{B}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{B}^{-1}\end{pmatrix},$$

where $\mathbf{B}$ is the Schur complement of $\mathbf{A}_{22}$, that is,

$$\mathbf{B} := \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}.$$

Since

$$\mathbf{A}_{11}^{-1} = O(\omega^{-n+1}), \quad \mathbf{A}_{11}^{-1}\mathbf{A}_{12} = O(\omega^{-2n}), \quad \mathbf{A}_{21}\mathbf{A}_{11}^{-1} = O(\omega) \quad \text{and} \quad \mathbf{B}^{-1} = O(\omega^{n}),$$

it follows that

$$\mathbf{M}_{n,j}^{-1} = \begin{pmatrix} O(\omega^{-n+1}) & O(\omega^{-n}) \\ O(\omega^{n+1}) & O(\omega^{n})\end{pmatrix} \quad \text{as } \omega \to 0. \tag{5.14}$$

Inserting (5.12), (5.13) and (5.14) into the expression (5.8) of $\mathbf{Q}^{(n)}$ and then making use of the series expansions of J_n, Y_n, J_n' and Y_n' we find out that

$$\mathbf{Q}_{21}^{(n)}(\lambda, \mu, \rho\omega^2) = \omega^n\left(\mathbf{G}_{n,0}(\lambda,\mu,\rho) + \sum_{l=1}^{N-n}\sum_{j=0}^{L+1}\mathbf{G}_{n,l}^{(j)}(\lambda,\mu,\rho)\omega^{2l}(\ln\omega)^j + o\left(\omega^{2(N-n)}\right)\right)$$

$$\mathbf{Q}_{22}^{(n)}(\lambda, \mu, \rho\omega^2) = \omega^{-n}\left(\mathbf{H}_{n,0}(\lambda,\mu,\rho) + \sum_{l=1}^{N-n}\sum_{j=0}^{L+1}\mathbf{H}_{n,l}^{(j)}(\lambda,\mu,\rho)\omega^{2l}(\ln\omega)^j + o\left(\omega^{2(N-n)}\right)\right),$$

which together with (5.9) yields (5.10). Here, the remaining terms $o(\omega^{2(N-n)})$ are understood element-wisely for the matrices. □

In order to construct a nearly S-vanishing structure of order N at low frequencies, thanks to Theorem 5.2 we need to determine the parameters λ_j, μ_j and ρ_j from the equations

$$\mathbf{V}_{n,0}^{\alpha,\beta}(\lambda,\mu,\rho) = \mathbf{V}_{n,l,j}^{\alpha,\beta}(\lambda,\mu,\rho) = 0,$$

for all $0 \le n \le N$, $1 \le l \le (N-n)$, $1 \le j \le (L+1)l$ and $\alpha, \beta \in \{P, S\}$. Numerically, this can be achieved by applying, for example, the gradient descent method to the minimization problem

$$\min_{\lambda_j,\mu_j,\rho_j} \sum_{\alpha,\beta\in\{P,S\}} \left\{ \left|\mathbf{V}_{n,0}^{\alpha,\beta}\right|^2 + \sum_{l=0}^{N-n} \sum_{j=0}^{(L+1)l} \left|\mathbf{V}_{n,l,j}^{\alpha,\beta}\right|^2 \right\}.$$

5.2 Enhancement of Near Cloaking in Elasticity

The aim of this section is to show that the nearly S-vanishing structures constructed in Section 5.1 can be used to enhance cloaking effect in elasticity. The enhancement of near cloaking is based on the idea of transformation optics (also called the scheme of changing variables) used in [26, 27, 36, 37]. Let (λ,μ,ρ) be a nearly S-vanishing structure of order N at low frequencies, taking the form of (5.1). This implies that for some fixed $\omega > 0$ there exists $\epsilon_0 > 0$ such that

$$\left|W_{m,n}^{\alpha,\beta}[\lambda,\mu,\rho,\epsilon\omega]\right| = o(\epsilon^{2N}), \quad |n| \le N, \quad \epsilon \le \epsilon_0.$$

On the other hand, recall from the proof of Lemma 3.3 that

$$\left|W_{n}^{\alpha,\beta}[\lambda,\mu,\rho,\epsilon\omega]\right| \le \frac{C_{\alpha,\beta}^{2|n|-2}}{|n|^{2|n|-2}}\,\epsilon^{2|n|-2} \le \frac{C_{\alpha,\beta}^{2|n|-2}}{|n|^{2|n|-2}}\,\epsilon^{2N-2} \quad \text{for all} \quad |n| \ge N,\ \epsilon \le \epsilon_0. \tag{5.15}$$

Hence, by Theorem 3.4, the far-field elastic scattering amplitudes can be estimated by

$$\mathbf{u}_\alpha^\infty[\lambda,\mu,\rho,\epsilon\omega](\hat{\mathbf{x}},\hat{\mathbf{x}}') = o(\epsilon^{2N-2}), \quad \alpha = P, S, \quad \text{as} \quad \epsilon \to 0 \tag{5.16}$$

uniformly in all observation directions $\hat{\mathbf{x}}$ and incident directions $\hat{\mathbf{x}}'$. Introduce the transformation on $\mathbb{R}^2$:

$$\Psi_\epsilon(\mathbf{x}) := \frac{1}{\epsilon}\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^2.$$

Then arguing as in the acoustic and electromagnetic case [11, 12] we have

$$\mathbf{u}_\alpha^\infty[\lambda\circ\Psi_\epsilon,\mu\circ\Psi_\epsilon,\rho\circ\Psi_\epsilon,\omega] = \mathbf{u}_\alpha^\infty[\lambda,\mu,\rho,\epsilon\omega] = o(\epsilon^{2N-2}) \quad \text{for all} \quad \epsilon \le \epsilon_0.$$

Note that the medium $(\lambda\circ\Psi_\epsilon,\mu\circ\Psi_\epsilon,\rho\circ\Psi_\epsilon)$ is a homogeneous multi-coated structure of radius 2ϵ.

We now apply the transformation invariance of the Lamé system to the medium $(\lambda\circ\Psi_\epsilon,\mu\circ\Psi_\epsilon,\rho\circ\Psi_\epsilon)$. Recall that the elastic wave propagation in such a homogeneous isotropic medium can be restated as

$$\nabla\cdot(\mathfrak{C}:\nabla\mathbf{u}) + \omega^2(\rho\circ\Psi_\epsilon)\mathbf{u} = 0 \quad \text{in} \quad \mathbb{R}^2,$$

where $\mathfrak{C} = (C_{ijkl})_{i,j,k,l=1}^N$ is the fourth-rank stiffness tensor defined by

$$C_{ijkl}(\mathbf{x}) = (\lambda\circ\Psi_\epsilon)\,\delta_{i,j}\delta_{k,l} + (\mu\circ\Psi_\epsilon)\,(\delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k}), \tag{5.17}$$

and the action of $\mathfrak{C}$ on a matrix $\mathbf{A} = (a_{ij})_{i,j=1,2}$ is defined as

$$\mathfrak{C} : \mathbf{A} = (\mathfrak{C} : \mathbf{A})_{i,j=1}^{2} = \left(\sum_{k,l=1,2} C_{ijkl}\, a_{kl} \right)_{i,j=1,2}. \tag{5.18}$$

In the case of a generic anisotropic elastic material, the stiffness tensor satisfies the following symmetries

$$\text{major symmetry:} \quad C_{ijkl} = C_{klij}, \qquad \text{minor symmetries:} \quad C_{ijkl} = C_{jikl} = C_{ijlk}, \tag{5.19}$$

for all $i, j, k, l = 1, 2$. Let $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2) = F_\epsilon(\mathbf{x}) : \mathbb{R}^2 \to \mathbb{R}^2$ be a bi-Lipschitz and orientation-preserving transformation such that $F_\epsilon(\{|\mathbf{x}| < \epsilon\}) = \{|\tilde{\mathbf{x}}| < 1\}$ and that the region $|\mathbf{x}| \geq 2$ remains invariant under the transformation. This implies that we have blown up a small traction-free disk of radius $\epsilon < 1$ to the unit disk centered at the origin. The push-forwards of $\mathfrak{C}$ and ρ are defined respectively by

$$(F_\epsilon)_*\mathfrak{C} := \widehat{\mathfrak{C}} = \left(\widehat{C}_{iqkp}(\tilde{x})\right)_{i,q,k,p=1}^{2} = \left(\frac{1}{\det(\mathbf{M})} \left\{ \sum_{l,j=1,2} C_{ijkl} \frac{\partial \tilde{x}_p}{\partial x_l} \frac{\partial \tilde{x}_q}{\partial x_j} \right\} \Big|_{x=F_\epsilon^{-1}(\tilde{x})} \right)_{i,q,k,p=1,2},$$

$$(F_\epsilon)_*\rho := \widehat{\rho} = \left(\frac{\rho}{\det(\mathbf{M})} \right) \Big|_{x=F_\epsilon^{-1}(\tilde{x})}, \quad \mathbf{M} = \left(\frac{\partial \tilde{x}_i}{\partial x_j} \right)_{i,j=1,2}.$$

We need the following lemma (see, for instance, [28, 31]).

Lemma 5.3. *The function* $\mathbf{u}$ *is a solution to* $\nabla \cdot (\mathfrak{C} : \nabla u) + \omega^2 \rho \mathbf{u} = 0$ *in* $\mathbb{R}^2$ *if and only if* $\widehat{\mathbf{u}} = \mathbf{u} \circ (F_\epsilon)^{-1}$ *satisfies* $\widehat{\nabla} \cdot (\widehat{\mathfrak{C}} : \widehat{\nabla} \widehat{\mathbf{u}}) + \omega^2 \widehat{\rho} \widehat{\mathbf{u}} = 0$ *in* $\mathbb{R}^2$*, where* $\widehat{\nabla}$ *denotes the gradient operator w.r.t to transformed variable* $\tilde{\mathbf{x}}$.

Applying the above lemma to the Lamé system (5.17) we obtain the following result.

Theorem 5.4. *If* (λ, μ, ρ) *is a nearly S-vanishing structure of order* N *at low frequencies, there exists* $\epsilon_0 > 0$ *such that*

$$\mathbf{u}_\alpha^\infty[(F_\epsilon)_*\mathfrak{C}, (F_\epsilon)_*(\rho \circ \Psi_\epsilon), \omega](\mathbf{x}, \mathbf{x}') = o(\epsilon^{2N-2}), \quad \alpha = P, S,$$

for all $\epsilon < \epsilon_0$*, uniformly in all* $\mathbf{x}$ *and* $\mathbf{x}'$*. Here the stiff tensor* $\mathfrak{C}$ *is defined by (5.17). Moreover, the elastic medium* $((F_\epsilon)_*\mathfrak{C}, (F_\epsilon)_*(\rho \circ \Psi_\epsilon))$ *in* $1 < |\mathbf{x}| < 2$ *is a nearly cloaking device for the* hidden *region* $|\mathbf{x}| < 1$.

Theorem 5.4 implies that for any frequency ω and any integer number N there exist $\epsilon_0 = \epsilon_0(\omega, N) > 0$ and the elastic medium $((F_\epsilon)_*\mathfrak{C}, (F_\epsilon)_*(\rho \circ \Psi_\epsilon))$ with $\epsilon < \epsilon_0$ such that the nearly cloaking enhancement can be achieved at the order $o(\epsilon^{2N-2})$. We finish this section with the following remarks.

Remark 5.5. *Unlike the acoustic and electromagnetic case, the transformed elastic tensor* $(F_\epsilon)_*\mathfrak{C}$ *is not anisotropic, since it possesses the major symmetry only. Note that the transformed mass density* $(F_\epsilon)_*(\rho \circ \Psi_\epsilon)$ *is still isotropic. In fact, it has been pointed out by Milton, Briane and Willis [31] that the invariance of the Lamé system can be achieved only if one relaxes the assumption on the minor symmetry of the transformed elastic tensor. This has led Norris and Shuvalov [34] and Parnell [36] to explore the elastic cloaking by using Cosserat material or by employing non-linear pre-stress in a neo-Hookean elastomeric material.*

Remark 5.6. *We have designed an enhanced nearly cloaking device for general incoming elastic plane waves. A device for cloaking only compressional or shear wave can be analogously constructed by using the corresponding elastic scattering coefficients.*

6 Discussion

In this article elastic scattering coefficients (ESC) of characteristically small inclusions are introduced using surface vector harmonics based cylindrical solutions to Lamé equations and the multipolar expansions of elastic fields based on them. It is established that the scattered field and the far field scattering amplitudes due to the incidence of a general plane wave admit natural expansions in terms of ESC. This connection substantiates their utility in direct and inverse elastic scattering. The scattering coefficients of a three dimensional elastic inclusion can be analogously defined using three dimensional vector spherical harmonics and specially constructed vector wave functions. An added complication in three dimensions is that there are three wave-modes (P, SV and SH modes) which cannot be completely decoupled. It can be easily proved that the ESC possess similar properties in three dimensions.

The decay rates and symmetry of the ESC with respect to indices and wave-modes are also discussed. These properties indicate that only first few coefficients are significant and sufficient to cater to a variety of scattering problems. The higher order ESC contain fine details of shape oscillations and geometric features of the inclusion. Thus, the largest order of stably recoverable ESC determines the maximal resolving power of the imaging setup and determines the resolution limit in feature extraction frameworks.

For reconstructing significant ESC from multi-static response data we have formulated a truncated linear system of equations where the truncation parameter can be tuned depending on the requirements of the actual physical problem, stability constraints, truncation error and the measurement noise. This truncated system is converted to a matrix system wherein all the ESC upto truncation order are arranged in a matrix which appears to be Hermitian and ill-conditioned (in the sense of rapidly decaying singular values) thanks to decay, symmetry and stability results proved herein. This observation is pertinent to designing subspace migration type shape identification frameworks in elastic media. Moreover, shape descriptors and invariants of elastic objects can also be designed using ESC.

Finally, as an application of ESC we constructed the scattering coefficients vanishing structure and elucidated that such structures can be used to enhance the performance of nearly elastic cloaking devices. The results present in the article are not restricted to only two dimensions and can be easily extended to three dimensional media.

In future studies, the role of ESC in mathematical imaging, especially from the perspectives of designing shape invariants and descriptors in elastic media, will be investigated. Moreover, in order to handle inverse elastic mediums scattering problems and to understanding the super-resolution phenomena in elastic media, the concept of inhomogeneous ESC will be discussed.

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