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Stabilized Galerkin for Transient Advection of Differential Forms

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Abstract

We deal with the discretization of generalized transient advection problems for differential forms on bounded spatial domains. We pursue an Eulerian method of lines approach with explicit time-stepping. Concerning spatial discretization we extend the jump stabilized Galerkin discretization proposed in [H. Heumann and R. Hiptmair, *Stabilized Galerkin methods for magnetic advection*, Math. Modelling Numer. Analysis, 47 (2013), pp. 1713–1732] to forms of any degree and, in particular, advection velocities that may have discontinuities resolved by the mesh. A rigorous a priori convergence theory is established for Lipschitz continuous velocities, conforming meshes and standard finite element spaces of discrete differential forms. However, numerical experiments furnish evidence of the good performance of the new method also in the presence of jumps of the advection velocity.

1 Introduction

The equations of magneto-hydrodynamics (MHD) [16, Section 3.8], [22, Section 4.1] provide a consistent description of the interaction of electromagnetic fields and conducting non-magnetic fluids like plasmas. The standard model for resistive MHD under a quasi-neutrality assumption comprises balance equations for mass, momentum and energy together with material laws and Maxwell's equations in their magneto-quasistatic reduction (eddy current model) for the electromagnetic fields.

The traditional formulation of the linear eddy current model in the presence of a conducting fluid moving with velocity $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{x}, t)$ boils down to the evolution PDE

$$\partial_t \mathbf{u} + \mathbf{curl}\,\varepsilon\,\mathbf{curl}\,\mathbf{u} + \alpha\mathbf{u} + \mathbf{curl}\,\mathbf{u} \times \boldsymbol{\beta} + \mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{u}) = \mathbf{f}, \tag{1.1}$$

governing the evolution of the unknown magnetic vector potential $\mathbf{u}$ and with ε being the magnetic diffusion coefficient. Alternatively, one may rely on the magnetic induction field as unknown, again denoted by $\mathbf{u}$, which yields

$$\partial_t \mathbf{u} - \mathbf{grad}\,\varepsilon\,\mathrm{div}\mathbf{u} + \alpha\mathbf{u} + \boldsymbol{\beta}\mathrm{div}\mathbf{u} + \mathbf{curl}\,(\mathbf{u} \times \boldsymbol{\beta}) = \mathbf{f}. \tag{1.2}$$

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1.1 Generalized advection-diffusion evolution problem

Both (1.1) and (1.2) belong to a single family of second order evolution problems, which we have dubbed *generalized advection-diffusion* problems. For a unified statement we rely on the language of exterior calculus. In that notation, the strong form of the generalized advection-diffusion equation in the space-time domain $\Omega \times I$, $I := [0, T]$, written in terms of differential forms reads

$$\begin{aligned} \star\partial_t\omega(t) + (-1)^{k+1}\mathsf{d}\varepsilon \star \mathsf{d}\omega(t) + \star\alpha\omega(t) + \star\mathrm{L}_{\boldsymbol{\beta}}\omega(t) &= f(t), && \text{in } \Omega \times I\,, \\ \operatorname{tr}\omega(t) &= \operatorname{tr} g(t), && \text{on } (\Gamma_{\text{in}} \cup \Gamma_0) \times I\,, \\ \operatorname{tr}(\mathsf{i}_{\mathbf{n}}\omega(t)) &= \operatorname{tr} s(t), && \text{on } \Gamma_{\text{in}} \times I\,, \\ \omega(0) &= \omega_0, && \text{in } \Omega\,. \end{aligned} \tag{1.3}$$

where $\omega(t)$ is a time-dependent differential k-form on the bounded domain $\Omega \subset \mathbb{R}^n$, $\boldsymbol{\beta} : \overline{\Omega} \times I \to \mathbb{R}^n$ is a *given* velocity field and $f(t) \in \Lambda^{n-k}(\Omega)$ a source term. The scalar diffusivity parameter ε and the reaction coefficient α are non-negative and bounded functions $\Omega \to \mathbb{R}$, and the boundary conditions are imposed at the inflow boundary $\Gamma_{\text{in}} := \{\mathbf{x} \in \partial\Omega : \boldsymbol{\beta} \cdot \mathbf{n}(\mathbf{x}) < 0\}$ and at the "elliptic boundary" (where the diffusion parameter $\varepsilon > 0$). All other notations are borrowed from [24, Section 2] and summarized in Table 1.1.

Symbol	Meaning in exterior calculus
$\Lambda^k(\Omega)$	: space of (smooth) differential forms on a bounded domain $\Omega \subset \mathbb{R}^n$
d	: exterior derivative operator $\Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ [37, Definition 1.2.2 e)]
δ	: adjoint of the exterior derivative $\star\delta = (-1)^k\mathsf{d}\star$, [37, Definition 1.2.2 f)]
$\mathsf{i}_{\boldsymbol{\beta}}$	: contraction $\Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$ with vector field $\boldsymbol{\beta}$ [37, Definition 1.2.2 d)]
$\mathsf{j}_{\boldsymbol{\beta}}\omega$	: adjoint of the contraction $\mathsf{i}_{\boldsymbol{\beta}}$, $\star\mathsf{j}_{\boldsymbol{\beta}} = (-1)^k\mathsf{i}_{\boldsymbol{\beta}}\star$, [24, Definition 2.2 (10)]
$\mathrm{L}_{\boldsymbol{\beta}}$	: Lie derivative $\Lambda^k(\Omega) \to \Lambda^k(\Omega)$ associated with vector field $\boldsymbol{\beta}$
$\mathcal{L}_{\boldsymbol{\beta}}$	: adjoint of the Lie derivative operator $\mathrm{L}_{\boldsymbol{\beta}}$
tr	: trace operator $\Lambda^k(\Omega) \to \Lambda^k(\partial\Omega)$ [37, p. 26]
$\wedge$	: wedge or exterior product $\Lambda^k(\Omega) \times \Lambda^\ell(\Omega) \to \Lambda^{k+\ell}(\Omega)$ [37, Definition 1.2.2 a)]
$\star$	: Euclidean Hodge operator $\Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ [37, Definition 1.2.2 c)]
$H\Lambda^k(\Omega)$	: Sobolev (Hilbert) space of k-forms, [5, Section 2.2]

Table 1.1: Notations from exterior calculus; for details see [24, Section 2] or [26, Sections 2.1 and 2.2] or compendia on differential geometry.

The so-called vector proxies[1] establish the connection between (1.1), (1.2) and (1.3). Indeed, in $\mathbb{R}^3$ endowed with the Euclidean inner product natural isomorphisms between $\Lambda^k(\mathbb{R}^3)$ and $\mathbb{R}$ or $\mathbb{R}^3$ can be defined. The fields associated to differential forms are called *proxy fields* for the forms and exterior calculus operations on forms correspond to operations on scalar functions and vector fields, see [24, Section 2], [5, Table 2.1], or Table 1.2 and Table 1.3. From these identifications we see that (1.1), (1.2) correspond to (1.3) for $k = 1$ and $k = 2$, respectively.

For $k = 0$ the evolution operator in (1.3) written in vector proxies becomes the familiar and widely studied second order advection-diffusion equation for the unknown scalar function u

$$\partial_t u - \operatorname{div}\varepsilon\,\mathbf{grad}u + \alpha u + \boldsymbol{\beta} \cdot \mathbf{grad}u = f. \tag{1.4}$$

[1] The term "vector proxy" was coined in [7]

$\omega \in \Lambda^k(\Omega)$	$k=0$	$k=1$	$k=2$	$k=3$
$\mathsf{d}\omega$	$\mathbf{grad}u$	$\mathbf{curl}\,\mathbf{u}$	$\mathrm{div}\mathbf{u}$	$-$
$\mathsf{i}_{\boldsymbol{\beta}}\omega$	$-$	$\boldsymbol{\beta}\cdot\mathbf{u}$	$\mathbf{u}\times\boldsymbol{\beta}$	$u\boldsymbol{\beta}$
$\delta\omega$	$-$	$-\mathrm{div}\mathbf{u}$	$\mathbf{curl}\,\mathbf{u}$	$-\mathbf{grad}u$
$\mathsf{j}_{\boldsymbol{\beta}}\omega$	$u\boldsymbol{\beta}$	$-\mathbf{u}\times\boldsymbol{\beta}$	$\boldsymbol{\beta}\cdot\mathbf{u}$	$-$
$\mathrm{L}_{\boldsymbol{\beta}}\omega$	$\boldsymbol{\beta}\cdot\mathbf{grad}u$	$\mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u})+\mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta}$	$\mathbf{curl}(\mathbf{u}\times\boldsymbol{\beta})+\boldsymbol{\beta}\mathrm{div}\mathbf{u}$	$\mathrm{div}(u\boldsymbol{\beta})$
$\mathcal{L}_{\boldsymbol{\beta}}\omega$	$-\mathrm{div}(u\boldsymbol{\beta})$	$\mathbf{curl}(\boldsymbol{\beta}\times\mathbf{u})-\boldsymbol{\beta}\mathrm{div}\mathbf{u}$	$\boldsymbol{\beta}\times\mathbf{curl}\,\mathbf{u}-\mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u})$	$-\boldsymbol{\beta}\cdot\mathbf{grad}u$
tr	$u(x)$	$\mathbf{n}(x)\times\mathbf{u}(x)$	$\mathbf{u}(x)\cdot\mathbf{n}(x)$	$-$
$H\Lambda^k(\Omega)$	$\mathrm{H}^1(\Omega)$	$\mathrm{H}(\mathbf{curl},\Omega)$	$\mathrm{H}(\mathrm{div},\Omega)$	$\mathrm{L}^2(\Omega)$

Table 1.2: Exterior calculus notations and corresponding expressions for vector proxies. For details see [24, Table 2], [26, Section 2.2], [5, Table 2.2].

	Exterior calculus	Proxy calculus
$\wedge$ product	$\wedge : \Lambda^1(\mathbb{R}^3)\times\Lambda^1(\mathbb{R}^3)\to\Lambda^2(\mathbb{R}^3)$ $\wedge : \Lambda^1(\mathbb{R}^3)\times\Lambda^2(\mathbb{R}^3)\to\Lambda^3(\mathbb{R}^3)$	$\times : \mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ (cross product) $\cdot : \mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ (dot product)
Hodge operator $\star$	$\star : \Lambda^0(\mathbb{R}^3)\to\Lambda^3(\mathbb{R}^3)$ $\star : \Lambda^1(\mathbb{R}^3)\to\Lambda^2(\mathbb{R}^3)$	id: $\mathbb{R}\to\mathbb{R}$ id: $\mathbb{R}^3\to\mathbb{R}^3$

Table 1.3: Correspondence between wedge product and Hodge operator for differential forms and vector proxies. More details in [26, Section 2.2], [5, Table 2.1].

By analogy we conclude that in the generalized advection-diffusion problem (1.3), the diffusion operator is $\mathsf{d}\star\mathsf{d}$, the zero-th order term amounts to a reaction term and the advection operator is the Lie derivative $\mathrm{L}_{\boldsymbol{\beta}}$ associated to the velocity field $\boldsymbol{\beta}$.

It is well known that for scalar advection-diffusion equation (1.4) straightforward Galerkin discretization with Lagrangian finite elements will break down in the singular perturbation limit of vanishing diffusion. Thus, robustness for $\varepsilon \searrow 0$ will also be a key issue for the spatial discretization of (1.1) and (1.2). In this article we tackle the challenge of robust Eulerian spatial finite element discretization for the general advection-diffusion problem (1.3). In fact, we will focus on the pure advection problem obtained from (1.3) for $\varepsilon = 0$; if a scheme performs well in this case, it will also be suitable for (1.3) when augmented with a standard Galerkin discretization of the diffusion term.

1.2 Pure advection problem: Statement and well-posedness

We introduce the spaces

$$\begin{aligned} V &:= \{\omega \in L^2\Lambda^k(\Omega) :\ \mathrm{L}_{\boldsymbol{\beta}}\omega \in L^2\Lambda^k(\Omega),\ \int_{\Gamma_{\mathrm{in}}} \mathrm{tr}\,\mathsf{i}_{-\boldsymbol{\beta}}(\omega\wedge\star\omega) < \infty\}, \\ W &:= \{\omega \in V :\ \mathrm{tr}\,\omega = g,\ \mathrm{tr}\,\mathsf{i}_{\mathbf{n}}\omega = s \text{ on } \Gamma_{\mathrm{in}},\ g(t)\in L^2\Lambda^k(\Gamma_{\mathrm{in}}),\ s(t)\in L^2\Lambda^{k-1}(\Gamma_{\mathrm{in}})\}. \end{aligned}$$

and state the pure advection initial boundary value problem: For $f \in C^0(I; L^2\Lambda^k(\Omega))$ and $\omega_0 \in W_{|t=0}$ find $\omega \in C^1(I; L^2\Lambda^k(\Omega)) \cap C^0(I; W)$ such that

$$\begin{aligned} \partial_t\omega(t) + \alpha\omega(t) + \mathrm{L}_{\boldsymbol{\beta}}\omega(t) &= f(t), \quad \text{in } \Omega\times I\,, \\ \omega(0) &= \omega_0, \quad \text{in } \Omega\,. \end{aligned} \tag{1.5}$$

If $\boldsymbol{\beta}$ does not depend on t and $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$, the Hille-Yosida theorem in [19, Theorem 6.52] can be directly applied to show that the variational problem associated to (1.5): find $\omega \in C^1(I; L^2\Lambda^k(\Omega)) \cap C^0(I; W)$ such that, for all $\eta \in L^2\Lambda^k(\Omega)$

$$\begin{aligned}(\partial_t\omega,\eta)_\Omega + (\alpha\omega,\eta)_\Omega + (\mathrm{L}_{\boldsymbol{\beta}}\omega,\eta)_\Omega &= (f,\eta)_\Omega\\ (\omega(0),\eta)_\Omega &= (\omega_0,\eta)_\Omega\end{aligned}$$

is well-posed. Here $(\cdot,\cdot)_\Omega$ denotes the $L^2\Lambda^k(\Omega)$ inner product $(\omega,\eta)_\Omega := \int_\Omega \omega \wedge \star\eta$.

Further, for velocity fields uniformly continuous in time and Lipschitz continuous in space, that is, $\boldsymbol{\beta} \in C^0(I; W^{1,\infty}(\Omega))$, it can be shown [24, Lemma 3.4] that the monotonicity condition

$$\int_\Omega \big(\alpha + \frac{1}{2}(\mathrm{L}_{\boldsymbol{\beta}(\cdot,t)} + \mathcal{L}_{\boldsymbol{\beta}(\cdot,t)})\big)\omega \wedge \star\omega \geq \alpha_0 \int_\Omega \omega \wedge \star\omega \quad \forall\omega \in L^2\Lambda^k(\Omega)\,,\ \forall t \in I\,, \tag{1.6}$$

for some constant $\alpha_0 > 0$ and with $\mathcal{L}_{\boldsymbol{\beta}} = -(-1)^{k(n-k)} \star \mathrm{L}_{\boldsymbol{\beta}}\star$, ensures that the operator $\alpha\,\mathrm{id} + \mathrm{L}_{\boldsymbol{\beta}} : W \to L^2\Lambda^k(\Omega)$ is uniformly maximal and monotone. Hence, it can be established [34, Theorems 2.2 and 2.3] that the Lie advection operator $\mathrm{L}_{\boldsymbol{\beta}}$ is *stable* in the sense of Kato [34, Definition 2.1, p. 130]. We can therefor revert to known results from semi-group theory for hyperbolic evolution systems [34, Chapter 5.2-5.4] for well-posedness statements of (1.5).

A coordinate-based representation of Lie derivatives (see Appendix B) highlights that (1.5) falls into the class of evolution problems for the so-called Friedrichs' symmetric operators [20] and then [29, pp. 143-145] gives well-posedness of (1.5) if $\Omega = \mathbb{R}^n$.

These results require $\boldsymbol{\beta}$ to be Lipschitz continuous in space. However, MHD solutions feature shocks that give rise to *discontinuous velocities*; discontinuous transport velocities are relevant in the context of magneto-quasistatic Maxwell's equations, also in the limit of small diffusion.

A well-posedness theory for velocity fields with less regularity is available only for scalar advection. In [18] DiPerna and Lions showed well-posedness of the scalar advection problem for velocity fields $\boldsymbol{\beta} \in \mathrm{L}^1_{\mathrm{loc}}(0,T; W^{1,1}(\mathbb{R}^n))$ with $\mathrm{div}\boldsymbol{\beta} \in \mathrm{L}^1(0,T; L^\infty(\mathbb{R}^n))$ through the concept of renormalized solutions. More recently, Ambrosio in [1] provided an extension of this breakthrough to transport velocity fields in $\mathrm{L}^1_{\mathrm{loc}}(0,T; \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n))$ and $\mathrm{div}\boldsymbol{\beta} \in \mathrm{L}^1(0,T; \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n))$. Moreover, a notion of generalized flow associated with low regular velocity fields (the regular Lagrangian flow) and an extension of the characteristics theory to beyond the smooth context have been subject of investigation of several authors, see [15], [2], [8] and the references therein. To the best of our knowledge, beside the case of scalar transport, a well-posedness theory for the generalized transport problem (1.5) with low regular advection velocities has not been developed.

Even though the above mentioned results have been established for nearly incompressible velocity fields (see [17] for a detailed overview), the assumption on the boundedness of the divergence of the velocity (absolute continuity with respect to the Lebesgue measure in the BV case) is of crucial importance for the well-posedness of the scalar advection problem. In the context of the generalized transport problem for a differential k-form, this corresponds to require the operator $\mathrm{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ to be bounded in space, which conceals a rather strong assumption on the regularity of the velocity itself, when $k = 1, 2$.

1.3 Novelty and outline

A full discretization of (1.5) was already presented in [24]. There, the authors introduced a semi-Lagrangian approach. Conversely, in the present paper we pursue a mesh based Eulerian method of lines approach to (1.5), employing a (jump) stabilized Galerkin discretization and piecewise polynomial discrete differential forms for spatial discretization. Our new methods will be constructed to accommodate discontinuous velocities aligned with the mesh.

A jump-stabilized discontinuous Galerkin method for the stationary advection problem for 0-forms in $\mathbb{R}^3$ and Lipschitz continuous velocities $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$, was introduced and theoretically analyzed in [12]. An extension of these results to the magnetic advection problem (1-forms in $\mathbb{R}^3$, *cf.* (1.1)) was proposed in [25], where a priori convergence rates were derived for both fully discontinuous piecewise polynomial functions and $\mathrm{H}(\mathbf{curl}, \Omega)$-conforming finite elements. Discontinuous velocity fields were not taken into account. We remark that for discontinuous velocities, even the spatial discretization of the scalar transport problem (1.4), for which existence and uniqueness of weak solutions are known, is discussed only rarely ([9], [38]).

The remainder of the paper is organized as follows. In Section 2, we devise a stabilized Galerkin spatial semi-discretization for the generalized stationary advection problem (1.5) for merely piecewise smooth velocity $\boldsymbol{\beta}$. It is an extension of the method introduced in [12] for 0-forms and in [25] for 1-forms. Trial and test functions are polynomial discrete differential forms, which will be introduced in Section 2.5. The new method is a substantial extension of the scheme presented in [25] to forms of arbitrary degree, any spatial dimension and velocities with jumps.

Next, Section 3 establishes stability a priori convergence estimates for the stabilized Galerkin discretization in the stationary setting. For want of well-posedness results for the generalized advection problem in case of discontinuous $\boldsymbol{\beta}$, these investigations are confined to Lipschitz continuous velocities $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$. The stability and consistency results obtained in that Section are instrumental for the convergence analysis of the fully discrete scheme in Section 4. We study explicit time-stepping following the approach of [31] and [13].

Finally, in Section 5 and Section 6 the performance of the new method is tested in various numerical experiments for both the stationary and transient generalized advection problem (1.5) in 2D. The tests cover both continuous and discontinuous velocities and employ tensor product grids and triangular meshes.

2 Spatial discretization

2.1 Stationary generalized advection problem

The Eulerian method of lines policy applies time-stepping after discretization in space. Therefore, we will first address the spatial discretization of (1.5) and we start from the *stationary* generalized advection boundary value problem for a k-form ω on the bounded computational domain $\Omega \subset \mathbb{R}^n$:

$$\omega + \mathrm{L}_{\boldsymbol{\beta}}\omega = f, \quad \text{in } \Omega\,, \tag{2.1a}$$

$$\operatorname{tr}\omega = g, \quad \text{on } \Gamma_{\text{in}}\,, \tag{2.1b}$$

$$\operatorname{tr}\mathrm{i}_{\mathbf{n}}\omega = s, \quad \text{on } \Gamma_{\text{in}}\,, \tag{2.1c}$$

with $f \in L^2\Lambda^k(\Omega)$, $g \in L^2\Lambda^k(\Gamma_{\text{in}})$, $s \in L^2\Lambda^{k-1}(\Gamma_{\text{in}})$, and *piecewise* Lipschitz continuous velocity field $\boldsymbol{\beta}$. As stated in [23, p. 59], if $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$ problem (2.1) is well-posed in V under the assumption (1.6).

2.2 Transmission conditions

We aim for stabilized Galerkin methods that, crudely speaking, involve a penalization of suitable jumps across interfaces inside Ω. In order to select the right jump terms, we have to understand the natural transmission conditions across an internal interface $f \subset \Omega$ satisfied by a solution ω of (2.1).

For smooth velocity $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$, the requirement $\mathrm{L}_{\boldsymbol{\beta}}\omega \in L^2\Lambda^k(\Omega)$ read in distributional sense, involves the transmission condition

$$\operatorname{tr}\left[\mathsf{i}_{\boldsymbol{\beta}}(\omega \wedge \star\eta)\right]_f = 0 \qquad \forall\, \eta \in C_0^\infty\Lambda^k(\Omega), \tag{2.2}$$

for any oriented (piecewise) smooth $n{-}1$-dimensional surface $f \subset \Omega$. Here, we wrote $[\cdot]_f$ for the jump of a function across the surface f. This formula is a consequence of the integration by parts formula for the Lie derivative

$$(\mathrm{L}_{\boldsymbol{\beta}}\omega, \eta)_\Omega - (\omega, \mathcal{L}_{\boldsymbol{\beta}}\eta)_\Omega = \int_{\partial\Omega} \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}}(\omega \wedge \star\eta) \quad \forall \omega, \eta \in C^\infty\Lambda^k(\Omega). \tag{2.3}$$

The transmission conditions (2.2) carry over to Lipschitz continuous velocity $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$. Clearly, no transmission conditions are imposed across surfaces tangential to $\boldsymbol{\beta}$ (characteristic surfaces).

In case of discontinuous velocity $\boldsymbol{\beta}$, an interpretation of $\mathrm{L}_{\boldsymbol{\beta}}\omega$ in the sense of distributions is no longer available. Therefore, at jumps of $\boldsymbol{\beta}$ resort to a strong interpretation of $\mathrm{L}_{\boldsymbol{\beta}}\omega$. Appealing to Cartan's homotopy formula (see for example [37, Equation 2.3] or [30, Theorem 14.35]) $\mathrm{L}_{\boldsymbol{\beta}} = \mathsf{d}\mathsf{i}_{\boldsymbol{\beta}} + \mathsf{i}_{\boldsymbol{\beta}}\mathsf{d}$, we conclude the strong transmission conditions

$$\operatorname{tr}[\omega]_f = \operatorname{tr}\left[\mathsf{i}_{\boldsymbol{\beta}}\omega\right]_f = 0 \qquad \forall \text{ oriented surfaces } f \subset \Omega,\ [\boldsymbol{\beta}]_f \neq 0, \tag{2.4}$$

from demanding $\omega \in L^2\Lambda^k(\Omega)$, $\mathsf{i}_{\boldsymbol{\beta}}\omega \in L^2\Lambda^{k-1}(\Omega)$ and $\mathsf{d}\omega \in L^2\Lambda^{k+1}(\Omega)$.

2.3 Stabilized Galerkin variational formulation

In the following, let $\mathcal{T}_h = \{T\}$ be a cellular partition (generalized triangulation) of $\Omega \subset \mathbb{R}^n$ into (curved) polyhedra T. Denote by $\mathcal{F}^\circ$ and $\mathcal{F}^\partial$ the set of interior and boundary $n{-}1$-faces of $\mathcal{T}_h$ (named facets) and $\mathcal{F} = \mathcal{F}^\circ \cup \mathcal{F}^\partial$. The set of facets at the inflow boundary is defined as $\mathcal{F}^\partial_- := \{f \in \mathcal{F}^\partial :\ f \subset \Gamma_{\text{in}}\}$ and $\overline{\Gamma}_{\text{in}} = \cup_{f\in\mathcal{F}^\partial_-} f$, whereas $\mathcal{F}^\partial \setminus \mathcal{F}^\partial_-$ is the set of facets at the outflow boundary. An oriented facet f has a distinguished normal $\mathbf{n}_f$. Any facet f, as part of the boundary of some element $T \in \mathcal{T}_h$, has either $\mathbf{n}_f = \mathbf{n}_{T|_f}$ or $\mathbf{n}_f = -\mathbf{n}_{T|_f}$. Then, given $\omega \in \Lambda^k(\Omega)$, its two different restrictions to f are denoted by ω^+ and ω^-, e.g. $\omega^+ := \omega_{|_{T^+}}$ where element T^+ has outward normal $\mathbf{n}_f$. Hence, we can introduce the notion of jump and average across a facet $f \in \mathcal{F}^\circ$ as

$$[\omega]_f := \omega^+ - \omega^-, \qquad \{\omega\}_f := \frac{1}{2}(\omega^+ + \omega^-).$$

For $f \subset \partial\Omega$ we assume f to be oriented such that $\mathbf{n}_f$ points outwards and $[\omega]_f = \{\omega\}_f := \omega$. We also write $h_T := \operatorname{diam} T$ and $h := \max_{T\in\mathcal{T}_h} h_T$.

Further, let $\Lambda_h^k(\mathcal{T}_h)$ denote some piecewise polynomial approximation space for differential k-forms. Here $\Lambda_h^k(\mathcal{T}_h)$ could be either a $H\Lambda^k(\Omega)$-conforming space $\Lambda_h^k(\mathcal{T}_h) \subset H\Lambda^k(\Omega)$ or a non-conforming space $\Lambda_h^k(\mathcal{T}_h) \subset L^2\Lambda^k(\Omega)$ for which $\Lambda_h^k(\mathcal{T}_h) \not\subset H\Lambda^k(\Omega)$.

The method is formulated in the general framework of time-dependent velocity fields $\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{x}, t)$and relies on the assumption that the possible (space) discontinuities of the velocity are resolved by the mesh:

Assumption 2.1. For every $t \in I$ we have $\boldsymbol{\beta}(\cdot,t)_{|T} \in W^{1,\infty}(T)$ for each $T \in \mathcal{T}_h$, that is the velocity field is assumed to be $\mathcal{T}_h$-piecewise Lipschitz continuous.

This may seem to be a severe limitation but for our purposes it represents a reasonable condition in view of the fact that the velocity field is obtained from numerically solving the MHD system.

Next, multiplying equation (2.1a) by a test form $\eta_h \in \Lambda_h^k(\mathcal{T}_h)$ and applying the integration by parts rule (2.3), results in

$$(\alpha\omega_h, \eta_h)_\Omega + \sum_{T\in\mathcal{T}_h} (\omega_h, \mathcal{L}_{\boldsymbol{\beta}}\eta_h)_T + \sum_{T\in\mathcal{T}_h} \int_{\partial T} \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}}(\omega_h \wedge \star\eta_h) = (f, \eta_h)_\Omega \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T}_h).$$

Let $\mathsf{j}_{\boldsymbol{\beta}}$ be the formal adjoint of the contraction operator $\mathsf{i}_{\boldsymbol{\beta}}$ as in Table 1.1. Applying the following product rule

$$\mathsf{i}_{\boldsymbol{\beta}}(\omega \wedge \star\eta) = \mathsf{i}_{\boldsymbol{\beta}}\omega \wedge \star\eta + (-1)^{k+\ell}\omega \wedge \star\mathsf{j}_{\boldsymbol{\beta}}\eta \quad \forall\, \omega \in \Lambda^k(\Omega), \eta \in \Lambda^\ell(\Omega) \tag{2.5}$$

to the boundary terms, results in, for all $\eta_h \in \Lambda_h^k(\mathcal{T}_h)$

$$(\alpha\omega_h, \eta_h)_\Omega + \sum_{T\in\mathcal{T}_h} (\omega_h, \mathcal{L}_{\boldsymbol{\beta}}\eta_h)_T + \sum_{f\in\mathcal{F}} \int_f \operatorname{tr}\left[\mathsf{i}_{\boldsymbol{\beta}}\omega_h \wedge \star\eta_h\right]_f + \int_f \operatorname{tr}\left[\omega_h \wedge \star\mathsf{j}_{\boldsymbol{\beta}}\eta_h\right]_f = (f, \eta_h)_\Omega.$$

Moreover, it can be easily verified that, for all $\mu_h, \eta_h \in \Lambda_h^k(\mathcal{T}_h)$, it holds

$$\sum_{f\in\mathcal{F}} \int_f \operatorname{tr}\left[\mu_h \wedge \star\eta_h\right]_f = \sum_{f\in\mathcal{F}} \int_f \operatorname{tr}(\{\mu_h\}_f \wedge \star[\eta_h]_f) + \sum_{f\in\mathcal{F}^\circ} \int_f \operatorname{tr}([\mu_h]_f \wedge \star\{\eta_h\}_f).$$

For $\omega \in W$ solution of problem (2.1), the transmission conditions (2.4) at the mesh facets

$$\operatorname{tr}[\omega]_f = \operatorname{tr}[\mathsf{i}_{\boldsymbol{\beta}}\omega]_f = 0 \qquad \forall\, f \in \mathcal{F}^\circ,$$

yield the variational formulation: find $\omega_h \in \Lambda_h^k(\mathcal{T}_h)$ such that $\mathsf{a_h}(\omega_h, \eta_h) = l(\eta_h)$ for all $\eta_h \in \Lambda_h^k(\mathcal{T}_h)$, where

$$l(\eta_h) := (f, \eta_h)_\Omega - \sum_{f\in\mathcal{F}_-^\partial} \int_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}}(g \wedge \star\eta_h) \tag{2.6}$$

$$\begin{aligned}\mathsf{a_h}(\omega_h, \eta_h) := (\alpha\omega_h, \eta_h)_\Omega + \sum_{T\in\mathcal{T}_h} (\omega_h, \mathcal{L}_{\boldsymbol{\beta}}\eta_h)_T + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}_-^\partial} \int_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}}(\omega_h \wedge \star\eta_h) + \\ + \sum_{f\in\mathcal{F}^\circ} \int_f \operatorname{tr}(\{\mathsf{i}_{\boldsymbol{\beta}}\omega_h\}_f \wedge \star[\eta_h]_f) + \int_f \operatorname{tr}(\{\omega_h\}_f \wedge \star[\mathsf{j}_{\boldsymbol{\beta}}\eta_h]_f).\end{aligned} \tag{2.7}$$

As it is well-known, classical Galerkin finite element discretization of advection problems suffer from instabilities. Therefore, devising stabilization techniques to counteract this limitation has been investigated widely. We consider the following stabilization operator

$$\mathsf{s_h}(\omega_h, \eta_h) := \sum_{f\in\mathcal{F}^\circ} \int_f c_f \operatorname{tr}([\mathsf{i}_{\boldsymbol{\beta}}\omega_h]_f \wedge \star[\eta_h]_f) + \int_f \bar{c}_f \operatorname{tr}([\omega_h]_f \wedge \star[\mathsf{j}_{\boldsymbol{\beta}}\eta_h]_f) \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T}_h) \tag{2.8}$$

where the stabilization scalar functions $c_f(\mathbf{x})$ and $\bar{c}_f(\mathbf{x})$ may depend on the velocity field and on the facets diameter h_f. Throughout, the stabilization parameters are assumed to satisfy the following:

Assumption 2.2. We assume that $c_f(\mathbf{x})$ and $\bar{c}_f(\mathbf{x})$ satisfy: $c_f\boldsymbol{\beta}\cdot\mathbf{n}_f \geq c_0 > 0$ and $\bar{c}_f\boldsymbol{\beta}\cdot\mathbf{n}_f \geq \bar{c}_0 > 0$ uniformly for all facets $f \in \mathcal{F}^\circ$.

In particular, by considering the direction of the numerical fluxes as given by the average of the velocity field, the choice

$$c_f = \bar{c}_f = \frac{1}{2}\frac{\{\boldsymbol{\beta}\}_f\cdot\mathbf{n}_f}{|\{\boldsymbol{\beta}\}_f\cdot\mathbf{n}_f|}, \qquad f\in\mathcal{F}^\circ, \tag{2.9}$$

gives a scheme with *upwind fluxes* (see [23, Remark 4.1.2] in the case $\boldsymbol{\beta}\in W^{1,\infty}(\Omega)$). Indeed, from (2.7) together with (2.8) the facets contribution, for $\omega_h,\eta_h\in\Lambda_h^k(\mathcal{T}_h)$, reads

$$\begin{aligned}
&\sum_{f\in\mathcal{F}^\circ}\int_f \operatorname{tr}(\{\mathsf{i}_{\boldsymbol{\beta}}\omega_h\}_f\wedge\star[\eta_h]_f) + \int_f c_f \operatorname{tr}([\mathsf{i}_{\boldsymbol{\beta}}\omega_h]_f\wedge\star[\eta_h]_f) + \\
&\qquad + \sum_{f\in\mathcal{F}^\circ}\int_f \operatorname{tr}(\{\omega_h\}_f\wedge\star[\mathsf{j}_{\boldsymbol{\beta}}\eta_h]_f) + \int_f c_f \operatorname{tr}([\omega_h]_f\wedge\star[\mathsf{j}_{\boldsymbol{\beta}}\eta_h]_f) = \\
&\qquad = \frac{1}{2}\sum_{f\in\mathcal{F}^\circ}\int_f \operatorname{tr}\Big((1+2c_f)(\mathsf{i}_{\boldsymbol{\beta}}\omega_h)^+\wedge\star[\eta_h]_f\Big) + \operatorname{tr}\Big((1-2c_f)(\mathsf{i}_{\boldsymbol{\beta}}\omega_h)^-\wedge\star[\eta_h]_f\Big) + \\
&\qquad + \frac{1}{2}\sum_{f\in\mathcal{F}^\circ}\int_f \operatorname{tr}\Big((1+2c_f)\omega_h^+\wedge\star[\mathsf{j}_{\boldsymbol{\beta}}\eta_h]_f\Big) + \operatorname{tr}\Big((1-2c_f)\omega_h^-\wedge\star[\mathsf{j}_{\boldsymbol{\beta}}\eta_h]_f\Big).
\end{aligned}$$

Note that, since the velocity field is discontinuous, the upwind direction at the mesh facets may not be well defined. Here we consider the direction of the stream as the one given by the average of the velocity. However, other possibilities are feasible: an upwind direction given locally by the velocity field can be used, even if this choice will lead to non-unique numerical fluxes at mesh facets.

The evaluation of the terms in (2.7) involving the Lie derivative $\mathcal{L}_{\boldsymbol{\beta}}\eta_h$ requires the knowledge of the first order derivatives of the velocity field $\boldsymbol{\beta}$. Note that since the velocity is assumed to be a smooth function in all elements $T\in\mathcal{T}_h$, the quantity $(\omega_h,\mathcal{L}_{\boldsymbol{\beta}}\eta_h)_T$ is well defined for all $T\in\mathcal{T}_h$. However, as suggested in [25], a different equivalent formulation of the bilinear form $\mathsf{a_h}(\cdot,\cdot)$ is convenient for implementation purposes.

Proposition 2.3. *The following equality holds for all $\omega_h,\eta_h\in\Lambda_h^k(\mathcal{T}_h)$,*

$$\begin{aligned}
\mathsf{a_h}(\omega_h,\eta_h) = &\ (\alpha\omega_h,\eta_h)_\Omega + \sum_{T\in\mathcal{T}_h}(\mathsf{i}_{\boldsymbol{\beta}}\mathsf{d}\omega_h,\eta_h)_T + (\omega_h,\mathsf{j}_{\boldsymbol{\beta}}\delta\eta_h)_T + \\
&+ \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-}\int_f \operatorname{tr}(\mathsf{i}_{\boldsymbol{\beta}}\omega_h\wedge\star\eta_h) - \sum_{f\in\mathcal{F}^\partial_-}\int_f \operatorname{tr}(\omega_h\wedge\star\mathsf{j}_{\boldsymbol{\beta}}\eta_h) + \\
&+ \sum_{f\in\mathcal{F}^\circ}\int_f \operatorname{tr}(\{\mathsf{i}_{\boldsymbol{\beta}}\omega_h\}_f\wedge\star[\eta_h]_f) - \int_f \operatorname{tr}([\omega_h]_f\wedge\star\{\mathsf{j}_{\boldsymbol{\beta}}\eta_h\}_f).
\end{aligned}\tag{2.10}$$

Proof. By using a Leibniz rule for the exterior derivative with respect to the wedge product

$$\mathsf{d}(\omega\wedge\star\eta) = \mathsf{d}\omega\wedge\star\eta + (-1)^{k+\ell}\omega\wedge\star\delta\eta \qquad \omega\in\Lambda^k(\Omega),\ \eta\in\Lambda^\ell(\Omega)$$

and Stokes' theorem [37, Theorem 1.2.7], it easily follows that

$$\int_{\partial\Omega}\operatorname{tr}(\omega\wedge\star\mu) = (\mathsf{d}\omega,\mu)_\Omega - (\omega,\delta\mu)_\Omega \qquad \forall\omega\in\Lambda^k(\Omega),\ \mu\in\Lambda^{k+1}(\Omega). \tag{2.11}$$

Hence, using (2.11) together with Cartan's homotopy formula for the adjoint of the Lie derivative $\mathcal{L}_{\boldsymbol{\beta}}$ results in

$$(\omega_h, \delta \mathsf{j}_{\boldsymbol{\beta}} \eta_h)_\Omega = (\mathsf{i}_{\boldsymbol{\beta}} \mathsf{d}\omega_h, \eta_h)_\Omega - \int_{\partial\Omega} \operatorname{tr}(\omega_h \wedge \star \mathsf{j}_{\boldsymbol{\beta}} \eta_h).$$

Furthermore, exploiting (2.5), yields

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} (\omega_h, \delta \mathsf{j}_{\boldsymbol{\beta}} \eta_h)_T = &\sum_{T\in\mathcal{T}_h} (\mathsf{i}_{\boldsymbol{\beta}} \mathsf{d}\omega_h, \eta_h)_T - \sum_{f\in\mathcal{F}^\partial} \int_f \operatorname{tr}(\omega_h \wedge \star \mathsf{j}_{\boldsymbol{\beta}} \eta_h) + \\ &+ \sum_{f\in\mathcal{F}^\circ} - \int_f \operatorname{tr}(\{\omega_h\}_f \wedge \star [\mathsf{j}_{\boldsymbol{\beta}} \eta_h]_f) - \int_f \operatorname{tr}([\omega_h]_f \wedge \star \{\mathsf{j}_{\boldsymbol{\beta}} \eta_h\}_f).\end{aligned} \tag{2.12}$$

where the outflow boundary terms can be recast as

$$\int_f \operatorname{tr}(\mathsf{i}_{\boldsymbol{\beta}} \omega_h \wedge \star \eta_h) = \int_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}} (\omega_h \wedge \star \eta_h) - \int_f \operatorname{tr}(\omega_h \wedge \star \mathsf{j}_{\boldsymbol{\beta}} \eta_h) \quad \forall\, f \in \mathcal{F}^\partial \setminus \mathcal{F}^\partial_- . \tag{2.13}$$

Therefore, substituting (2.12) and (2.13) into the bilinear form (2.7) yields the conclusion. □

Note that if $\Lambda^k_h(\mathcal{T}_h)$ is a space of $H\Lambda^k(\Omega)$-conforming discrete differential forms, the terms $\operatorname{tr}([\omega_h]_f \wedge \star \{\mathsf{j}_{\boldsymbol{\beta}} \eta_h\}_f)$ in (2.10) and $\bar{c}_f \operatorname{tr}([\omega_h]_f \wedge \star [\mathsf{j}_{\boldsymbol{\beta}} \eta_h]_f)$ in (2.8) vanish for all $f \in \mathcal{F}^\circ$ and every $\omega_h, \eta_h \in \Lambda^k_h(\mathcal{T}_h)$.

Remark 2.4. (Lipschitz continuous velocity fields $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$)
Let us consider the particular case of velocity fields that feature Lipschitz continuity in space, that is $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$. An easy computation allows to write, for all $\omega_h, \eta_h \in \Lambda^k_h(\mathcal{T}_h)$

$$\sum_{f\in\mathcal{F}^\circ} \int_f \operatorname{tr}(\{\mathsf{i}_{\boldsymbol{\beta}} \omega_h\}_f \wedge \star [\eta_h]_f) = \sum_{f\in\mathcal{F}^\circ} \int_f \operatorname{tr}(\mathsf{i}_{\{\boldsymbol{\beta}\}_f} \{\omega_h\}_f \wedge \star [\eta_h]_f) + \frac{1}{4} \int_f \operatorname{tr}(\mathsf{i}_{[\boldsymbol{\beta}]_f} [\omega_h]_f \wedge \star [\eta_h]_f)$$

$$\sum_{f\in\mathcal{F}^\circ} \int_f \operatorname{tr}(\{\omega_h\}_f \wedge \star [\mathsf{j}_{\boldsymbol{\beta}} \eta_h]_f) = \sum_{f\in\mathcal{F}^\circ} \int_f \operatorname{tr}(\{\omega_h\}_f \wedge \star \mathsf{j}_{\{\boldsymbol{\beta}\}_f} [\eta_h]_f) + \int_f \operatorname{tr}(\{\omega_h\}_f \wedge \star \mathsf{j}_{[\boldsymbol{\beta}]_f} \{\eta_h\}_f)$$

and similarly for the stabilization terms in (2.8). Since trivially $[\boldsymbol{\beta}]_f \equiv 0$ for all $f \in \mathcal{F}^\circ$, all the terms involving the jump of the velocity can be dropped and the variational problem reduces to: find $\omega_h \in \Lambda^k_h(\mathcal{T}_h)$ such that $\mathsf{a_h}(\omega_h, \eta_h) + \mathsf{s_h}(\omega_h, \eta_h) = l(\eta_h)$ for all $\eta_h \in \Lambda^k_h(\mathcal{T}_h)$, where $l(\eta_h)$ is as in (2.6) while the stabilized bilinear form reads

$$\begin{aligned}\mathsf{a_h}(\omega_h, \eta_h) + \mathsf{s_h}(\omega_h, \eta_h) = &(\alpha\omega_h, \eta_h)_\Omega + \sum_{T\in\mathcal{T}_h} (\omega_h, \mathcal{L}_{\boldsymbol{\beta}} \eta_h)_{T\in\mathcal{T}_h} + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \int_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}} (\omega_h \wedge \star \eta_h) + \\ &+ \sum_{f\in\mathcal{F}^\circ} \int_f \operatorname{tr}(\mathsf{i}_{\boldsymbol{\beta}} \{\omega_h\}_f \wedge \star [\eta_h]_f) + \int_f \operatorname{tr}(\{\omega_h\}_f \wedge \star \mathsf{j}_{\boldsymbol{\beta}} [\eta_h]_f) + \\ &+ \sum_{f\in\mathcal{F}^\circ} \int_f c_f \operatorname{tr}(\mathsf{i}_{\boldsymbol{\beta}} [\omega_h]_f \wedge \star [\eta_h]_f) + \int_f \bar{c}_f \operatorname{tr}([\omega_h]_f \wedge \star \mathsf{j}_{\boldsymbol{\beta}} [\eta_h]_f).\end{aligned}$$

If $c_f = \bar{c}_f$, using (2.5) the bilinear form above can be recast as

$$\begin{aligned}\mathsf{a_h}(\omega_h, \eta_h) + \mathsf{s_h}(\omega_h, \eta_h) = &(\alpha\omega_h, \eta_h)_\Omega + \sum_{T\in\mathcal{T}_h} (\omega_h, \mathcal{L}_{\boldsymbol{\beta}} \eta_h)_T + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \int_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}} (\omega_h \wedge \star \eta_h) + \\ &+ \sum_{f\in\mathcal{F}^\circ} \int_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}} (\{\omega_h\}_f \wedge \star [\eta_h]_f) + \int_f c_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}} ([\omega_h]_f \wedge \star [\eta_h]_f)\end{aligned} \tag{2.14}$$

and the formulation in [23, Equation 4.8, p. 61] is recovered. Note that, owing to the fact that $\{\boldsymbol{\beta}\}_f = \boldsymbol{\beta}|_f$, the choice of stabilization given in (2.9) yields a scheme with genuine upwind fluxes. Moreover, since the stabilization terms vanish for $\omega \in W$ solution of (2.1), the variational formulation with stabilized bilinear form given by (2.7) and (2.8) is consistent with (2.1), namely Galerkin orthogonality

$$\mathsf{a_h}\left(\omega - \omega_h, \eta_h\right) + \mathsf{s_h}\left(\omega - \omega_h, \eta_h\right) = 0 \qquad \forall \eta_h \in \Lambda_h^k\left(\mathcal{T}_h\right) \tag{2.15}$$

holds for $\omega_h \in \Lambda_h^k\left(\mathcal{T}_h\right)$ numerical solution of the discretized problem. Observe that the stabilized Galerkin formulation (2.6), (2.14) for Lipschitz continuous velocities $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$ can be equivalently derived by imposing on the mesh facets the transmission conditions (2.2).

Analogously, the bilinear form corresponding to the reformulated variational problem (2.10) for $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$ and $\omega_h, \eta_h \in \Lambda_h^k\left(\mathcal{T}_h\right)$ reads:

$$\begin{aligned} \mathsf{a_h}\left(\omega_h, \eta_h\right) = & \left(\alpha\omega_h, \eta_h\right)_\Omega + \sum_{T\in\mathcal{T}_h} \left(\mathsf{i}_{\boldsymbol{\beta}}\mathsf{d}\omega_h, \eta_h\right)_T + \left(\omega_h, \mathsf{j}_{\boldsymbol{\beta}}\delta\eta_h\right)_T + \\ & + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \int_f \operatorname{tr}(\mathsf{i}_{\boldsymbol{\beta}}\omega_h \wedge \star\eta_h) - \sum_{f\in\mathcal{F}^\partial_-} \int_f \operatorname{tr}(\omega_h \wedge \star\mathsf{j}_{\boldsymbol{\beta}}\eta_h) + \\ & + \sum_{f\in\mathcal{F}^\circ} \int_f \operatorname{tr}(\mathsf{i}_{\boldsymbol{\beta}}\left\{\omega_h\right\}_f \wedge \star\left[\eta_h\right]_f) - \int_f \operatorname{tr}(\left[\omega_h\right]_f \wedge \star\mathsf{j}_{\boldsymbol{\beta}}\left\{\eta_h\right\}_f). \end{aligned} \tag{2.16}$$

2.4 Stabilized Galerkin formulation in terms of vector proxies

For the sake of completeness, we present the vector proxy representation of the stabilized reformulated bilinear form (2.10), (2.8) corresponding to the variational formulation associated with the transport problem of the corresponding k-form. Table 1.2 and Table 1.3 are used to establish the correspondences. Let V_h be finite element spaces of vector proxies associated to the spaces $\Lambda_h^k\left(\mathcal{T}_h\right)$ of polynomial differential k-forms on the mesh $\mathcal{T}_h$. Let $u, v \in V_h$ or $\mathbf{u}, \mathbf{v} \in V_h$ be the vector proxy representations of the k-forms $\omega_h, \eta_h \in \Lambda_h^k\left(\mathcal{T}_h\right)$:

$$\begin{aligned} k = 0: \quad \mathsf{a_h}\left(u, v\right) + \mathsf{s_h}\left(u, v\right) = & \int_\Omega \alpha uv\, d\mathbf{x} + \sum_{T\in\mathcal{T}_h} \int_T \boldsymbol{\beta}\cdot\mathbf{grad}uv\, d\mathbf{x} - \sum_{f\in\mathcal{F}^\partial_-} \int_f \boldsymbol{\beta}\cdot\mathbf{n}_f uv\, dS + \\ & + \sum_{f\in\mathcal{F}^\circ} - \int_f \left[u\right]_f \left\{\boldsymbol{\beta}v\right\}_f \cdot \mathbf{n}_f\, dS + \int_f \bar{c}_f \left[u\right]_f \left[\boldsymbol{\beta}v\right]_f \cdot \mathbf{n}_f\, dS. \end{aligned}$$

$$\begin{aligned} k = 1: \quad \mathsf{a_h}\left(\mathbf{u}, \mathbf{v}\right) + \mathsf{s_h}\left(\mathbf{u}, \mathbf{v}\right) = & \int_\Omega \alpha\mathbf{u}\cdot\mathbf{v}\, d\mathbf{x} + \sum_{T\in\mathcal{T}_h} \int_T (\mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta})\cdot\mathbf{v}\, d\mathbf{x} - \int_T \mathbf{u}\cdot\boldsymbol{\beta}\operatorname{div}\mathbf{v}\, d\mathbf{x} + \\ & + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \int_f (\mathbf{u}\cdot\boldsymbol{\beta})(\mathbf{v}\cdot\mathbf{n}_f)\, dS + \sum_{f\in\mathcal{F}^\partial_-} \int_f (\mathbf{u}\times\mathbf{n}_f)\cdot(\boldsymbol{\beta}\times\mathbf{v})\, dS + \\ & + \sum_{f\in\mathcal{F}^\circ} \int_f \left\{\mathbf{u}\cdot\boldsymbol{\beta}\right\}_f \left[\mathbf{v}\right]_f \cdot \mathbf{n}_f\, dS + \int_f (\left[\mathbf{u}\right]_f \times \mathbf{n}_f)\cdot\left\{\boldsymbol{\beta}\times\mathbf{v}\right\}_f\, dS + \\ & + \sum_{f\in\mathcal{F}^\circ} \int_f c_f \left[\mathbf{u}\cdot\boldsymbol{\beta}\right]_f \left[\mathbf{v}\right]_f \cdot \mathbf{n}_f\, dS - \int_f \bar{c}_f (\left[\mathbf{u}\right]_f \times \mathbf{n}_f)\cdot\left[\boldsymbol{\beta}\times\mathbf{v}\right]_f\, dS. \end{aligned}$$

$$k=2: \quad \mathsf{a_h}(\mathbf{u},\mathbf{v})+\mathsf{s_h}(\mathbf{u},\mathbf{v}) = \int_\Omega \alpha\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x} + \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\beta}\mathrm{div}\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x} + \int_T \mathbf{u}\cdot(\boldsymbol{\beta}\times\mathbf{curl}\,\mathbf{v})\,d\mathbf{x}+$$
$$+\sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-}\int_f(\mathbf{u}\times\boldsymbol{\beta})\cdot(\mathbf{v}\times\mathbf{n}_f)\,dS - \sum_{f\in\mathcal{F}^\partial_-}\int_f(\mathbf{u}\cdot\mathbf{n}_f)(\mathbf{v}\cdot\boldsymbol{\beta})\,dS+$$
$$+\sum_{f\in\mathcal{F}^\circ}\int_f\{\mathbf{u}\times\boldsymbol{\beta}\}_f\cdot([\mathbf{v}]_f\times\mathbf{n}_f)\,dS - \int_f[\mathbf{u}]_f\cdot\mathbf{n}_f\,\{\boldsymbol{\beta}\cdot\mathbf{v}\}_f\;dS+$$
$$+\sum_{f\in\mathcal{F}^\circ}\int_f c_f\,[\mathbf{u}\times\boldsymbol{\beta}]_f\cdot([\mathbf{v}]_f\times\mathbf{n}_f)\,dS + \int_f \bar{c}_f\,[\mathbf{u}]_f\cdot\mathbf{n}_f\,[\boldsymbol{\beta}\cdot\mathbf{v}]_f\;dS.$$

$$k=3: \quad \mathsf{a_h}(u,v)+\mathsf{s_h}(u,v) = \int_\Omega \alpha uv\,d\mathbf{x} - \sum_{T\in\mathcal{T}_h}\int_T u\boldsymbol{\beta}\cdot\mathbf{grad}\,v\,d\mathbf{x} - \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-}\int_f\boldsymbol{\beta}\cdot\mathbf{n}_f uv\,dS+$$
$$+\sum_{f\in\mathcal{F}^\circ}\int_f\{\boldsymbol{\beta}u\}_f\cdot\mathbf{n}_f\,[v]_f\;dS + \int_f c_f\,[\boldsymbol{\beta}u]_f\cdot\mathbf{n}_f\,[v]_f\;dS.$$

2.5 Trial and test spaces of discrete differential forms

From now, we restrict ourselves to special types of meshes:

- $\mathcal{T}_h$ is either a simplicial decomposition of $\Omega\subset\mathbb{R}^n$ as defined in [5, Section 4.1 and Section 5.3],
- or a tensor product mesh, namely a compatible, locally quasi-uniform, affine mesh partition of Ω into non-degenerate axiparallel parallelotopes.

On such meshes various families of piecewise polynomial discrete differential forms of any degree have been constructed, see [5], [3], [26, Section 3] and [4] for a detailed overview.

For a simplicial decomposition $\mathcal{T}_h$, the spaces of polynomial totally discontinuous discrete differential k-forms on $\mathcal{T}_h$ are defined as

$$\mathcal{P}^{\mathrm{d}}_r\Lambda^k(\mathcal{T}_h) := \{\omega\in L^2\,\Lambda^k(\Omega)\,,\ \omega_{|T}\in\mathcal{P}_r\Lambda^k(T),\ T\in\mathcal{T}_h\}$$

where $\mathcal{P}_r\Lambda^k(T)$ is the space of differential k-forms with polynomial coefficients of degree at most r on the n-cell $T\in\mathcal{T}_h$, obtained as the restriction of $\mathcal{P}_r\Lambda^k(\mathbb{R}^n)$ to T. The corresponding space of $H\Lambda^k(\Omega)$-conforming discrete differential forms $\mathcal{P}_r\Lambda^k(\mathcal{T}_h) := \{\omega\in H\Lambda^k(\Omega)\,,\omega_{|T}\in\mathcal{P}_r\Lambda^k(T), T\in\mathcal{T}_h\}$ allows to introduce another family of polynomial differential k-forms on $\mathcal{T}_h$, namely

$$\mathcal{P}^-_r\Lambda^k(\mathcal{T}_h) = \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h)\oplus\kappa\mathcal{H}_{r-1}\Lambda^{k+1}(\mathcal{T}_h) = \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h)+\kappa\mathcal{P}_{r-1}\Lambda^{k+1}(\mathcal{T}_h)$$

or equivalently $\mathcal{P}^-_r\Lambda^k(\mathcal{T}_h) := \{\omega\in\mathcal{P}_r\Lambda^k(\mathcal{T}_h) : \kappa\omega\in\mathcal{P}_r\Lambda^{k-1}(\mathcal{T}_h)\}$, where $\mathcal{H}_r\Lambda^k(\mathcal{T}_h)$ is the space of homogeneous polynomial differential k-forms of degree r and $\kappa:\mathcal{H}_r\Lambda^k(\mathcal{T}_h)\to\mathcal{H}_{r+1}\Lambda^{k-1}(\mathcal{T}_h)$ denotes the Koszul differential [5, Section 3.2]. The so-called "first family" of finite element differential k-forms is hence defined as $\mathcal{P}^-_r\Lambda^k(\mathcal{T}_h) = \{\omega\in H\Lambda^k(\Omega) : \omega_{|T}\in\mathcal{P}^-_r\Lambda^k(T), T\in\mathcal{T}_h\}$ whose functions satisfy the continuity requirement that the trace $\mathrm{tr}\,\omega$ is single-valued on all $n{-}1$-cells which in turn ensures inclusion in $H\Lambda^k(\Omega)$.

The family $\mathcal{Q}^-_r\Lambda^k(\mathcal{T}_h)$ of finite element differential forms on a tensor product mesh can be constructed by iteratively applying a tensor product strategy from the 1-dimensional interval.

This tensor product construction allows to build a subcomplex of the de Rham complex. We refer the interested reader to [4, Section 5] for details on such constructions.

The $H\Lambda^k(\Omega)$-conforming finite element spaces presented above over a cell complex $\mathcal{T}_h$ form a discrete de Rham sequence as a cochain projection from the de Rham complex through projection operators Π_h^k, namely the following diagram commutes

$$\begin{array}{ccccccc} H\Lambda^0(\Omega) & \xrightarrow{\mathsf{d}} & H\Lambda^1(\Omega) & \xrightarrow{\mathsf{d}} & \dots & \xrightarrow{\mathsf{d}} & H\Lambda^n(\Omega) \\ \downarrow \Pi_h^0 & & \downarrow \Pi_h^1 & & & & \downarrow \Pi_h^n \\ \Lambda_h^0(\mathcal{T}_h) & \xrightarrow{\mathsf{d}} & \Lambda_h^1(\mathcal{T}_h) & \xrightarrow{\mathsf{d}} & \dots & \xrightarrow{\mathsf{d}} & \Lambda_h^n(\mathcal{T}_h) \end{array} \tag{2.17}$$

where each $\Lambda_h^k(\mathcal{T}_h) \to \Lambda_h^{k+1}(\mathcal{T}_h)$ can be substituted by $\mathcal{P}_r\Lambda^k(\mathcal{T}_h) \to \mathcal{P}_{r-1}\Lambda^{k+1}(\mathcal{T}_h)$, or $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) \to \mathcal{P}_r^-\Lambda^{k+1}(\mathcal{T}_h)$, or $\mathcal{Q}_r^-\Lambda^k(\mathcal{T}_h) \to \mathcal{Q}_r^-\Lambda^{k+1}(\mathcal{T}_h)$ for every $r \geq 1$ (see [5, Section 5.5] and [3]).

In the subsequent analysis we will make use of pairs of $H\Lambda^k(\Omega)$-conforming spaces $\Lambda_h^k(\mathcal{T}_h)$ and non-conforming spaces $\Lambda_h^{\mathrm{d},k}(\mathcal{T}_h)$ as in the following:

(I) $\Lambda_h^{\mathrm{d},k}(\mathcal{T}_h) = \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$ and $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ with $\mathcal{T}_h$ simplicial mesh;

(II) $\Lambda_h^{\mathrm{d},k}(\mathcal{T}_h) = \{\omega \in L^2\Lambda^k(\Omega),\ \omega_{|T} \in \mathcal{P}_{r+1}^-\Lambda^k(T),\ T \in \mathcal{T}_h\}$ and $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T}_h)$ with $\mathcal{T}_h$ simplicial mesh;

(III) $\Lambda_h^{\mathrm{d},k}(\mathcal{T}_h) = \{\omega \in L^2\Lambda^k(\Omega),\ \omega_{|T} \in \mathcal{Q}_{r+1}^-\Lambda^k(T),\ T \in \mathcal{T}_h\}$ and $\Lambda_h^k(\mathcal{T}_h) = \mathcal{Q}_{r+1}^-\Lambda^k(\mathcal{T}_h)$ with $\mathcal{T}_h$ tensor product mesh.

Remark 2.5. We pay particular attention to $H\Lambda^k(\Omega)$-conforming trial/test spaces because they allow the straightforward Galerkin discretization of the diffusion form $\mathsf{d} \star \mathsf{d}$ present in (1.3).

3 Stationary transport: Estimates for continuous velocity fields

As explained in Section 1.2, a rigorous convergence analysis is only possible in the case $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$, for want of a well-posedness result for (2.1) with discontinuous velocity fields. Hence, all theoretical results will rely on the assumption $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$. Moreover, without loss of generality we can assume the scaling $\|\boldsymbol{\beta}\|_{L^\infty(\Omega)} = 1$.

Also, in this section, we admit only the same special classes of meshes as in Section 2.5. We recall the notion of shape regularity of a mesh as presented, e.g., in [26, Section 3.6] following [14, Section 3.1]. Throughout, let $\Lambda_h^k(\mathcal{T}_h)$ denote some piecewise polynomial approximation space for differential k-forms. If not otherwise specified, $\Lambda_h^k(\mathcal{T}_h)$ could be either a $H\Lambda^k(\Omega)$-conforming approximation space $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ or $\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T}_h)$, but also the totally discontinuous space $\mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$ on a simplicial mesh and $\mathcal{Q}_{r+1}^-\Lambda^k(\mathcal{T}_h)$ on a tensor product mesh. Note that we will always assume that $\Lambda_h^n(\mathcal{T}_h) = \mathcal{P}_r^{\mathrm{d}}\Lambda^n(\mathcal{T}_h)$.

Let $V(h) := \Lambda_h^k(\mathcal{T}_h) + V$, we introduce the discrete operators $A_h, S_h : V(h) \to \Lambda_h^k(\mathcal{T}_h)$ such that for all $\omega \in V(h)$, $\eta_h \in \Lambda_h^k(\mathcal{T}_h)$,

$$(A_h\omega, \eta_h)_\Omega := \mathsf{a_h}(\omega, \eta_h) \quad \text{and} \quad (S_h\omega, \eta_h)_\Omega := \mathsf{s_h}(\omega, \eta_h);$$

with $\mathsf{a_h}(\cdot,\cdot)$ and $\mathsf{s_h}(\cdot,\cdot)$ as in (2.14). Note that the bilinear form $\mathsf{s_h}(\cdot,\cdot)$ associated to the stabilization operator is symmetric and nonnegative on $V(h) \times V(h)$. Moreover, for all $\eta_h \in$

$\Lambda_h^k(\mathcal{T}_h)$, applying (2.3) to $\mathsf{a_h}(\eta_h, \eta_h)$, results in

$$
\begin{aligned}
\mathsf{a_h}(\eta_h, \eta_h) = {} & (\alpha\eta_h, \eta_h)_\Omega + \sum_{T\in\mathcal{T}_h} (\eta_h, \mathcal{L}_{\boldsymbol{\beta}}\eta_h)_T + \\
& + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \int_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}}(\eta_h \wedge \star\eta_h) + \sum_{f\in\mathcal{F}^\circ} \int_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}}(\{\eta_h\}_f \wedge \star[\eta_h]_f) = \\
= {} & \sum_{T\in\mathcal{T}_h} \left(\eta_h, \alpha + \frac{1}{2}(\mathrm{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})\eta_h\right)_T + \\
& + \frac{1}{2} \sum_{f\in\mathcal{F}\setminus\mathcal{F}^\partial_-} \int_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}}(\eta_h \wedge \star\eta_h) - \frac{1}{2} \sum_{f\in\mathcal{F}^\partial_-} \int_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}}(\eta_h \wedge \star\eta_h) = \\
= {} & \frac{1}{2} \int_{\partial\Omega} |\boldsymbol{\beta} \cdot \mathbf{n}_{\partial\Omega}| \operatorname{tr} \mathsf{i}_{\mathbf{n}_{\partial\Omega}}(\eta_h \wedge \star\eta_h) - \frac{1}{2}(\Lambda\eta_h, \eta_h)_\Omega
\end{aligned}
\tag{3.1}
$$

where $\Lambda := -(2\alpha \operatorname{id} + \mathrm{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})$. Let us introduce the following norms on $V(h)$,

$$
\|\omega\|_h^2 := \|\omega\|^2_{L^2\Lambda^k(\Omega)} + |\omega|_h^2 \,;
$$

with

$$
\begin{aligned}
|\omega|_h^2 &:= \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \|\omega\|^2_{f,\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\partial_-} \|\omega\|^2_{f,-\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\circ} \|[\omega]_f\|^2_{f,c_f\boldsymbol{\beta}} \,, \qquad (3.2)\\
\|\omega\|^2_{f,\boldsymbol{\beta}} &:= \int_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}}(\omega \wedge \star\omega) \,, \quad \text{and} \quad \|\omega\|^2_{f,c_f\boldsymbol{\beta}} := \int_f c_f \operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}}(\omega \wedge \star\omega) \,.
\end{aligned}
$$

Note that the above norms are well-defined in view of the definition of inflow and outflow boundary and the fact that $\operatorname{tr} \mathsf{i}_{\boldsymbol{\beta}}(\omega \wedge \star\omega) = (\boldsymbol{\beta} \cdot \mathbf{n}_f) \operatorname{tr} \mathsf{i}_{\mathbf{n}_f}(\omega \wedge \star\omega)$ together with Assumption 2.2 on the stabilization coefficients c_f, $f \in \mathcal{F}^\circ$. The above norms are combined into

$$
\|\omega\|_* := h^{-1/2} \|\omega\|_{L^2\Lambda^k(\Omega)} + h^{1/2} |\omega|_{H^1\Lambda^k(\mathcal{T}_h)} + \left(\sum_{f\in\mathcal{F}} \|\omega\|^2_{L^2\Lambda^k(f)}\right)^{1/2} + |\omega|_h \tag{3.3}
$$

where $|\cdot|_{H^1\Lambda^k(\mathcal{T}_h)}$ stands for the broken $H\Lambda^k$-seminorm on $\mathcal{T}_h$. Convergence estimates for the spatial discretization of the stationary boundary value problem are key to analyzing the convergence of the fully discrete scheme. They hinge on stability results for the differential operator $L_h := A_h + S_h$. In order for these results to hold for both non-conforming and $H\Lambda^k(\Omega)$-conforming space discretization, we approximate discontinuous differential forms by differential forms in $H\Lambda^k(\mathcal{T}_h)$ as in the following:

Proposition 3.1. *Let $\Lambda_h^{\mathrm{d},k}(\mathcal{T}_h)$ and $\Lambda_h^k(\mathcal{T}_h)$ be defined as in either (I), (II) or (III). Then for every $\omega \in \Lambda_h^{\mathrm{d},k}(\mathcal{T}_h)$, there exists $\omega^c \in \Lambda_h^k(\mathcal{T}_h)$ such that*

$$
\|\omega - \omega^c\|^2_{L^2\Lambda^k(\Omega)} \le Ch \sum_{f\in\mathcal{F}} \|\operatorname{tr}[\omega]\|^2_{L^2\Lambda^k(f)}
$$

with constant $C > 0$ depending only on the polynomial degree and the shape regularity of the mesh.

The proof of this result is constructive and relegated to Appendix A. Note that the construction of the $H\Lambda^k(\Omega)$-conforming approximation is based on an averaging interpolation which is the extension to discrete differential k-forms of the operator introduced and studied for scalar functions in $\mathbb{R}^3$ in [28] and [27, Appendix].

Lemma 3.2. *There exists a constant C_S depending on the stabilization coefficients $|c_f|^{1/2}$, the polynomial degree r and the shape regularity of the mesh, such that*

$$|\omega_h|_h \leq C_S h^{-1/2} \|\omega_h\|_{L^2\Lambda^k(\Omega)} \qquad \forall\, \omega_h \in \Lambda_h^k(\mathcal{T}_h)\,. \tag{3.4}$$

Moreover, for every $\omega \in V(h)$

$$\|L_h\omega\|_{L^2\Lambda^k(\Omega)} \leq C_L \|\omega\|_{L^2\Lambda^k(\Omega)} + |\omega|_{H^1\Lambda^k(\mathcal{T}_h)} + C_L' h^{-1/2}|\omega|_h; \tag{3.5}$$

where the constant C_L depends on α and $|\boldsymbol{\beta}|_{W^{1,\infty}(\Omega)}$, and the constant C_L' depends on the stabilization coefficients $|c_f|^{1/2}$, $|c_f|^{-1/2}$, the polynomial degree r and the shape regularity of the mesh.

Proof. The first inequality (3.4) immediately follows by the definition of h-seminorm in (3.2) and inverse trace inequalities [14, p. 146].

In order to show (3.5), let $\eta_h \in \Lambda_h^k(\mathcal{T}_h)$. Integration by parts yields

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} (\omega, \mathsf{j}_{\boldsymbol{\beta}}\delta\eta_h)_T = \sum_{T\in\mathcal{T}_h} (\mathsf{d}\mathsf{i}_{\boldsymbol{\beta}}\omega, \eta_h)_T - \sum_{f\in\mathcal{F}^\partial} \int_f \operatorname{tr}(\mathsf{i}_{\boldsymbol{\beta}}\omega \wedge \star\eta_h) \\ + \sum_{f\in\mathcal{F}^\circ} - \int_f \operatorname{tr}(\mathsf{i}_{\boldsymbol{\beta}}[\omega]_f \wedge \star\{\eta_h\}_f) - \int_f \operatorname{tr}(\mathsf{i}_{\boldsymbol{\beta}}\{\omega\}_f \wedge \star[\eta_h]_f).\end{aligned}$$

Inserting into the expression of $\mathsf{a_h}(\omega, \eta_h)$ in (2.16) (given in Proposition 2.3) results in

$$\begin{aligned}(L_h\omega, \eta_h)_\Omega = (\alpha\omega, \eta_h)_\Omega + \sum_{T\in\mathcal{T}_h} (\mathrm{L}_{\boldsymbol{\beta}}\omega, \eta_h)_T - \sum_{f\in\mathcal{F}_-^\partial} \int_f \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}(\omega \wedge \star\eta_h) + \\ + \sum_{f\in\mathcal{F}^\circ} - \int_f \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}([\omega]_f \wedge \star\{\eta_h\}_f) + \int_f c_f \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}([\omega]_f \wedge \star[\eta_h]_f).\end{aligned} \tag{3.6}$$

Therefore, using Cauchy-Schwarz inequality gives

$$\begin{aligned}|\,(L_h\omega, \eta_h)_\Omega\,| \leq (\|\alpha\|_{L^\infty(\Omega)} + |\boldsymbol{\beta}|_{W^{1,\infty}(\Omega)}) \|\omega\|_{L^2\Lambda^k(\Omega)} \|\eta_h\|_{L^2\Lambda^k(\Omega)} + \\ + \|\boldsymbol{\beta}\|_{L^\infty(\Omega)} |\omega|_{H^1\Lambda^k(\mathcal{T}_h)} \|\eta_h\|_{L^2\Lambda^k(\Omega)} + \sum_{f\in\mathcal{F}_-^\partial} \|\omega\|_{f,-\boldsymbol{\beta}} \|\eta_h\|_{f,-\boldsymbol{\beta}} + \\ + \sum_{f\in\mathcal{F}^\circ} \|[\omega]\|_{f,c_f\boldsymbol{\beta}} \left\|[\eta_h] - c_f^{-1}\{\eta_h\}\right\|_{f,c_f\boldsymbol{\beta}}.\end{aligned} \tag{3.7}$$

The last term can be bounded using inverse trace inequalities as follows

$$
\begin{aligned}
\sum_{f\in\mathcal{F}^\circ} \left\| [\eta_h] - c_f^{-1}\{\eta_h\} \right\|^2_{f,c_f\boldsymbol{\beta}} \leq & \sum_{\substack{f\in\mathcal{F}^\circ \\ f=\partial T^+\cap\partial T^-}} |c_f| \, \|\boldsymbol{\beta}\|_{L^\infty(\Omega)} \Big(\|\eta_h\|^2_{L^2\Lambda^k(\partial T^+)} + \|\eta_h\|^2_{L^2\Lambda^k(\partial T^-)} + \\
& + \left\| c_f^{-1}\eta_h \right\|^2_{L^2\Lambda^k(\partial T^+)} + \left\| c_f^{-1}\eta_h \right\|^2_{L^2\Lambda^k(\partial T^-)} \Big) \\
\leq & \sum_{\substack{f\in\mathcal{F}^\circ \\ f=\partial T^+\cap\partial T^-}} |c_f| \, \|\boldsymbol{\beta}\|_{L^\infty(\Omega)} \max\{1, |c_f|^{-2}\} \, \|\eta_h\|^2_{L^2\Lambda^k(\partial T^+\cup\partial T^-)} \\
\leq & \, \|\boldsymbol{\beta}\|_{L^\infty(\Omega)} \max_{f\in\mathcal{F}^\circ} \max\{|c_f|, |c_f|^{-1}\} \sum_{T\in\mathcal{T}_h} h^{-1} \|\eta_h\|^2_{L^2\Lambda^k(T)} \, .
\end{aligned}
\tag{3.8}
$$

Combining (3.7) with (3.8) and using inverse trace inequalities on the boundary terms yields

$$
\begin{aligned}
|\,(L_h\omega, \eta_h)_\Omega\,| \leq \; & C_L \|\omega\|_{L^2\Lambda^k(\Omega)} \|\eta_h\|_{L^2\Lambda^k(\Omega)} + \|\boldsymbol{\beta}\|_{L^\infty(\Omega)} |\omega|_{H^1\Lambda^k(\mathcal{T}_h)} \|\eta_h\|_{L^2\Lambda^k(\Omega)} + \\
& + C h^{-1/2} \|\boldsymbol{\beta}\|^{1/2}_{L^\infty(\Omega)} \|\eta_h\|_{L^2\Lambda^k(\Omega)} \sum_{f\in\mathcal{F}^\partial_-} \|\omega\|_{f,-\boldsymbol{\beta}} + \\
& + C h^{-1/2} \|\boldsymbol{\beta}\|^{1/2}_{L^\infty(\Omega)} \max_{f\in\mathcal{F}^\circ} \max\{|c_f|^{1/2}, |c_f|^{-1/2}\} \|\eta_h\|_{L^2\Lambda^k(\Omega)} \sum_{f\in\mathcal{F}^\circ} \|[\omega]\|_{f,c_f\boldsymbol{\beta}} \\
\leq \; & \Big(C_L \|\omega\|_{L^2\Lambda^k(\Omega)} + |\omega|_{H^1\Lambda^k(\mathcal{T}_h)} + C'_L h^{-1/2} |\omega|_h \Big) \|\eta_h\|_{L^2\Lambda^k(\Omega)} \, ,
\end{aligned}
$$

with $C'_L > 0$ depending on $|c_f|^{1/2}$ and $|c_f|^{-1/2}$, the polynomial degree and the shape regularity of the mesh. Note that the above result applies to both $H\Lambda^k(\Omega)$-conforming and non-conforming space discretization. $\square$

For the sake of conciseness, the following bounds on orthogonal subscales are presented in the case of full polynomial spaces of discrete differential k-forms on simplices, case (I), namely for the spaces $\Lambda_h^{\mathrm{d},k}(\mathcal{T}_h) = \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$ and $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$. The proof for the cases (II) and on tensor product meshes (III), follows mutatis mutandis.

Lemma 3.3. *The following statements hold true:*

(i) If Π_h denotes the L^2-orthogonal projection onto $\Lambda_h^{\mathrm{d},k}(\mathcal{T}_h)$, then there exists a constant C_π such that for all $\omega \in \Lambda_h^{\mathrm{d},k}(\mathcal{T}_h) + V$, $\eta_h \in \Lambda_h^{\mathrm{d},k}(\mathcal{T}_h)$ with $r \geq 1$, it holds

$$
|(L_h(\omega - \Pi_h\omega), \eta_h)_\Omega| \leq C_\pi \|\omega - \Pi_h\omega\|_* \|\eta_h\|_h \, ;
$$

(ii) If Π_h denotes the global L^2-orthogonal projection onto $\Lambda_h^k(\mathcal{T}_h)$, then there exists a constant C'_π such that for all $\omega \in V(h)$, $\eta_h \in \Lambda_h^k(\mathcal{T}_h)$

$$
|(L_h(\omega - \Pi_h\omega), \eta_h)_\Omega| \leq C'_\pi \|\omega - \Pi_h\omega\|_* \|\eta_h\|_h \, ;
$$

where the constants C_π and C'_π depend on α, $|\boldsymbol{\beta}|_{W^{1,\infty}(\Omega)}$, the stabilization coefficients $|c_f|^{1/2}$, $|c_f|^{-1/2}$, the polynomial degree r and the shape regularity of the mesh.

Proof. In order to show (i), we proceed as in [25, Theorem 3.1], namely one can add to the bilinear form $(L_h(\omega - \Pi_h\omega), \eta_h)_\Omega$ given in (2.14) the zero term $\sum_{T\in\mathcal{T}_h} \left(\omega - \Pi_h\omega, \mathcal{L}_{\boldsymbol{\beta}_h}\eta_h\right)_T$ where $\boldsymbol{\beta}_h$ is the L^2-projection of $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$ onto piecewise constant vector fields. Hence, using estimates on the projection error for $\boldsymbol{\beta}$ in the L^∞-norm, Cauchy-Schwarz inequality and inverse inequalities, results in

$$\begin{aligned}
|(L_h(\omega-\Pi_h\omega), \eta_h)_\Omega| &\le \|\alpha\|_{L^\infty(\Omega)} \|\omega - \Pi_h\omega\|_{L^2\Lambda^k(\Omega)} \|\eta_h\|_{L^2\Lambda^k(\Omega)} + \\
&+ (|\boldsymbol{\beta} - \boldsymbol{\beta}_h|_{W^{1,\infty}(\Omega)} + C|\boldsymbol{\beta}|_{W^{1,\infty}(\Omega)}) \|\omega - \Pi_h\omega\|_{L^2\Lambda^k(\Omega)} \|\eta_h\|_{L^2\Lambda^k(\Omega)} + \\
&+ \|\boldsymbol{\beta}\|^{1/2}_{L^\infty(\Omega)} \max_{f\in\mathcal{F}^\circ} \max\{|c_f|^{1/2}, |c_f|^{-1/2}\} |\eta_h|_h \sum_{T\in\mathcal{T}_h} \|\omega - \Pi_h\omega\|_{L^2\Lambda^k(\partial T)} \\
&+ |\eta_h|_h \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \|\omega - \Pi_h\omega\|_{f,\boldsymbol{\beta}} \le C_\pi \|\omega - \Pi_h\omega\|_* \|\eta_h\|_h \qquad \forall\, \eta_h \in \Lambda^{\mathrm{d},k}_h(\mathcal{T}_h)\,,
\end{aligned} \tag{3.9}$$

where the interior facet terms have been bounded has in (3.8).

In the case (ii) of $H\Lambda^k(\Omega)$-conforming discretization, we can proceed as above, use estimate (3.9), but the non-zero term $\sum_{T\in\mathcal{T}_h} \left(\omega - \Pi_h\omega, -\mathcal{L}_{\boldsymbol{\beta}_h}\eta_h\right)_T$ has to be bounded. We show that for all $\omega \in V(h)$, $\eta_h \in \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$,

$$\sum_{T\in\mathcal{T}_h} \left| \left(\omega - \Pi_h\omega, \mathcal{L}_{\boldsymbol{\beta}_h}\eta_h\right)_T\right| \le Ch^{-1/2} \|\omega - \Pi_h\omega\|_{L^2\Lambda^k(\Omega)} \|\eta_h\|_h\,, \tag{3.10}$$

with the constant $C > 0$ depending only on the polynomial degree and the shape regularity of the mesh. In order to do that, we exploit the fact that since $\boldsymbol{\beta}_h$ is piecewise constant, for all $T \in \mathcal{T}_h$, $\mathrm{L}_{\boldsymbol{\beta}_h} = -\mathcal{L}_{\boldsymbol{\beta}_h}$ and we build $H\Lambda^k(\Omega)$-conforming approximations for each of the two terms appearing in Cartan's formula $\mathrm{L}_{\boldsymbol{\beta}_h} = \mathsf{i}_{\boldsymbol{\beta}_h}\mathsf{d} + \mathsf{d}\mathsf{i}_{\boldsymbol{\beta}_h}$. In particular, in view of Proposition 3.1, let $\gamma^{c,k} \in \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ be the $H\Lambda^k(\Omega)$-conforming approximation of $\mathsf{i}_{\boldsymbol{\beta}_h}\mathsf{d}\eta_h \in \mathcal{P}^{\mathrm{d}}_r\Lambda^k(\mathcal{T}_h)$ and let $\gamma^{c,k-1} \in \mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{T}_h)$ be the $H\Lambda^k(\Omega)$-conforming approximation of $\mathsf{i}_{\boldsymbol{\beta}_h}\eta_h \in \mathcal{P}^{\mathrm{d}}_{r+1}\Lambda^{k-1}(\mathcal{T}_h)$. Since $\gamma^{c,k}, \mathsf{d}\gamma^{c,k-1} \in \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$,

$$\left(\omega - \Pi_h\omega, \mathrm{L}_{\boldsymbol{\beta}_h}\eta_h\right)_T = \left(\omega - \Pi_h\omega, \mathsf{i}_{\boldsymbol{\beta}_h}\mathsf{d}\eta_h - \gamma^{c,k}\right)_T + \left(\omega - \Pi_h\omega, \mathsf{d}(\mathsf{i}_{\boldsymbol{\beta}_h}\eta_h - \gamma^{c,k-1})\right)_T,$$

and by Cauchy-Schwarz inequality and inverse inequality, one has

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} \left| \left(\omega - \Pi_h\omega, \mathrm{L}_{\boldsymbol{\beta}_h}\eta_h\right)_T\right| &\le \|\omega - \Pi_h\omega\|_{L^2\Lambda^k(\Omega)} \left\|\mathsf{i}_{\boldsymbol{\beta}_h}\mathsf{d}\eta_h - \gamma^{c,k}\right\|_{L^2\Lambda^k(\Omega)} + \\
&+ Ch^{-1} \|\omega - \Pi_h\omega\|_{L^2\Lambda^k(\Omega)} \left\|\mathsf{i}_{\boldsymbol{\beta}_h}\eta_h - \gamma^{c,k-1}\right\|_{L^2\Lambda^{k-1}(\Omega)}.
\end{aligned}$$

By the approximation results in Proposition 3.1, the projection errors can be bounded as

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} \left| \left(\omega - \Pi_h\omega, \mathrm{L}_{\boldsymbol{\beta}_h}\eta_h\right)_T\right| &\le Ch \|\omega - \Pi_h\omega\|_{L^2\Lambda^k(\Omega)} \left(\sum_{f\in\mathcal{F}} \left\|\mathrm{tr}\left[\mathsf{i}_{\boldsymbol{\beta}_h}\mathsf{d}\eta_h\right]_f\right\|^2_{L^2\Lambda^k(f)}\right)^{1/2} + \\
&+ \|\omega - \Pi_h\omega\|_{L^2\Lambda^k(\Omega)} \left(\sum_{f\in\mathcal{F}} \left\|\mathrm{tr}\left[\mathsf{i}_{\boldsymbol{\beta}_h}\eta_h\right]_f\right\|^2_{L^2\Lambda^{k-1}(f)}\right)^{1/2}.
\end{aligned}$$

Upper bounds for the facet terms can be derived as follows: Let us decompose the velocity field $\boldsymbol{\beta}$ into its normal component $\boldsymbol{\beta}_n := (\boldsymbol{\beta}\cdot\mathbf{n})\mathbf{n}$ and its tangential component $\boldsymbol{\beta}_t := (\mathbf{n}\times\boldsymbol{\beta})\times\mathbf{n}$. Then we can write

$$\mathrm{tr}(\mathsf{i}_{\boldsymbol{\beta}}\eta_h) = \mathrm{tr}(\mathsf{i}_{\boldsymbol{\beta}_n}\eta_h + \mathsf{i}_{\boldsymbol{\beta}_t}\eta_h) = (\boldsymbol{\beta}\cdot\mathbf{n})\,\mathrm{tr}(\mathsf{i}_{\mathbf{n}}\eta_h) + \mathsf{i}_{\boldsymbol{\beta}_t}\,\mathrm{tr}\,\eta_h \quad \forall\eta_h \in \Lambda^k_h(\mathcal{T}_h)\,, \forall k. \tag{3.11}$$

If $f = \partial T^+ \cap \partial T^-$, using estimates on the projection error for $\boldsymbol{\beta}$, trace and inverse inequalities together with (3.11) and the fact that $\mathsf{d}\eta_h \in H\Lambda^{k+1}(\Omega)$ due to (2.17), results in

$$\begin{aligned}\left\|\operatorname{tr}\left[\mathsf{i}_{\boldsymbol{\beta}_h}\mathsf{d}\eta_h\right]_f\right\|_{L^2\Lambda^k(f)} &\leq \left\|\operatorname{tr}\left[\mathsf{i}_{(\boldsymbol{\beta}-\boldsymbol{\beta}_h)}\mathsf{d}\eta_h\right]_f\right\|_{L^2\Lambda^k(f)} + \left\|\operatorname{tr}\left[\mathsf{i}_{\boldsymbol{\beta}}\mathsf{d}\eta_h\right]_f\right\|_{L^2\Lambda^k(f)} \\ &\leq \|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_{L^\infty(\Omega)}\left\|\left[\mathsf{d}\eta_h\right]_f\right\|_{L^2\Lambda^{k+1}(f)} + \left\|(\boldsymbol{\beta}\cdot\mathbf{n}_f)\operatorname{tr}\mathsf{i}_{\mathbf{n}_f}\left[\mathsf{d}\eta_h\right]_f\right\|_{L^2\Lambda^k(f)} \\ &\leq Ch|\boldsymbol{\beta}|_{W^{1,\infty}(\Omega)}h^{-1/2}h^{-1}\|\eta_h\|_{L^2\Lambda^k(T^+\cup T^-)} + Ch^{-1}\left\|\left[\eta_h\right]_f\right\|_{f,\boldsymbol{\beta}}.\end{aligned}$$

Similarly, $\eta_h \in \mathcal{P}_r\Lambda^k(\mathcal{T}_h) \subset H\Lambda^k(\Omega)$ implies $\left[\mathsf{i}_{\boldsymbol{\beta}_t}\operatorname{tr}\eta_h\right]_f = 0$ for all $f \in \mathcal{F}$, hence

$$\begin{aligned}\left\|\operatorname{tr}\left[\mathsf{i}_{\boldsymbol{\beta}_h}\eta_h\right]_f\right\|_{L^2\Lambda^{k-1}(f)} &\leq \left\|\operatorname{tr}\left[\mathsf{i}_{(\boldsymbol{\beta}-\boldsymbol{\beta}_h)}\eta_h\right]_f\right\|_{L^2\Lambda^{k-1}(f)} + \left\|\operatorname{tr}\left[\mathsf{i}_{\boldsymbol{\beta}}\eta_h\right]_f\right\|_{L^2\Lambda^{k-1}(f)} \\ &\leq Ch|\boldsymbol{\beta}|_{W^{1,\infty}(\Omega)}h^{-1/2}\|\eta_h\|_{L^2\Lambda^k(T^+\cup T^-)} + \left\|\left[\eta_h\right]_f\right\|_{f,\boldsymbol{\beta}}\end{aligned}$$

which leads to the desired estimate (3.10). Finally, combining the estimates (3.9) and (3.10) yields

$$\begin{aligned}|\,(L_h(\omega-\Pi_h\omega),\eta_h)_\Omega\,| &\leq C_\pi\|\omega-\Pi_h\omega\|_*\|\eta_h\|_h + Ch^{-1/2}\|\omega-\Pi_h\omega\|_{L^2\Lambda^k(\Omega)}\|\eta_h\|_h \\ &\leq C'_\pi\|\omega-\Pi_h\omega\|_*\|\eta_h\|_h\,.\end{aligned}$$

□

Note that (3.5) together with the definition of $\|\cdot\|_*$ in (3.3), inverse and trace inequalities gives

$$\|L_h\omega\|_{L^2\Lambda^k(\Omega)} \leq Ch^{-1/2}\|\omega\|_* \leq Ch^{-1}\|\omega\|_{L^2\Lambda^k(\Omega)} \quad \forall\omega \in V(h), \tag{3.12}$$

and C depends only on the polynomial degree and the shape regularity of the mesh.

As it will be shown in the following, stability of second order Runge-Kutta schemes can be achieved with the standard CFL condition if the space discretization is performed with piecewise linear finite elements. Therefore, this case is tackled separately. In particular, we can establish the following estimate.

Lemma 3.4. *Let Π_h^0 denote the L^2-orthogonal projection onto $\mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$. In the case of space discretization with piecewise affine elements $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_1^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$, or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_1\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$, there exists a constant C_π which depends on α, $|\boldsymbol{\beta}|_{W^{1,\infty}(\Omega)}$, the stabilization coefficients $|c_f|^{1/2}$, $|c_f|^{-1/2}$ and the shape regularity of the mesh, such that for all $\omega_h, \eta_h \in \Lambda_h^k(\mathcal{T}_h)$*

$$|(L_h\omega_h, \eta_h - \Pi_h^0\eta_h)_\Omega| \leq C_\pi h^{-1/2}\|\omega_h\|_h\left\|\eta_h - \Pi_h^0\eta_h\right\|_{L^2\Lambda^k(\Omega)}.$$

Proof. The proof we propose is very similar to the proof in [13, Lemma 2.1]. Let us first consider the case of non-conforming truly discontinuous discretization, namely $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_1^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$. Using the formulation in (3.6) and proceeding similarly as in the proof of (i) from

Lemma 3.3, one has

$$
\begin{aligned}
|(L_h\omega_h,\eta_h-\Pi_h^0\eta_h)_\Omega| &\le \|\alpha\|_{L^\infty(\Omega)}\|\omega_h\|_{L^2\Lambda^k(\Omega)}\left\|\eta_h-\Pi_h^0\eta_h\right\|_{L^2\Lambda^k(\Omega)}+\\
&\quad+(|\boldsymbol{\beta}-\boldsymbol{\beta}_h|_{W^{1,\infty}(\Omega)}+C|\boldsymbol{\beta}|_{W^{1,\infty}(\Omega)})\|\omega_h\|_{L^2\Lambda^k(\Omega)}\left\|\eta_h-\Pi_h^0\eta_h\right\|_{L^2\Lambda^k(\Omega)}+\\
&\quad+C\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}^{1/2}\max_{f\in\mathcal{F}^\circ}\max\{|c_f|^{1/2},|c_f|^{-1/2}\}h^{-1/2}\left\|\eta_h-\Pi_h^0\eta_h\right\|_{L^2\Lambda^k(\Omega)}\sum_{f\in\mathcal{F}^\circ}\|[\omega_h]\|_{f,c_f\boldsymbol{\beta}}+\\
&\quad+C\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}^{1/2}h^{-1/2}\left\|\eta_h-\Pi_h^0\eta_h\right\|_{L^2\Lambda^k(\Omega)}\sum_{f\in\mathcal{F}_-^\partial}\|\omega_h\|_{f,-\boldsymbol{\beta}}\\
&\le C_\pi h^{-1/2}\|\omega_h\|_h\left\|\eta_h-\Pi_h^0\eta_h\right\|_{L^2\Lambda^k(\Omega)}
\end{aligned}
\tag{3.13}
$$

where we have used the fact that, for $\eta_h\in\mathcal{P}_1^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$, $\Pi_h^0\eta_h\in\mathcal{P}_1^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$ and $\mathrm{L}_{\boldsymbol{\beta}_h}\omega_h\in\mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$ thus $(\mathrm{L}_{\boldsymbol{\beta}_h}\omega_h,\eta_h-\Pi_h^0\eta_h)_T=0$ for all $T\in\mathcal{T}_h$.

In the case of $H\Lambda^k(\Omega)$-conforming discretization, we denote $y_h:=\eta_h-\Pi_h^0\eta_h$. Then,

$$
\begin{aligned}
|(L_h\omega_h,\Pi_h y_h)_\Omega| &= \Big|(\alpha\omega_h,\Pi_h y_h)_\Omega+\sum_{T\in\mathcal{T}_h}\big((\mathrm{L}_{\boldsymbol{\beta}}-\mathrm{L}_{\boldsymbol{\beta}_h})\omega_h,\Pi_h y_h\big)_T+\\
&\quad+\sum_{T\in\mathcal{T}_h}\big(\mathrm{L}_{\boldsymbol{\beta}_h}\omega_h,\Pi_h y_h\big)_T+\sum_{f\in\mathcal{F}_-^\partial}\int_f\operatorname{tr}\mathsf{i}_{-\boldsymbol{\beta}}(\omega_h\wedge\star\Pi_h y_h)+\\
&\quad+\sum_{f\in\mathcal{F}^\circ}\int_f c_f\operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}([\omega_h]_f\wedge\star[\Pi_h y_h]_f)-\int_f\operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}([\omega_h]_f\wedge\star\{\Pi_h y_h\}_f)\Big|\\
&\le C_\pi h^{-1/2}\|\omega_h\|_h\|\Pi_h y_h\|_{L^2\Lambda^k(\Omega)}+\sum_{T\in\mathcal{T}_h}|(\mathrm{L}_{\boldsymbol{\beta}_h}\omega_h,\Pi_h y_h)_T|\\
&\le C_\pi h^{-1/2}\|\omega_h\|_h\|y_h\|_{L^2\Lambda^k(\Omega)}+\sum_{T\in\mathcal{T}_h}|(\mathrm{L}_{\boldsymbol{\beta}_h}\omega_h,\Pi_h y_h-y_h)_T|,
\end{aligned}
$$

where we have used the estimate derived in (3.13) and again the fact that $\big(\mathrm{L}_{\boldsymbol{\beta}_h}\omega_h,y_h\big)_T=0$ for all $T\in\mathcal{T}_h$. Moreover, using the bound (3.10), there exists $C>0$ depending on the shape regularity of the mesh such that

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h}\big(\mathrm{L}_{\boldsymbol{\beta}_h}\omega_h,\Pi_h y_h-y_h\big)_T &\le Ch^{-1/2}\|\omega_h\|_h\|y_h-\Pi_h y_h\|_{L^2\Lambda^k(\Omega)}\\
&\le Ch^{-1/2}\|\omega_h\|_h\|y_h\|_{L^2\Lambda^k(\Omega)}
\end{aligned}
$$

which concludes the proof. □

As a consequence of the estimates shown in Lemma 3.2, Lemma 3.3 and Lemma 3.4, we present a convergence result for the stationary advection problem with Lipschitz continuous velocity fields. Analogous estimates were proposed in [23, Theorem 4.1.8] for non-conforming differential forms in $\mathbb{R}^n$. The present result extends to $H\Lambda^k(\Omega)$-conforming discrete differential forms in $\mathbb{R}^n$ the a priori convergence estimates derived in [12, Section 5] for 0-forms in $\mathbb{R}^3$ and in [23, Theorem 4.1.13 and 4.1.14] for 1- and 2-forms in $\mathbb{R}^3$. Note that the numerical experiments presented in Section 6 for non-Lipschitz velocities indicate that the following convergence result might hold in a more general setting.

Theorem 3.5. *Let $\alpha \in L^\infty(\Omega)$ and $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$ in (1.5) satisfy the monotonicity condition (1.6). Furthermore, let the stabilization parameters c_f fulfill the non-negativity Assumption 2.2. Then*

$$\mathsf{a_h}(\omega, \omega) + \mathsf{s_h}(\omega, \omega) \geq \min\left\{\frac{1}{2}\alpha_0, 1\right\} \|\omega\|_h^2 \quad \forall \omega \in \Lambda_h^k(\mathcal{T}_h). \tag{3.14}$$

Moreover, if $\omega \in H^{r+1}\Lambda^k(\Omega)$ is solution of the advection problem (1.5) and $\omega_h \in \Lambda_h^k(\mathcal{T}_h)$ is solution of the discrete variational formulation with bilinear form given in (2.14), then

$$\|\omega - \omega_h\|_h \leq Ch^{r+1/2} \|\omega\|_{H^{r+1}\Lambda^k(\Omega)}$$

with the constant $C > 0$ depending on $|c_f|$, $|c_f|^{-1}$, α, $\boldsymbol{\beta}$, the polynomial degree r and the shape regularity of the mesh.

Proof. The proof of stability (3.14) immediately follows by (3.1), the positivity condition (1.6) and the definition of the h-norm.

Let Π_h denote the L^2-projection into $\Lambda_h^k(\mathcal{T}_h)$. By stability and consistency (2.15), one has

$$\min\left\{\frac{1}{2}\alpha_0, 1\right\} \|\omega - \Pi_h\omega\|_h^2 \leq |(L_h(\omega - \Pi_h\omega), \eta_h)_\Omega|$$

where $\eta_h := \omega_h - \Pi_h\omega$. We proceed as in the proof of Lemma 3.3 (i) to get (3.9) and use a multiplicative trace inequality (see [10, Theorem 1.6.6]) for the interior facets terms, i.e.

$$\|\omega - \Pi_h\omega\|_{L^2\Lambda^k(\partial T)}^2 \leq C\left(h_T^{-1}\|\omega - \Pi_h\omega\|_{L^2\Lambda^k(T)}^2 + h_T|\omega - \Pi_h\omega|_{H^1\Lambda^k(T)}^2\right)$$

with C depending only on the shape of T. Moreover, in the case of $H\Lambda^k(\Omega)$-conforming discrete differential forms the extra non-zero terms are bounded as in (3.10). The approximation estimates [5, Theorem 5.3]

$$\inf_{\mu_h \in \mathcal{P}_r\Lambda^k(T)} \|\omega - \mu_h\|_{L^2\Lambda^k(T)} \leq Ch^{r+1}\|\omega\|_{H^{r+1}\Lambda^k(T)};$$
$$\inf_{\mu_h \in \mathcal{P}_r\Lambda^k(T)} \|\omega - \mu_h\|_{H^1\Lambda^k(T)} \leq Ch^{r}\|\omega\|_{H^{r+1}\Lambda^k(T)};$$

for $C > 0$ independent of h, yield the conclusion. □

Note that in the non-stabilized case ($c_f = 0$), the bilinear form in (2.14) is coercive in the $L^2\Lambda^k$-norm but only sub-optimal convergence is attained, namely $\|\omega - \omega_h\|_{L^2\Lambda^k(\Omega)} \leq Ch^r\|\omega\|_{H^{r+1}\Lambda^k(\Omega)}$ holds with $C > 0$ independent of the mesh width h.

4 Fully discrete problem

In the present section, we formulate the fully discrete advection problem for a differential k-form by coupling the stabilized Galerkin spatial discretization introduced in Section 2 with explicit time-stepping schemes. In particular, the forward Euler method and explicit second-order and third-order Runge-Kutta (RK) schemes are investigated.

On the time interval $I = [0, T]$, we consider a uniform partition $\bigcup_{n=0}^{N-1}[t^n, t^{n+1}]$ for a given positive integer N and $t^n = n\tau$ with uniform time step τ such that $T = N\tau$. The semi-discrete problem reads: find $\omega_h(t) \in \Lambda_h^k(\mathcal{T}_h)$ such that

$$\begin{aligned} (\partial_t\omega_h(t), \eta_h)_\Omega + (L_h(t)\omega_h(t), \eta_h)_\Omega &= l(t)(\eta_h) && \forall \eta_h \in \Lambda_h^k(\mathcal{T}_h) \\ (\omega_h(0), \eta_h)_\Omega &= \left(\omega^0, \eta_h\right)_\Omega && \forall \eta_h \in \Lambda_h^k(\mathcal{T}_h) \end{aligned} \tag{4.1}$$

where the bilinear forms $\mathsf{a_h}(\cdot,\cdot)$, $\mathsf{s_h}(\cdot,\cdot)$ and the linear functional $l(\cdot)$ are obtained at each time step through spatial discretization as in (2.7), (2.8) and (2.6) with forcing term $f(\mathbf{x},t)$ and velocity field $\boldsymbol{\beta}(\mathbf{x},t)$. The semi-discrete problem (4.1) can equivalently be recast as the finite dimensional operator evolution equation

$$\partial_t \omega_h(t) + L_h(t)\omega_h(t) = F_h(t) \quad \forall t \in [0,T] \tag{4.2}$$

where $F_h(t) \in \Lambda_h^k(\mathcal{T}_h)$ is such that $(F_h(t), \eta_h)_\Omega = l(t)(\eta_h)$ for all $\eta_h \in \Lambda_h^k(\mathcal{T}_h)$.

In light of the results established in Lemma 3.2, Lemma 3.3 and Lemma 3.4, quasi-optimal convergence rates for the $L^2\Lambda^k$-error in space L^∞-error in time can be proven for smooth solutions of the problem (4.1), along the lines of the analysis proposed by Burman et al. in [13]. In particular, under CFL-type conditions, the efficacy of the proposed space-time discretization lies in the fact that the anti-diffusive nature of explicit RK schemes is compensated by the artificial dissipation introduced through the stabilized spatial discretization.

In the following paragraphs, we introduce the fully discrete problem for (4.2) by explicitly stating the stages corresponding to the time-stepping. Moreover, we present the convergence results corresponding to each fully discrete scheme.

4.1 Explicit Euler Scheme

The first order explicit Euler scheme for the problem (4.2) reads

$$\omega_h^{n+1} = \omega_h^n - \tau L_h^n \omega_h^n + \tau F_h^n \tag{4.3}$$

where $\omega_h^n = \omega_h(t^n)$, $L_h^n := L_h(t^n)$ and $F_h^n := F_h(t^n)$.

Theorem 4.1. *Let $\omega \in C^0(0,T; H^{r+1}\Lambda^k(\Omega)) \cap C^2(0,T; L^2\Lambda^k(\Omega))$ be the exact solution of* (1.5) *and let $\{\omega_h^n\}_{n=1}^N \subset \Lambda_h^k(\mathcal{T}_h)$ be the discrete solution of problem* (4.3)*. Let Assumption 2.2 and the monotonicity condition* (1.6) *for $\boldsymbol{\beta} \in C^0(0,T; W^{1,\infty}(\Omega))$ hold true. Consider the trial spaces $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{Q}_{r+1}^-\Lambda^k(\mathcal{T}_h)$. Then there exist constants C, $C_{\scriptscriptstyle CFL} > 0$ depending only on the constants in Lemma 3.2, Lemma 3.3 and Lemma 3.4 and the trial/test spaces $\Lambda_h^k(\mathcal{T}_h)$ such that, if*

- *(i) $\tau \le C_{\scriptscriptstyle CFL} h$, for $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$;*
- *(ii) $\tau \le C_{\scriptscriptstyle CFL} h^2$, for all other choices of $\Lambda_h^k(\mathcal{T}_h)$;*

then

$$\max_{0\le n\le N} \|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k(\Omega)} \le C(\tau + h^{r+1/2}).$$

Proof. For the sake of clarity we briefly sketch here the underlying idea proposed in [31] and [13], valid also for the higher order time schemes introduced below. For $\omega^n = \omega(t^n)$ exact solution at the n-th time step, the proof starts with writing the error generated at each stage (here one stage) of the scheme as

$$\omega^n - \omega_h^n = (\omega^n - \Pi_h\omega^n) - (\omega_h^n - \Pi_h\omega^n) =: e_\pi^n - e_h^n$$

and bounding the error e_h^n by the approximation error e_π^n. In order to do that, by deriving the equations governing the time evolution of the error e_h^n, an energy identity associated with the time scheme can be identified. Starting from this identity, the desired estimate is obtained by deriving upper bounds on each terms via the estimates established in Lemma 3.2, Lemma 3.3 and Lemma 3.4. Note that (3.12) is crucial in order to achieve stability under the CFL-condition in (ii), while the bound on the orthogonal subscales inferred in Lemma 3.3 and Lemma 3.4 is instrumental in the derivation of quasi-optimal error estimates. The conclusion follows by a Gronwall type argument and standard estimates on the projection error e_π^n. □

Remark 4.2. Note that the mild time step constraint in Theorem 4.1 (i) valid for spatial approximations based on piecewise constants discontinuous elements (finite volume) $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$ hinges on the trivial observation that the bound (3.5) in Lemma 3.2 reduces to $\|L_h\omega\|_{L^2\Lambda^k(\Omega)} \leq C_L\,\|\omega\|_{L^2\Lambda^k(\Omega)} + C_L' h^{-1/2}|\omega|_h$ for all $\omega \in V(h)$. This is the standard CFL-condition for upwind finite volume or finite difference schemes for scalar advection problems.

4.2 Explicit RK2 Schemes

We consider, as in [13, Section 3.1], explicit Runge-Kutta scheme of order two (RK2) for the problem (4.2) of the form

$$\mu_h^n = \omega_h^n - \tau L_h^n\omega_h^n + \tau F_h^n \tag{4.4}$$

$$\omega_h^{n+1} = \frac{1}{2}(\mu_h^n + \omega_h^n) - \frac{1}{2}\tau L_h^n\mu_h^n + \frac{1}{2}\tau\psi_h^n \tag{4.5}$$

where $\psi_h^n := F_h^n + \tau(\partial_t F_h)(t^n) + \delta_h^n$, for f in (1.5) sufficiently smooth in time and δ_h^n such that $\|\delta_h^n\|_{L^2\Lambda^k(\Omega)} \leq C\tau^2$.

Similarly to the explicit Euler scheme, convergence of the fully discrete problem with second order two-stage Runge-Kutta schemes of the form (4.4), (4.5) can be established. A detailed proof of the following theorem can be found in [13, Theorem 3.1 and Theorem 3.2].

Theorem 4.3. *Let $\omega \in C^0(0,T;H^{r+1}\Lambda^k(\Omega)) \cap C^3(0,T;L^2\Lambda^k(\Omega))$ be the exact solution of* (1.5) *and let $\{\omega_h^n\}_{n=1}^N \subset \Lambda_h^k(\mathcal{T}_h)$ be the discrete solution of problem* (4.4)*-*(4.5)*. Let Assumption 2.2 and the monotonicity condition* (1.6) *for $\boldsymbol{\beta} \in C^0(0,T;W^{1,\infty}(\Omega))$ hold true. Consider the trial spaces $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{Q}_{r+1}^-\Lambda^k(\mathcal{T}_h)$. Then there exist constants C, $C_{\text{CFL}} > 0$ depending only on the constants in Lemma 3.2, Lemma 3.3 and Lemma 3.4 and the trial/test spaces $\Lambda_h^k(\mathcal{T}_h)$ such that, if*

- *(i) $\tau \leq C_{\text{CFL}}h$, for a non-conforming spatial discretization with $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_1^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$ or for a $H\Lambda^k(\Omega)$-conforming approximation with spaces $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_1\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{Q}_1^-\Lambda^k(\mathcal{T}_h)$;*

- *(ii) $\tau \leq C_{\text{CFL}}h^{4/3}$, for all other choices of $\Lambda_h^k(\mathcal{T}_h)$;*

then

$$\max_{0\leq n\leq N}\|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k(\Omega)} \leq C(\tau^2 + h^{r+1/2}).$$

4.3 Explicit RK3 Schemes

The explicit Runge-Kutta scheme of order three (RK3) for the problem (4.2) as in [13, Section 4.1] reads

$$\mu_h^n = \omega_h^n - \tau L_h^n\omega_h^n + \tau F_h^n \tag{4.6}$$

$$\gamma_h^n = \frac{1}{2}(\mu_h^n + \omega_h^n) - \frac{1}{2}\tau L_h^n\mu_h^n + \frac{1}{2}\tau(F_h^n + \tau(\partial_t F_h)(t^n)) \tag{4.7}$$

$$\omega_h^{n+1} = \frac{1}{3}(\mu_h^n + \gamma_h^n + \omega_h^n) - \frac{1}{3}\tau L_h^n\mu_h^n + \frac{1}{3}\tau\psi_h^n \tag{4.8}$$

where $\psi_h^n := F_h^n + \tau(\partial_t F_h)(t^n) + \frac{1}{2}\tau^2(\partial_{tt}F_h)(t^n) + \delta_h^n$ with f in (1.5) sufficiently smooth in time and $\|\delta_h^n\|_{L^2\Lambda^k(\Omega)} \le C\tau^3$.

A detailed proof of the following theorem can be found in [13, Theorem 4.1].

Theorem 4.4. *Let $\omega \in C^0(0,T;H^{r+1}\Lambda^k(\Omega)) \cap C^4(0,T;L^2\Lambda^k(\Omega))$ be the exact solution of (1.5) and let $\{\omega_h^n\}_{n=1}^N \subset \Lambda_h^k(\mathcal{T}_h)$ be the discrete solution of problem (4.6), (4.7) and (4.8). Let Assumption 2.2 and the monotonicity condition (1.6) for $\boldsymbol{\beta} \in C^0(0,T;W^{1,\infty}(\Omega))$ hold true. Consider the trial spaces $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T}_h)$ or $\Lambda_h^k(\mathcal{T}_h) = \mathcal{Q}_{r+1}^-\Lambda^k(\mathcal{T}_h)$. Then there exist constants C, $C_{CFL} > 0$ depending only on the constants in Lemma 3.2, Lemma 3.3 and Lemma 3.4 and the trial/test spaces $\Lambda_h^k(\mathcal{T}_h)$ such that, under the 1-CFL condition $\tau \le C_{CFL}h$ it holds*

$$\max_{0\le n\le N} \|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k(\Omega)} \le C(\tau^3 + h^{r+1/2}).$$

5 Numerical experiments in 2D: continuous velocity

The two dimensional transport problem written for a 1-form amounts of solving (as in [25, Section 5])

$$\begin{aligned} \partial_t \mathbf{u} + \alpha\mathbf{u} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) - R\boldsymbol{\beta}\mathrm{div}(R\mathbf{u}) &= \mathbf{f} \quad && \text{in } \Omega\times[0,T] \\ \mathbf{u} &= \mathbf{g} && \text{on } \Gamma_{\mathrm{in}}\times[0,T] \\ \mathbf{u}(0) &= \mathbf{u}_0 && \text{in } \Omega \end{aligned} \tag{5.1}$$

where R corresponds to a clockwise rotation of $\pi/2$. We consider the pure transport ($\alpha = 0$) problem in the domain $\Omega = [-1,1]^2$ and in the time interval $[0,1]$. The time-dependent Lipschitz continuous velocity field is

$$\boldsymbol{\beta}(\mathbf{x},t) := \sin(2\pi t)(1-x^2)(1-y^2)\begin{pmatrix} y \\ -x \end{pmatrix}$$

so that there is no inflow boundary. The initial condition is defined as the "bump"

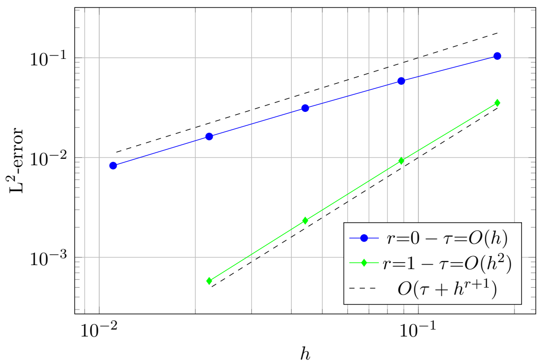
$$\mathbf{u}_0 := \begin{cases} (\varphi,\varphi)^\top & \text{if } x^2 + (y-0.25)^2 < 0.25 \\ (0,0)^\top & \text{otherwise} \end{cases} \tag{5.2}$$

with

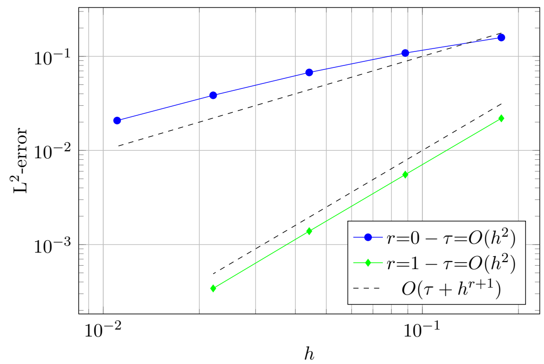
$$\varphi(x,y) := \begin{pmatrix} \cos(\pi\sqrt{x^2+(y-0.25)^2})^4 \\ \cos(\pi\sqrt{x^2+(y-0.25)^2})^4 \end{pmatrix}$$

and the forcing term is set to zero, $\mathbf{f} = (0,0)^\top$.

Since at time $t = 1$ the exact solution coincides with the initial condition $\mathbf{u}_0$ in (5.2), we compare the numerical solution obtained at the final time $T = 1$ with $\mathbf{u}_0$. Figure 5.1 shows the L^2-error at the final time, for a numerical spatial discretization based on non-conforming fully discontinuous piecewise polynomial vector-valued functions (a) and H($\mathbf{curl}$, Ω)-conforming rotated Raviart-Thomas [36] elements (b) of polynomial degree r and explicit Euler time-stepping. The CFL-condition is chosen in order to fulfill the hypotheses of Theorem 4.1. Note that the error of the fully discrete scheme attains the convergence rates predicted by the theory. Analogously, the convergence rates reported for the Heun method (Figure 5.2) and for the RK3 scheme (Figure 5.3) comply with the error behavior derived in Theorem 4.3 and Theorem 4.4, respectively.

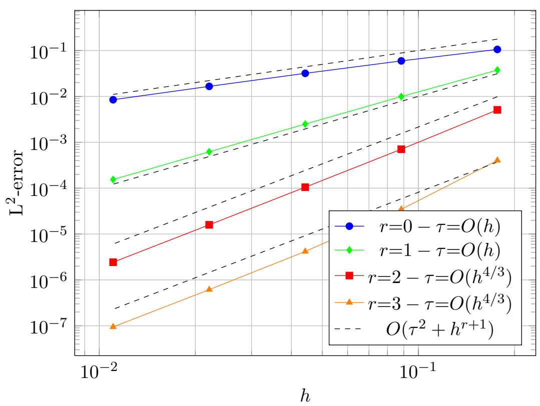


(a) $\Lambda^1_h(\mathcal{T}_h) = \mathcal{P}^{\mathrm{d}}_r\Lambda^1(\mathcal{T}_h)$

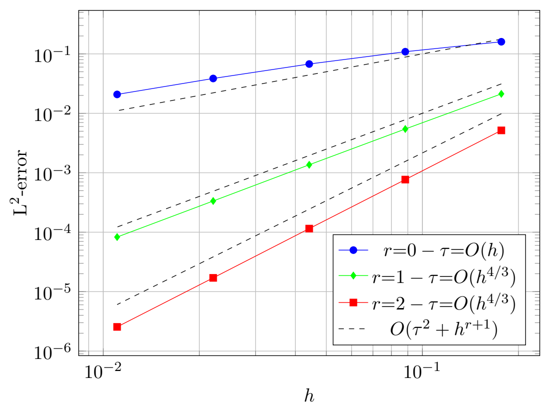


(b) $\Lambda^1_h(\mathcal{T}_h) = \mathcal{P}^{-}_{r+1}\Lambda^1(\mathcal{T}_h)$

Figure 5.1: Stabilized Galerkin discretization with upwind stabilization, polynomial differential 1-forms $\Lambda^1_h(\mathcal{T}_h)$ and explicit Euler time-stepping.

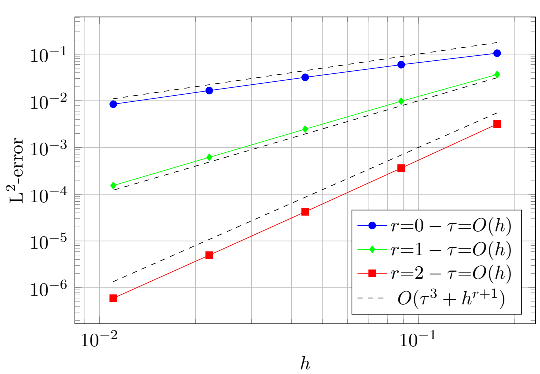


(a) $\Lambda^1_h(\mathcal{T}_h) = \mathcal{P}^{\mathrm{d}}_r\Lambda^1(\mathcal{T}_h)$

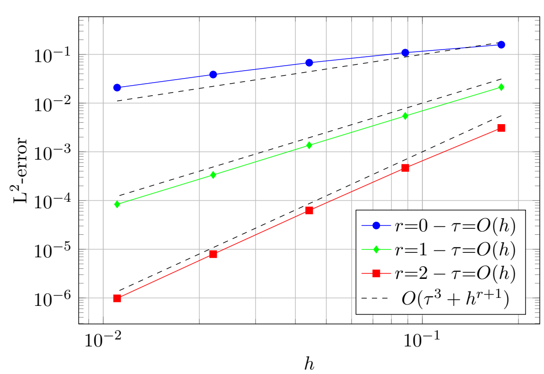


(b) $\Lambda^1_h(\mathcal{T}_h) = \mathcal{P}^{-}_{r+1}\Lambda^1(\mathcal{T}_h)$

Figure 5.2: Stabilized Galerkin discretization with upwind stabilization, polynomial differential 1-forms $\Lambda^1_h(\mathcal{T}_h)$ and second order Heun time-stepping.



(a) $\Lambda^1_h(\mathcal{T}_h) = \mathcal{P}^{\mathrm{d}}_r\Lambda^1(\mathcal{T}_h)$



(b) $\Lambda^1_h(\mathcal{T}_h) = \mathcal{P}^{-}_{r+1}\Lambda^1(\mathcal{T}_h)$

Figure 5.3: Stabilized Galerkin discretization with upwind stabilization, polynomial differential 1-forms $\Lambda^1_h(\mathcal{T}_h)$ and third order explicit Runge-Kutta time-stepping.

Remark 5.1. Though we observed an increased convergence rate of the spatial discretization (also in the stationary case) compared to the predictions of the theory, our results are probably sharp. In the case of scalar transport it is well-known [35] that on very special meshes, sometimes called Peterson meshes, the L^2-norm estimates are sharp.

6 Numerical experiments in 2D: discontinuous velocity

Lacking a sound convergence theory for the generalized advection problem in the presence of discontinuous velocity fields, the first set of experiments aims at testing the numerical performances of the stabilized Galerkin spatial discretization (proposed in Section 2) for the stationary advection problem.

6.1 Stationary problem: Test of convergence

Let us consider the stationary pure transport ($\alpha = 0$) problem corresponding to (5.1) in the unit square $\Omega = [0,1]^2$. We perform a set of numerical simulations on unstructured meshes $\{\mathcal{T}_h\}_h$ obtained by uniform refinement of an initial partition $\mathcal{T}_0$ which resolves the jump discontinuity in the velocity field $\boldsymbol{\beta} = (\beta_1, \beta_2)^\top$. The velocity is assumed to be piecewise polynomial with respect to the open subdomain partition $\Omega_1 = (0, 0.5) \times (0, 1)$ and $\Omega_2 = (0.5, 1) \times (0, 1)$. Namely,

$$\beta_1(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \Omega_1 \\ 3 & \mathbf{x} \in \Omega_2 \end{cases} \qquad \beta_2(x, y) = 2y + 1 \ \text{ in } \Omega.$$

The data $\mathbf{f}$ and $\mathbf{g}$ in (5.1) are chosen such that the strong solution of the BVP is given by the discontinuous vector field $\mathbf{u}$ with components

$$u_1(\mathbf{x}) = \begin{cases} 3 & \mathbf{x} \in \Omega_1 \\ 1 & \mathbf{x} \in \Omega_2 \end{cases} \qquad u_2(x, y) = (1 - x^2)(1 - y^2) \ \text{ in } \Omega.$$

Note that the exact solution is tangentially continuous and such that the forcing term $\mathbf{f}$ belongs to $\mathrm{L}^2(\Omega)$. We perform a numerical discretization based on:

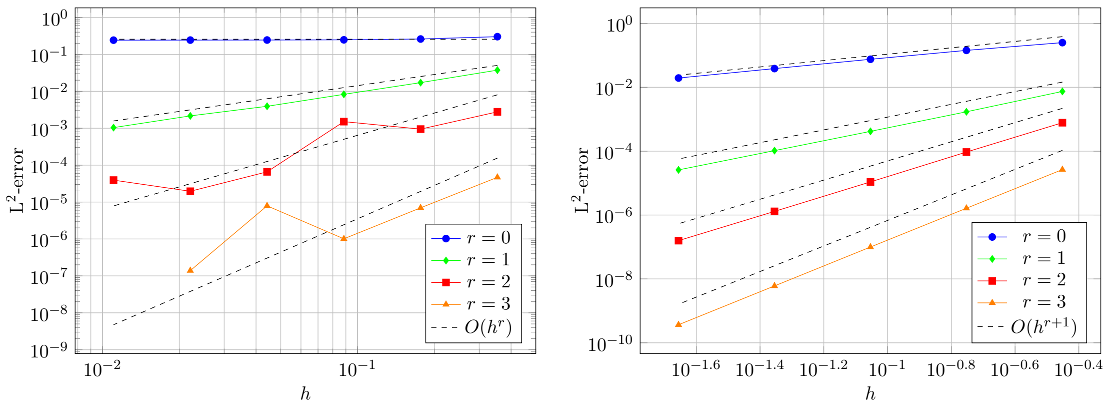
(i) $\Lambda^1_h(\mathcal{T}_h) = \mathcal{P}^-_{r+1}\Lambda^1(\mathcal{T}_h)$, rotated Raviart-Thomas elements (Figure 6.1);

(ii) $\Lambda^1_h(\mathcal{T}_h) = \mathcal{P}_r\Lambda^1(\mathcal{T}_h)$, rotated Brezzi-Douglas-Marini (BDM) elements [11] (Figure 6.2);

(iii) $\Lambda^1_h(\mathcal{T}_h) = \mathcal{P}^{\mathrm{d}}_r\Lambda^1(\mathcal{T}_h)$, piecewise polynomial discontinuous 1-forms (Figure 6.3);

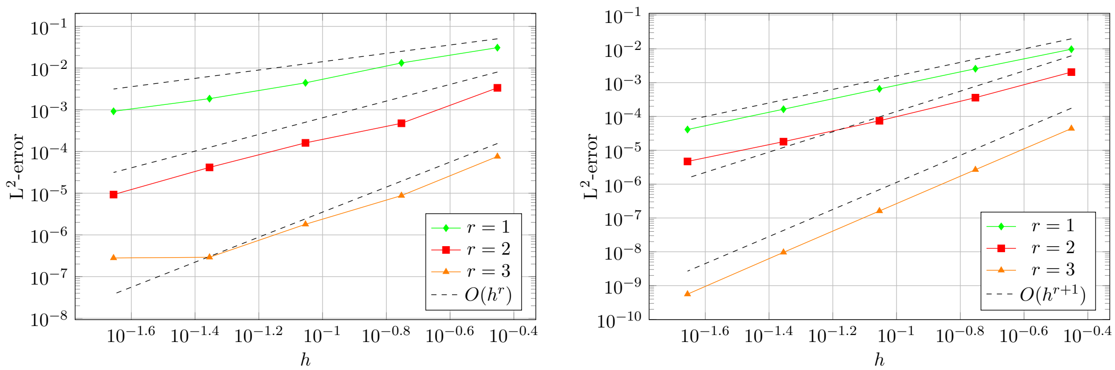
Figure 6.1, Figure 6.2 and Figure 6.3 show the behavior of the L^2-error as the mesh is refined for the non-stabilized and stabilized Galerkin spatial scheme introduced in Section 2. As with Lipschitz continuous velocity fields, convergence rate $r+1$ of the L^2-error is attained by the stabilized scheme with edge elements of polynomial degree r. For lowest order conforming elements, the rate deteriorates by a factor of 1 when the non-stabilized scheme is applied, whereas higher order polynomial discretization yields numerical solutions which suffer of large oscillations.

Note that a discretization based on the full polynomial space (case (ii), Figure 6.2) the error behaves as in the case of Lipschitz continuous velocity fields when the polynomial degree r is odd. Even polynomial degrees lead to a deteriorated convergence rate of the error in the L^2-norm.



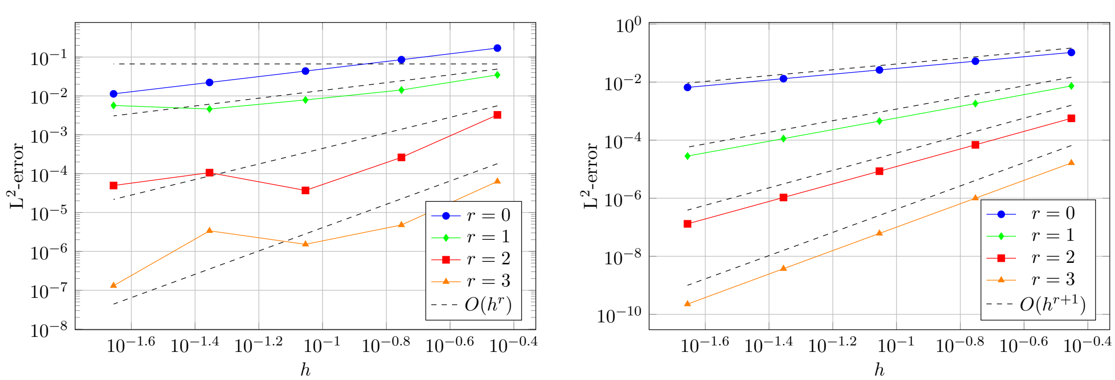
(a) Non-stabilized scheme $c_f = 0$. (b) Stabilized scheme, c_f as in (2.9).

Figure 6.1: $\mathrm{H}(\mathbf{curl}, \Omega)$-conforming finite elements of the first family, $\Lambda_h^1(\mathcal{T}_h) = \mathcal{P}_{r+1}^{-}\Lambda^1(\mathcal{T}_h)$.



(a) Non-stabilized scheme $c_f = 0$. (b) Stabilized scheme, c_f as in (2.9).

Figure 6.2: $\mathrm{H}(\mathbf{curl}, \Omega)$-conforming finite elements of the second family, $\Lambda_h^1(\mathcal{T}_h) = \mathcal{P}_r\Lambda^1(\mathcal{T}_h)$.



(a) Non-stabilized scheme $c_f = 0$. (b) Stabilized scheme, c_f as in (2.9).

Figure 6.3: Non-conforming DG finite elements $\Lambda_h^1(\mathcal{T}_h) = \mathcal{P}_r^{\mathrm{d}}\Lambda^1(\mathcal{T}_h)$.

6.2 Stationary problem: Velocity with non-resolved discontinuities

The derivation of the method, as of Section 2, relies on the assumption that the mesh resolves the possible discontinuities of the velocity field. In the following experiment we investigate an example of normal jump discontinuity in the velocity field not resolved by the mesh or any of its refinements and observe that the instabilities arising downstream of the discontinuity irremediably compromise the accuracy of the numerical solution and wreck the performance of the method. The failure of the numerical scheme in this test case may be ascribed to the fact that, since across the mesh facets the jump of the velocity vanishes, the scheme itself does not capture the discontinuity of the velocity and hence of the solution. Jump discontinuities are only taken into account through numerical quadrature.

In more details, the pure magnetic transport problem is solved in the domain $\Omega = [0,1]^2$ with a tensor product mesh and velocity field $\boldsymbol{\beta} = (\beta_1, \beta_2)$ defined component-wise as

$$\beta_1(x,y) = \begin{cases} 1 & x < y \\ 3 & x > y \end{cases} \qquad \beta_2 \equiv 1 \text{ in } \Omega.$$

The data $\mathbf{f}$ and $\mathbf{g}$ are such that the strong solution of the stationary problem corresponding to (5.1) is given by $\mathbf{u} = (u_1, u_2)$ with

$$u_1(x,y) = \begin{cases} 3\sin(\pi x) & x < y \\ \sin(\pi x) & x > y \end{cases} \qquad u_2 \equiv (1-x^2)(1-y^2) \text{ in } \Omega,$$

as shown in Figure 6.4 (first column). The numerical discretization is performed with lowest order edge elements i.e. $\Lambda^1_h(\mathcal{T}_h) = \mathcal{Q}_1^- \Lambda^1(\mathcal{T}_h)$ and upwind stabilization.

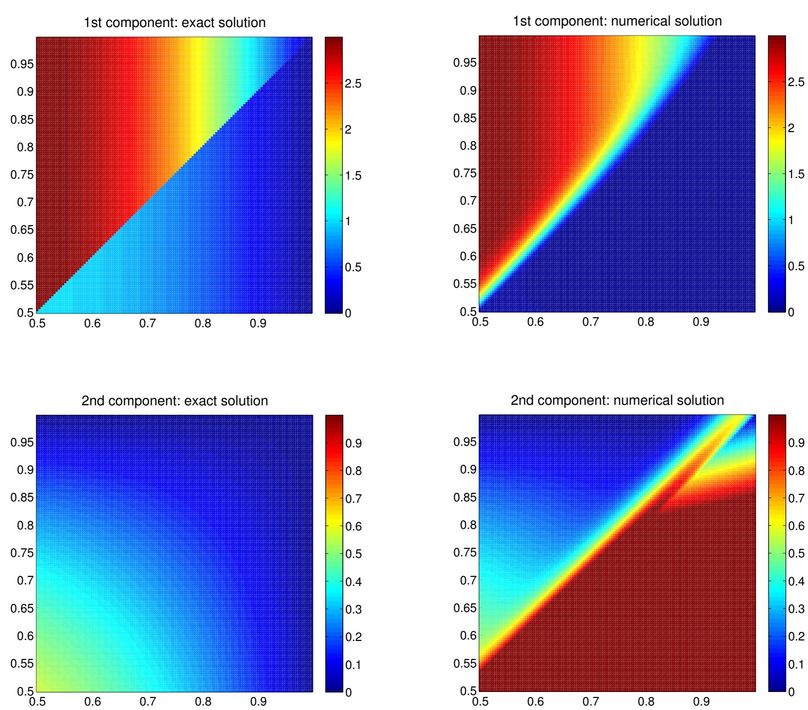


Figure 6.4: Vector components of the exact and numerical solutions for jumps of the velocity $\boldsymbol{\beta}$ not resolved by the mesh.

On the basis of Figure 6.4, it can be inferred that the numerical solution obtained with the upwind stabilized scheme is not affected by spurious oscillations but fails to reproduce the exact solution. An error analysis provides evidence of large errors along the discontinuity and no convergence is achieved.

6.3 Transient problem: Test cases, shear and collisional velocities

Encouraged by the promising performances of the scheme presented in Section 6.1 for the stationary advection problem in the presence of resolved discontinuous velocities, we tackle the full discretization of the transient problem.

Let us consider the two dimensional transient magnetic advection problem (5.1) in the space domain $\Omega = [0,2]^2$ with periodic boundary conditions at the boundary $\partial\Omega$ and time domain $I = [0,2]$. The numerical discretization is performed on a tensor product mesh with lowest order rotated Raviart-Thomas elements $\Lambda^1_h(\mathcal{T}_h) = \mathcal{Q}_1^-\Lambda^1(\mathcal{T}_h)$ and the second order Heun method as time integrator (in order to avoid restrictive time steps). Let $\mathbb{1}_S$ be the characteristic function on the subset $S \subset \Omega$, we consider different velocity fields whose discontinuities are resolved by the domain partition $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ with $\Omega_1 = (0,1)\times(0,2)$ and $\Omega_2 = (1,2)\times(0,2)$. Numerical simulations have been conducted with the following velocity fields and initial conditions:

(i) $\boldsymbol{\beta} = (0, \mathbb{1}_{\Omega_1} - \mathbb{1}_{\Omega_2})^\top$ (see Figure 6.5 (a)) and $\mathbf{u}_0(x,y) = (x(2-x)y(2-y), \sin(2\pi x))^\top$ as initial condition: The initial condition is in $\mathrm{H}(\mathbf{curl},\Omega)$, and its component in the direction of the velocity field vanishes along the discontinuity;

(ii) $\boldsymbol{\beta} = (0, \mathbb{1}_{\Omega_1} - \mathbb{1}_{\Omega_2})^\top$ (see Figure 6.5 (a)) and initial condition $\mathbf{u}_0(x,y) = (\sin(2\pi x), 4)^\top$: The initial condition is curl-free, while its contraction with the velocity field, namely the vector component in the velocity direction is not in $\mathrm{H}^1(\Omega)$;

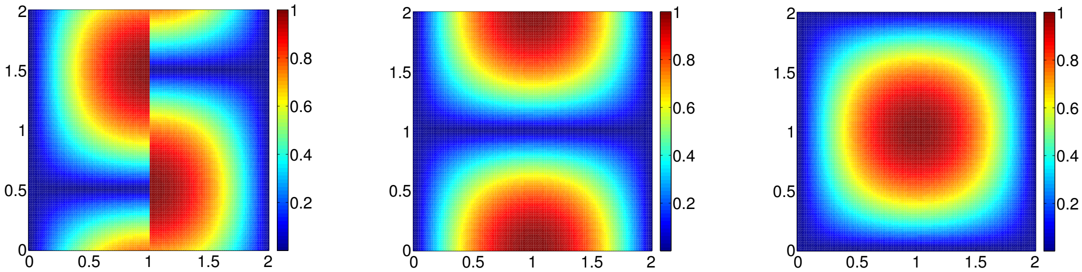
(iii) $\boldsymbol{\beta} = (\mathbb{1}_{\Omega_1} - \mathbb{1}_{\Omega_2}, 0)^\top$ (see Figure 6.5 (b)) and initial condition $\mathbf{u}_0(x,y) = (\sin(2\pi x), 4)^\top$: The initial condition is curl-free, and its component in the direction of the velocity field vanishes along the discontinuity;

(iv) $\boldsymbol{\beta} = (\mathbb{1}_{\Omega_1} - \mathbb{1}_{\Omega_2}, 0)^\top$ (see Figure 6.5 (b)) and $\mathbf{u}_0(x,y) = (x(2-x)y(2-y), \sin(2\pi x))^\top$ as initial condition: $\mathbf{u}_0$ is in $\mathrm{H}(\mathbf{curl},\Omega)$, and its component in the direction of the velocity field is not in $\mathrm{H}^1(\Omega)$.

(a) $\boldsymbol{\beta} = (0, \mathbb{1}_{\Omega_1} - \mathbb{1}_{\Omega_2})^\top$ (b) $\boldsymbol{\beta} = (\mathbb{1}_{\Omega_1} - \mathbb{1}_{\Omega_2}, 0)^\top$

Figure 6.5: Sketch of shear velocity (a) and collisional velocity (b).

For the case (i), we run the simulation for an entire period, namely until $T = 2$ and compare the solution with the initial condition. Figure 6.6 shows that the initial datum is transported in the two different domains and the L^2-error computed at the final time reaches the expected first order convergence (see Figure 6.7).



(a) Time $t = 0.5$ (b) Time $t = 1$ (c) Time $t = 2$

Figure 6.6: First component of the numerical solution obtained from the stabilized scheme (c_f as in (2.9)) with $\Lambda_h^1(\mathcal{T}_h) = \mathcal{Q}_1^- \Lambda^1(\mathcal{T}_h)$ and Heun time-stepping ($\tau = 0.1h$), for shear velocity and initial condition as in (i).

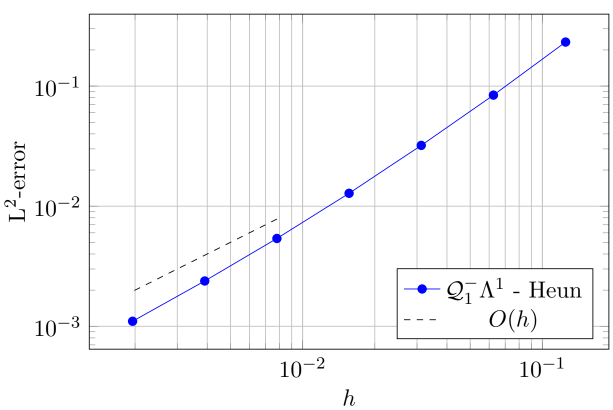
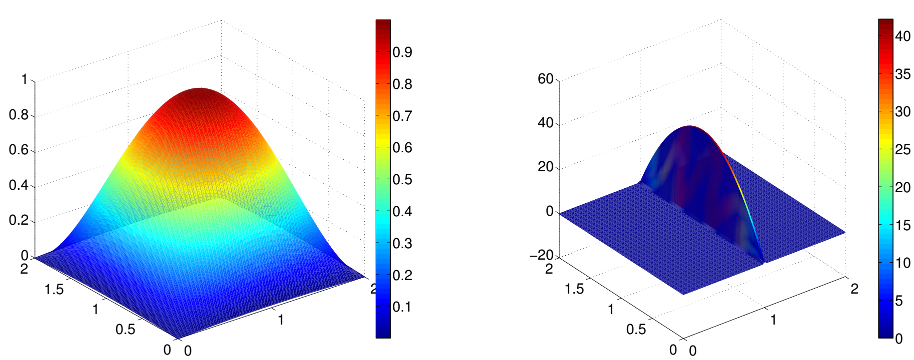


Figure 6.7: L^2-error at time T=2 of the fully discrete problem: $\Lambda_h^1(\mathcal{T}_h) = \mathcal{Q}_1^- \Lambda^1(\mathcal{T}_h)$ in space and Heun time-stepping. Shear velocity and initial condition as in (i).

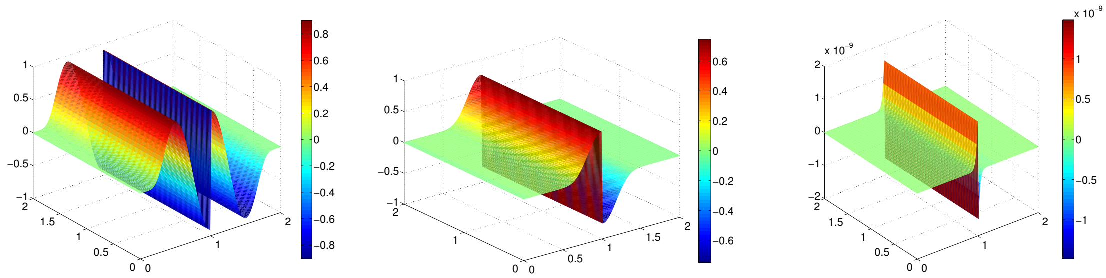
In case (iv), even if the initial condition is smooth, the magnetic advection with normally discontinuous collisional velocity yields the formation of a shock along the discontinuity (Figure 6.8 (b)) until complete blow up. An analogous behavior of the numerical magnetic potential obtained from the stabilized scheme can be reported in the case (ii), where instantaneous blow up of the solution along the discontinuity is observed. However, we expect a blow-up of the solution in these situations: the observed behavior of the numerical solution is not engendered by instabilities produced within the numerical scheme, but accurately reflects "physical reality".



(a) Time $t = 0$ (b) Time $t = 1$

Figure 6.8: First component of the numerical solution obtained from the stabilized scheme (c_f as in (2.9)) with $\Lambda_h^1(\mathcal{T}_h) = \mathcal{Q}_1^- \Lambda^1(\mathcal{T}_h)$ and Heun method ($\tau = 0.1h$), for collisional velocity and smooth initial condition as in (iv).

A similar conclusion can be drawn for the case (iii) in Figure 6.9.



(a) Time $t = 0.25$ (b) Time $t = 0.75$ (c) Time $t = 2$

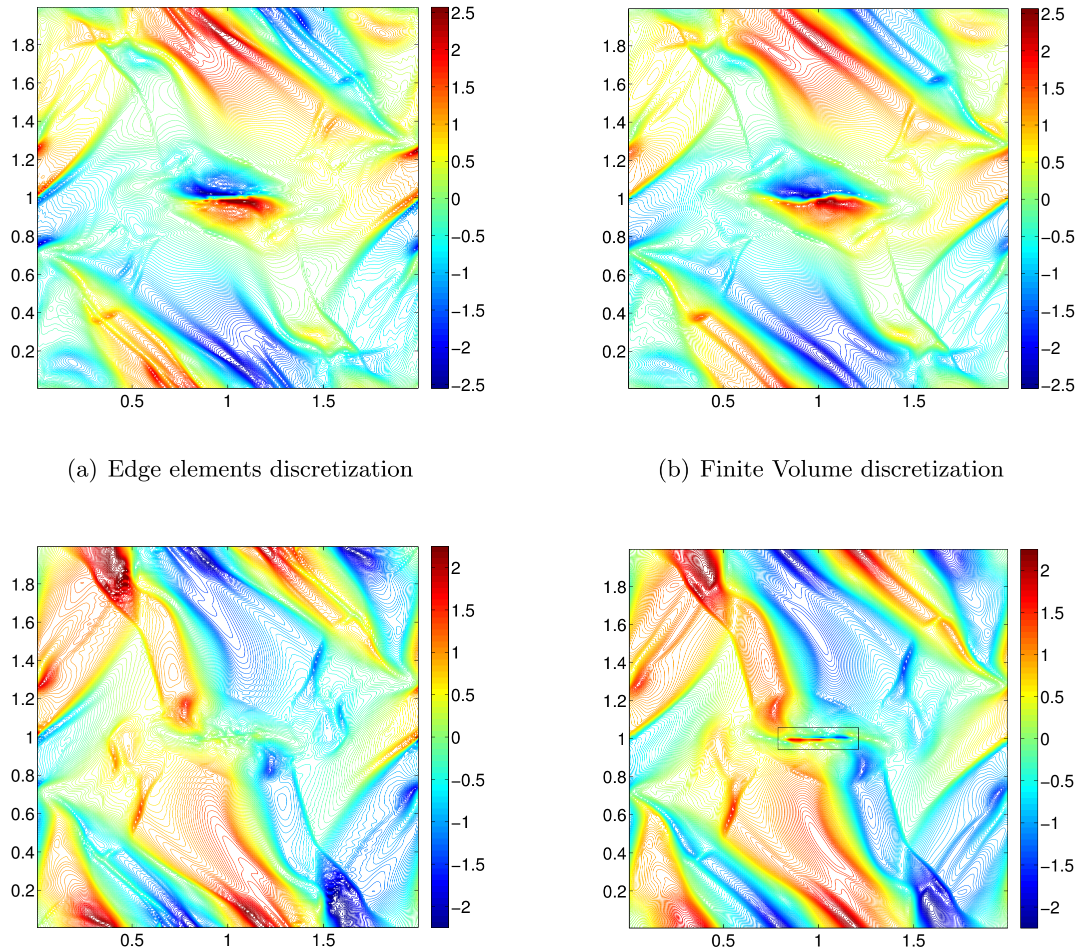
Figure 6.9: First component of the magnetic potential obtained from the stabilized scheme (c_f as in (2.9)) with $\Lambda_h^1(\mathcal{T}_h) = \mathcal{Q}_1^- \Lambda^1(\mathcal{T}_h)$ and Heun method ($\tau = 0.1h$), for collisional velocity and smooth initial condition as in (iii).

6.4 Orszag-Tang benchmark

As last numerical experiment, we rely on a widely used two dimensional benchmark problem, the so-called Orszag–Tang vortex system [33], which describes the transition to supersonic turbulence in the MHD equations. The development of shock waves and the complex interaction between various shocks with different speed which characterized the solution, makes the Orszag-Tang benchmark a challenging test for numerical methods.

Let us consider the two dimensional magnetic advection problem in $\Omega = [0, 2]^2$ with periodic boundary conditions at the boundary $\partial\Omega$. The time interval is $I = [0, 1]$ with uniform time step $\tau = 5 \cdot 10^{-3}$. The initial condition is the smooth vector field $\mathbf{u}_0(x, y) = (\sin(2\pi x), \sin(\pi y))^\top$ and the velocity field is piecewise constant with respect to the mesh. In particular, it is given at each time step as the outcome of a second order Finite Volume simulation of the full MHD system (from [32] and [21]). Note that even if the initial velocity is smooth, complex structures such as shocks and shock interactions develop in time. Concerning the discretization, the numerical scheme has been implemented on a tensor product mesh with 200×200 elements and $\mathrm{H}(\mathbf{curl}, \Omega)$-conforming lowest order rotated Raviart-Thomas elements $\Lambda_h^1(\mathcal{T}_h) = \mathcal{Q}_1^- \Lambda^1(\mathcal{T}_h)$ have been used for the spatial discretization, while a second order two-stage Runge-Kutta time-stepping is deployed in order to exploit the mild CFL-condition of the scheme. The stabilization parameter is as in (2.9), i.e. the upwind direction is assumed to be given by the average of the velocity field.

The rotated magnetic potential obtained using the scheme (4.1) has been compared with the magnetic induction derived from the full MHD system and a second order Finite Volume discretization (from [32] and [21]). The numerical method we have proposed well resolves shocks, is stable and no spurious oscillations occur. As can be inferred from Figure 6.10, the current sheet characterizing the second component of the magnetic induction (black box in Figure 6.10 (d)) is not captured by the $\mathrm{H}(\mathbf{curl}, \Omega)$-conforming scheme due to the low order polynomial space discretization which is highly diffusive.



(a) Edge elements discretization

(b) Finite Volume discretization

(c) Edge elements discretization

(d) Finite Volume discretization.

Figure 6.10: Orszag-Tang benchmark: Comparison at time $T = 1$ between the magnetic potential obtained with the stabilized scheme (2.7)-(2.8), lowest order edge elements and second order Heun time-stepping (first column (a)-(c)) and the corresponding components of the magnetic field obtained from the MHD simulation with a finite volume scheme in [21] (second column (b)-(d)). The current sheet in Figure (d) is framed in the black box.

A Conforming Approximation of Non-Conforming Discrete Differential Forms

This Appendix pertains to the proof of Proposition 3.1.

Let us consider the same types of meshes as in Section 2.5. Furthermore, let $\Delta_j(\mathcal{T}_h)$ denote the set of all j-faces of $\mathcal{T}_h$, with $\Delta_j(\mathcal{T}_h) = \Delta_j^\circ(\mathcal{T}_h) \cup \Delta_j^\partial(\mathcal{T}_h)$ and $\Delta_j^\partial(\mathcal{T}_h)$ being the set of j-faces of $\mathcal{T}_h$ belonging to the domain boundary $\partial\Omega$. With a little abuse of notation, given a m-face $f \in \Delta_m(\mathcal{T}_h)$, let $\Delta_j(f)$ denote the set of all j-dimensional faces of f if $j < m$ and the set of all j-faces containing f if $j > m$.

We briefly introduce the degrees of freedom on the spaces $\Lambda_h^k(\mathcal{T}_h)$ needed for the subsequent analysis. We refer to [6, Section 5] and [4] for a detailed discussion of spaces of finite element differential forms on simplicial and tensor product meshes, respectively. On a n-simplex

$T \in \mathcal{T}_h$, the degrees of freedom of $\mathcal{P}_r\Lambda^k(T)$ are defined on every j-simplex $f_j \in \Delta_j(T)$, with $k \le j \le \min\{n, r+k-1\}$, as

$$\omega \in \mathcal{P}_r\Lambda^k(T) \longmapsto W^\ell_{f_j}(\omega) := \int_{f_j} \operatorname{tr}\omega \wedge \eta^\ell_j \qquad \forall \ell = 1, \dots, N_j, \tag{A.1}$$

where $\{\eta^\ell_j\}_{\ell=1}^{N_j}$ is a basis of $\mathcal{P}^-_{r-j+k}\Lambda^{j-k}(f_j)$ (see for example [26, Sections 3.2 and 3.4]). Similarly, the degrees of freedom of $\mathcal{P}^-_r\Lambda^k(T)$ are defined on every j-simplex $f_j \in \Delta_j(T)$, with $k \le j \le \min\{n, r+k-1\}$, as

$$\omega \in \mathcal{P}^-_r\Lambda^k(T) \longmapsto W^\ell_{f_j}(\omega) := \int_{f_j} \operatorname{tr}\omega \wedge \eta^\ell_j \qquad \forall \ell = 1, \dots, N_j,$$

where $\{\eta^\ell_j\}_{\ell=1}^{N_j}$ is a basis of $\mathcal{P}_{r-j+k-1}\Lambda^{j-k}(f_j)$. Finally, on a n-dimensional hypercube $T \in \mathcal{T}_h$ the degrees of freedom of $\mathcal{Q}^-_r\Lambda^k(T)$ are defined on every j-face $f_j \in \Delta_j(T)$, with $k \le j \le \min\{n, r+k-1\}$, as

$$\omega \in \mathcal{Q}^-_r\Lambda^k(T) \longmapsto W^\ell_{f_j}(\omega) := \int_{f_j} \operatorname{tr}\omega \wedge \eta^\ell_j \qquad \forall \ell = 1, \dots, N_j,$$

where $\{\eta^\ell_j\}_{\ell=1}^{N_j}$ is a basis of $\mathcal{Q}^-_{r-1}\Lambda^{j-k}(f_j)$. Furthermore, if $f_{n-1} \in \Delta_{n-1}(\mathcal{T}_h)$ then we rely on a trace property of the shape function space to infer $\operatorname{tr}\omega \in \mathcal{P}_r\Lambda^k(f_{n-1})$ which is hence determined by the degrees of freedom corresponding to the j-faces $f_j \subset f_{n-1}$. Note that the interelement continuity required for the finite element space $\Lambda^k_h(\mathcal{T}_h)$ to belong to $H\Lambda^k(\Omega)$ is directly imposed through the choice of degrees of freedom associated to the j-dimensional faces in $\mathcal{T}_h$, with $k \le j \le n-1$.

Without loss of generality, in what follows we can assume that $n < r+k-1$. Moreover, for the sake of conciseness, we restrict the analysis to the case $\Lambda^k_h(\mathcal{T}_h) = \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$. The extension to the cases $\Lambda^k_h(\mathcal{T}_h) = \mathcal{P}^-_r\Lambda^k(\mathcal{T}_h)$ and $\Lambda^k_h(\mathcal{T}_h) = \mathcal{Q}^-_r\Lambda^k(\mathcal{T}_h)$ follows alike.

Lemma A.1. *Let $\widehat{T}$ be the reference n-simplex. Let $\phi : \widehat{T} \to T$ be an affine isomorphism. Then there exist $C_1, C_2 > 0$ such that, for all $\hat{\omega} \in \mathcal{P}_r\Lambda^k(\hat{T})$, it holds*

$$C_1 \|\hat{\omega}\|^2_{L^2\Lambda^k(\hat{T})} \le \sum_{j=k}^{n-1} \sum_{f_j \in \Delta^\circ_j(T)} \sum_{\ell=1}^{N_j} (W^\ell_{f_j,T})^2 + \sum_{\ell=1}^{N_n} (W^\ell_T)^2 \le C_2 \|\hat{\omega}\|^2_{L^2\Lambda^k(\hat{T})}.$$

Proof. The proof relies on standard transformation techniques. Details can be found in [26, Section 3.6]. □

Definition A.2. We define the averaging interpolation operator $P_h : \mathcal{P}^{\mathrm{d}}_r\Lambda^k(\mathcal{T}_h) \to \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ through the degrees of freedom of $\psi := P_h\omega$ for $\omega \in \mathcal{P}^{\mathrm{d}}_r\Lambda^k(\mathcal{T}_h)$ with degrees of freedom defined locally as in (A.1). In particular,

- Degrees of freedom on n-simplices:

$$\Psi^\ell_T := W^\ell_T, \quad \forall T \in \Delta_n(\mathcal{T}_h), \;\; \ell = 1, \dots, N_n; \tag{A.2}$$

- Degrees of freedom on $n-1$-simplices: for $\ell = 1, \dots, N_{n-1}$

$$\Psi^\ell_{f_{n-1}} := \begin{cases} \frac{1}{2} \sum\limits_{T \in \Delta_n(f_{n-1})} W^\ell_{f_{n-1},T}, & \forall f_{n-1} \in \Delta^\circ_{n-1}(\mathcal{T}_h); \\ W^\ell_{f_{n-1},T} & \forall f_{n-1} \in \Delta^\partial_{n-1}(\mathcal{T}_h). \end{cases} \tag{A.3}$$

- Degrees of freedom on j-simplices for $k \le j \le n-2$: for $\ell = 1, \dots, N_j$

$$\Psi^{\ell}_{f_j} := \begin{cases} \dfrac{1}{\sum_{T\in\Delta_n(f_j)} 1} \displaystyle\sum_{T\in\Delta_n(f_j)} W^{\ell}_{f_j,T}\,, & \forall f_j \in \Delta^{\circ}_j(\mathcal{T}_h); \\ W^{\ell}_{f_j,T} & \forall f_j \in \Delta^{\partial}_j(\mathcal{T}_h). \end{cases} \tag{A.4}$$

Note that if $f_{n-1} \in \Delta_{n-1}(T_1) \cap \Delta_{n-1}(T_2)$, then $\Psi^{\ell}_{f_{n-1}}|_{T_1} = \Psi^{\ell}_{f_{n-1}}|_{T_2}$ for all $\ell = 1, \dots, N_{n-1}$. This means that $\operatorname{tr}\psi|_{T_1} = \operatorname{tr}\psi|_{T_2}$ on every $f_{n-1} \in \Delta_{n-1}(\mathcal{T}_h)$.

The averaging operator introduced in Definition A.2 enables to approximate discontinuous non-conforming differential forms by $H\Lambda^k(\Omega)$-conforming ones through the following approximation result.

Proposition A.3. *Let $\omega \in \mathcal{P}^{\mathrm{d}}_r\Lambda^k(\mathcal{T}_h)$ and P_h be the averaging operator introduced in Definition A.2. Then there exists a constant $C > 0$ depending only on the polynomial degree r and the shape regularity of the mesh, such that*

$$\|\omega - P_h\omega\|^2_{L^2\Lambda^k(\Omega)} \le Ch \sum_{f_{n-1}\in\Delta_{n-1}(\mathcal{T}_h)} \left\| \operatorname{tr}[\omega]_{f_{n-1}} \right\|^2_{L^2\Lambda^k(f_{n-1})}.$$

Proof. Let $T^+ \in \Delta_n(\mathcal{T}_h)$, $T^+ = \phi(\widehat{T})$ be fixed. In view of Lemma A.1 and the definition of averaging operator, it holds

$$\left\| \hat{\omega} - \widehat{P_h\omega} \right\|^2_{L^2\Lambda^k(\widehat{T})} \simeq \sum_{j=k}^{n-1} \sum_{f_j\in\Delta_j(T^+)} \sum_{\ell=1}^{N_j} (W^{\ell}_{f_j,T^+} - \Psi^{\ell}_{f_j})^2.$$

Let us estimate separately each set of degrees of freedom on the j-subsimplices of $\mathcal{T}_h$. First, we consider the $n-1$-face $f_{n-1} = \Delta^{\circ}_{n-1}(T^+) \cap \Delta^{\circ}_{n-1}(T^-)$, then due to (A.3),

$$\sum_{\ell=1}^{N_{n-1}} (W^{\ell}_{f_{n-1},T^+} - \Psi^{\ell}_{f_{n-1}})^2 = \frac{1}{4} \sum_{\ell=1}^{N_{n-1}} (W^{\ell}_{f_{n-1},T^+} - W^{\ell}_{f_{n-1},T^-})^2.$$

Consider now the degrees of freedom on the j-subsimplices of $\mathcal{T}_h$, $f_j \in \Delta^{\circ}_j(T^+)$ with $k \le j \le n-2$. Let $M_j := \sum_{T\in\Delta_n(f_j)} 1$. Note that for every $k \le j \le n-2$, M_j is the cardinality of the set of n-simplices of $\mathcal{T}_h$ sharing f_j and, due to the shape regularity, this number is bounded uniformly in the mesh width h. By the definition in (A.4) follows

$$\begin{aligned} \sum_{\ell=1}^{N_j} (W^{\ell}_{f_j,T^+} - \Psi^{\ell}_{f_j})^2 &= \sum_{\ell=1}^{N_j} \Big(W^{\ell}_{f_j,T^+} - M_j^{-1} \sum_{T\in\Delta^{\circ}_n(f_j)} W^{\ell}_{f_j,T} \Big)^2 \\ &\le CM_j^{-2} \sum_{T\in\Delta^{\circ}_n(f_j)} \sum_{\ell=1}^{N_j} (W^{\ell}_{f_j,T^+} - W^{\ell}_{f_j,T})^2 \\ &\le C \sum_{\substack{f_{n-1}\in\Delta^{\circ}_{n-1}(f_j) \\ f_{n-1}\subset\partial T^+\cap\partial T^-}} \sum_{\ell=1}^{N_j} (W^{\ell}_{f_j,T^+} - W^{\ell}_{f_j,T^-})^2. \end{aligned}$$

Moreover, by definition of trace and jump across a face $f_{n-1} = \Delta^\circ_{n-1}(T^+) \cap \Delta^\circ_{n-1}(T^-)$, the degrees of freedom of $\operatorname{tr}[\omega]_{f_{n-1}}$ are given by the set $\{W^\ell_{f_j,T^+} - W^\ell_{f_j,T^-}\}$ for $j = k, \dots, n-1$ and $\ell = 1, \dots, j$. Hence, applying Lemma A.1 yields

$$\sum_{j=k}^{n-2} \sum_{f_j \in \Delta^\circ_j(f_{n-1})} \sum_{\ell=1}^{N_j} (W^\ell_{f_j,T^+} - W^\ell_{f_j,T^-})^2 + \sum_{\ell=1}^{N_{n-1}} (W^\ell_{f_{n-1},T^+} - W^\ell_{f_{n-1},T^-})^2 \le C \left\| \operatorname{tr}[\hat{\omega}]_{\hat{f}_{n-1}} \right\|^2_{L^2\Lambda^k(\hat{f}_{n-1})}.$$

Using a scaling argument yields the conclusion. □

B A coordinate based representation of Lie derivatives

Let $\{\mathbf{e}^k_I\}_{I=1}^{\binom{n}{k}}$ be the orthonormal basis of alternating k-linear forms, e.g. $\omega = \sum_{I=1}^{\binom{n}{k}} \omega_I \mathbf{e}^k_I$ with 0-forms ω_I for arbitrary $\omega \in \Lambda^k(\Omega)$ and $\mathbf{e}^k_I \wedge \star \mathbf{e}^k_J = \delta_{I,J}$. Then the projection of $\mathrm{L}_{\boldsymbol{\beta}}\omega$ onto $\mathbf{e}_J$ yields

$$\begin{aligned}
\mathrm{L}_{\boldsymbol{\beta}}\omega \wedge \star \mathbf{e}_J &= \mathrm{L}_{\boldsymbol{\beta}}(\omega \wedge \star \mathbf{e}_J) + \omega \wedge \star \mathcal{L}_{\boldsymbol{\beta}} \mathbf{e}_J \\
&= \mathsf{d}\mathsf{i}_{\boldsymbol{\beta}}(\omega \wedge \star \mathbf{e}_J) + \omega \wedge \star \mathcal{L}_{\boldsymbol{\beta}} \mathbf{e}_J \\
&= \mathsf{d}\mathsf{i}_{\boldsymbol{\beta}}\omega_J + \sum_{I=1}^{\binom{n}{k}} \omega_I \mathbf{e}^k_I \wedge \star \mathcal{L}_{\boldsymbol{\beta}} \mathbf{e}^k_J.
\end{aligned}$$

Hence, if $\mathbf{u} = (\omega_1, \dots, \omega_I, \dots)^\top$ is a vector proxy of $\omega \in \Lambda^k(\Omega)$ then $\sum_{i=1}^n \beta_i \partial_{x_i} \mathbf{u} + \mathbf{C}\mathbf{u}$ with $\mathbf{C}_{JI} := \delta_{J,I}(\sum_{i=1}^n \partial_{x_i}\beta_i) + \mathbf{e}^k_I \wedge \star \mathcal{L}_{\boldsymbol{\beta}} \mathbf{e}^k_J$ is a vector proxy of $\mathrm{L}_{\boldsymbol{\beta}}\omega$.

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