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ANALYSIS AND NUMERICAL APPROXIMATION OF BRINKMAN REGULARIZATION OF TWO PHASE FLOWS IN POROUS MEDIA

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ABSTRACT. We consider a hyperbolic-elliptic system of PDEs that arises in the modeling of two-phase flows in a porous medium. The phase velocities are modeled using a Brinkman regularization of the classical Darcy's law. We propose a notion of weak solutions for these equations and prove existence of these solutions. An efficient finite difference scheme is proposed and is shown to converge to the weak solutions of this system. The Darcy limit of the Brinkman regularization is studied numerically using the convergent finite difference scheme in two space dimensions as well as using both analytical and numerical tools in one space dimension.

1. The two Phase Flow Problem

Two phase flows in a porous medium model many interesting phenomena in geophysics. As examples, we mention water flooding of oil reservoirs and carbon dioxide sequestration in subsurface formations.

A prototypical situation of interest is the flow of two phases, say oil and water in a porous medium. The variables of interest are the phase saturations s_w and s_o representing the saturation (volume fraction) of the water and oil phase respectively. We have the identity:

$$s_w + s_o \equiv 1. \tag{1.1}$$

Hence, we can describe the dynamics in terms of the saturation of either of the two phases. We denote the water saturation as $s_w = s$ in the discussion below. Assuming a constant porosity ($\phi \equiv 1$), the two phases are transported by [3]

$$(s_r)_t + \mathrm{div}_x(\mathbf{v}_r) = 0, \quad r \in \{w, o\}. \tag{1.2}$$

Here, the phase velocities are denoted by $\mathbf{v}_w$ and $\mathbf{v}_o$ respectively. In view of the identity (1.1), the two phase velocities can be summed up to yield the *incompressibility* condition,

$$\mathrm{div}_x(\mathbf{v}) = 0, \quad \mathbf{v} = \mathbf{v}_w + \mathbf{v}_o. \tag{1.3}$$

The total velocity is denoted by $\mathbf{v}$.

The phase velocities in a homogeneous isotropic medium are described by the Darcy's law [3]:

$$\mathbf{v}_r = -\lambda_r \nabla_x p_r, \quad r \in \{w, o\}. \tag{1.4}$$

The quantity $\lambda_r = \lambda_r(s_r)$ is the phase mobility and p_r is the phase pressure. Note that we have neglected gravity in the above version of the Darcy's law (gravity can be readily considered, leading to an additional term, see [3]). Assume that the *capillary pressure* i.e, $p_c = p_w - p_o$ is zero, we can sum (1.4) for both phases and obtain

$$\mathbf{v} = -\lambda_T(s)\nabla_x p, \tag{1.5}$$

with $p = p_w = p_o$ being the pressure and $\lambda_T = \lambda_w + \lambda_o$ being the total mobility.

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Using (1.5), the gradient of pressure in (1.4) can be eliminated leading to

$$\mathbf{v}_w = \frac{\lambda_w(s)}{\lambda_T(s)}\mathbf{v}. \tag{1.6}$$

Denoting the fractional flow function f as

$$f(s) = \frac{\lambda_w(s)}{\lambda_T(s)} = \frac{\lambda_w(s)}{\lambda_w(s) + \lambda_o(s)}, \tag{1.7}$$

the saturation equation (1.2) for water can be written down as

$$s_t + \mathrm{div}_x(f(s)\mathbf{v}) = 0. \tag{1.8}$$

Combining the saturation equation with the incompressibility condition (1.3) and the pressure equation, we obtain the evolution equations for two phase flow in a porous medium:

$$\begin{aligned} s_t + \mathrm{div}_x(f(s)\mathbf{v}) &= 0,\\ \mathrm{div}_x(\mathbf{v}) &= 0,\\ \mathbf{v} &= -\lambda_T(s)\nabla_x p. \end{aligned} \tag{1.9}$$

The above equations have to be augmented by suitable initial and boundary conditions.

The phase mobility $\lambda_w : [0,1] \mapsto \mathbb{R}$ is a monotone increasing function with $\lambda_w(0) = 0$ and the phase mobility $\lambda_o : [0,1] \mapsto \mathbb{R}$ is a monotone decreasing function with $\lambda_o(1) = 0$. Furthermore, the total mobility is strictly positive i.e, $\lambda_T \geq \lambda_* > 0$ for some λ_*.

The above equations are a hyperbolic-elliptic system as the saturation equation in (1.9) is a scalar hyperbolic conservation law in several space dimensions with a coefficient given by the velocity $\mathbf{v}$. The velocity can be obtained by solving an elliptic equation for the pressure p.

It is well known that solutions of hyperbolic conservation laws can develop discontinuities, even for smooth initial data, [8]. The presence of these discontinuities or shock waves implies that solutions of conservation laws are sought in a weak sense and are augmented with additional admissibility criteria or *entropy conditions* in order to ensure uniqueness.

As the two phase flow equations involve a conservation law, we need to define a suitable concept of entropy solutions for these equations and show that these solutions are well-posed. The problem of proving well-posedness of global weak solutions of the two phase flow equations (1.9) has remained open for many decades. The main challenge in showing existence is the fact that the velocity field $\mathbf{v}$ acts as a coefficient in the saturation equations. Although conservation laws with coefficients have been studied extensively in recent years, see [1, 13, 6, 2] and references therein, the state of the art results require that the coefficient is a function of bounded variation. Many attempts at showing that the velocity field $\mathbf{v}$ in (1.8) is sufficiently regular, for example is a BV function or has enough Sobolev regularity, have failed. Partial results (with strong assumptions on the velocity field or on the solution) have been obtained in [17, 20] and references therein.

Another approach is to consider a modified version of the two phase flow equations. Recalling that the two phase flow equations (1.9) were derived under the assumption that the capillary pressure was zero. Adding small but non-zero capillary pressure leads to a viscous perturbation of the saturation equation, see [14]. The viscous problem has been shown to be well-posed in [14]. However, the fact that the coefficient of viscosity can be very small leads to difficulties in numerical approximation of these equations as the viscous scales have to resolved. Furthermore, sharp saturation fronts might be smeared due to the added viscosity.

A different approach to the above two considers the more fundamental question- is the Darcy's law (1.4) correct ? Many studies have focused on this question and have found that the Darcy's law maybe inadequate to explain the dynamics of fluid flow in porous media, even for a single phase [4]. It is plausible that the problems of showing well posedness for the full two-phase flow model can be attributed to the modeling deficiencies of the Darcy's law.

Several modifications of the Darcy's law have been proposed, see [5]. Of particular interest in this paper is the Brinkman modification [4]. It is well known that this modification explains the dynamics of flow in porous media, better than the Darcy model in many situations of interest, see [18, 15] and references therein. The Brinkman model for the phase velocity of each phase is given by,

$$-\mu\Delta_x \mathbf{v}_r + \mathbf{v}_r = -\lambda_r \nabla_x p_r, \quad r \in \{w, o\}. \tag{1.10}$$

Here, μ denotes a small scale parameter. Note that the Brinkman approximation adds a smoothing term to Darcy's law.

Adding the phase velocity relations (1.10) for both phases w, o and neglecting capillary pressure i.e, $p_w = p_o = p$, we obtain that the total velocity $v = v_w + v_o$ satisfies,

$$-\mu\Delta_x \mathbf{v} + \mathbf{v} = -\lambda_T(s)\nabla_x p,$$

Applying the divergence operator to both sides of the above equation and using incompressibility (1.3), we obtain the following elliptic equation for the pressure.

$$-\operatorname{div}_{\mathbf{x}}\left(\lambda_T(s)\nabla_{\mathbf{x}} p\right) = 0 \tag{1.11}$$

Here, we have labeled the water saturation $s = s_w$. Combining this equation with the conservation of mass for the water phase and with the Brinkman approximation (1.10) describing the velocity of the water phase and the fractional flow function defined in (1.7), we obtain the following complete system,

$$\begin{aligned} \partial_t s + \operatorname{div}_{\mathbf{x}}\left(\mathbf{v}_w\right) &= 0, \\ -\mu\Delta_{\mathbf{x}}\mathbf{v}_w + \mathbf{v}_w &= -f(s)\lambda_T(s)\nabla_{\mathbf{x}} p \\ -\operatorname{div}_{\mathbf{x}}\left(\lambda_T(s)\nabla_{\mathbf{x}} p\right) &= 0, \end{aligned} \tag{1.12}$$

that describes the flow of two phases in a porous medium, obeying the Brinkman's law. The system (1.12) is henceforth termed as the Brinkman regularization of two phase flows in a porous medium. We remark that the Darcy system (1.9) can be obtained from the Brinkman regularization (1.12) by setting $\mu = 0$ and rewriting the water phase velocity in terms of the fractional flow function.

The rest of this paper is concerned with the analysis and numerical approximation of the Brinkman regularization (1.12). Our aims are three fold:

- To define a suitable notion of solutions to the Brinkman regularization (1.12) and to show that these solutions exist.
- To design an efficient numerical scheme to approximate the Brinkman regularization for two phase flows and to prove that this scheme converges when the mesh is refined.
- To compare the solutions of the Brinkman regularization with those of the standard Darcy model for two phase flow (1.9) in order to ascertain whether the Brinkman regularization is a suitable approximation of the Darcy's law in the regime of two phase flows.

The rest of this paper provides answers to the above questions and is organized as follows: in Section 2, equivalent forms of the Brinkman regularization are stated, a suitable notion of solutions is defined and the main existence theorem is described. Section 3 deals with the proof of existence of the Brinkman regularization. A convergent numerical scheme for approximating (1.12) is presented in Section 4. Finally, we provide further comparisons between the Darcy and Brinkman models (particularly in one space dimension) in Section 5.

2. Statement of Problem

In this section, we will consider the following Darcy-Brinkman system (1.12) augmented with initial and boundary conditions,

$$
\begin{cases}
\partial_t s + \mathrm{div}_{\mathbf{x}}\,(\mathbf{v}_w) = 0, & t > 0,\ \mathbf{x} \in \Omega,\\
-\mu \Delta_{\mathbf{x}} \mathbf{v}_w + \mathbf{v}_w = -f(s)\lambda_T(s)\nabla_{\mathbf{x}} p, & t > 0,\ \mathbf{x} \in \Omega,\\
-\mathrm{div}_{\mathbf{x}}\,(\lambda_T(s)\nabla_{\mathbf{x}} p) = 0, & t > 0,\ \mathbf{x} \in \Omega,\\
\partial_\nu p(t,\mathbf{x}) = \pi(t,\mathbf{x}), & t > 0,\ \mathbf{x} \in \partial\Omega,\\
\mathbf{v}_w(t,\mathbf{x}) \cdot \nu(\mathbf{x}) = h(t,\mathbf{x}), & t > 0,\ \mathbf{x} \in \partial\Omega,\\
\int_\Omega p(t,\mathbf{x}) d\mathbf{x} = 0, & t > 0,\\
s(0,\mathbf{x}) = s_0(\mathbf{x}), & \mathbf{x} \in \Omega,
\end{cases}
\tag{2.1}
$$

where

- **(H.1)** Ω is an open connected subset of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary and ν is the unit outer normal;
- **(H.2)** f is a smooth Lipschitz bounded function, $0 < \mu \leq 1$ is a constants, and $h, \pi : (0,\infty) \times \partial\Omega \to \mathbb{R}$ are smooth bounded maps;
- **(H.3)** λ_T is smooth Lipschitz bounded such that $\lambda_T(\cdot) \geq \lambda_* $ for some constant $\lambda_* > 0$, and $\lambda_T f'$ and λ_T'/λ_T are bounded;
- **(H.4)** the initial datum $s_0 \in H^1(\Omega)$.

Note that all the above assumptions are consistent with the definitions of the phase mobilities in the Darcy's law.

Formally applying the Helmholtz operator $-\mu\Delta_{\mathbf{x}} + 1$ to the first equation in (2.1) we obtain the third order problem

$$
\begin{cases}
\partial_t s - \mu \Delta_{\mathbf{x}} \partial_t s - \mathrm{div}_{\mathbf{x}}\,(f(s)\lambda_T(s)\nabla_{\mathbf{x}} p) = 0, & t > 0,\ \mathbf{x} \in \Omega,\\
-\mathrm{div}_{\mathbf{x}}\,(\lambda_T(s)\nabla_{\mathbf{x}} p) = 0, & t > 0,\ \mathbf{x} \in \Omega,\\
\partial_\nu p(t,\mathbf{x}) = \pi(t,\mathbf{x}), & t > 0,\ \mathbf{x} \in \partial\Omega,\\
\int_\Omega p(t,\mathbf{x}) d\mathbf{x} = 0, & t > 0,\\
\mu \partial_\nu \partial_t s + f(s)\lambda_T(s)\pi = h, & t > 0,\ \mathbf{x} \in \partial\Omega,\\
s(0,\mathbf{x}) = s_0(\mathbf{x}), & \mathbf{x} \in \Omega.
\end{cases}
\tag{2.2}
$$

Since we can rewrite the first equation in the form

$$
\partial_t s - \mathrm{div}_{\mathbf{x}}\,(\mu \nabla_{\mathbf{x}} \partial_t s + f(s)\lambda_T(s)\nabla_{\mathbf{x}} p) = 0, \tag{2.3}
$$

the boundary condition on $\partial_\nu \partial_t s$ reads as the flux boundary condition on (2.3). Indeed the flux in (2.3) is $\mu \nabla_{\mathbf{x}} \partial_t s + f(s)\lambda_T(s)\nabla_{\mathbf{x}} p$, multiplying by the unit outer normal ν and using the fact that $\partial_\nu p = \pi$ we have

$$
\begin{aligned}
\left(\mu \nabla_{\mathbf{x}} \partial_t s + f(s)\lambda_T(s)\nabla_{\mathbf{x}} p\right) \cdot \nu\big|_{\partial\Omega} &= \big(\mu \partial_\nu \partial_t s + f(s)\lambda_T(s) \underbrace{\partial_\nu p}_{=\pi}\big)\big|_{\partial\Omega}\\
&= \mu \partial_\nu \partial_t s + f(s)\lambda_T(s)\pi.
\end{aligned}
\tag{2.4}
$$

On the other hand, comparing the first equations in (2.1) and (2.2) we get

$$
\mathbf{v}_w = \mu \nabla_{\mathbf{x}} \partial_t s + f(s)\lambda_T(s)\nabla_{\mathbf{x}} p \tag{2.5}
$$

therefore, (2.4) and the boundary condition on $\mathbf{v}_w$ give

$$\mu\partial_\nu\partial_t s + f(s)\lambda_T(s)\pi = h, \qquad (0,\infty)\times\partial\Omega. \tag{2.6}$$

Next, we introduce the notion of weak solutions to the Brinkman system (2.1) below.

Definition 2.1. *Let s, $p : [0,\infty)\times\Omega \to \mathbb{R}$, $\mathbf{v}_w : [0,\infty)\times\Omega \to \mathbb{R}^N$ be functions. We say that $(s, \mathbf{v}_w, p)$ is a solution of (2.1) if*

(**D.1**) *for every $T > 0$*

$$s \in W^{1,\infty}(0,T;H^1(\Omega)), \quad p \in L^\infty(0,T;H^1(\Omega)\cap W^{2,1}(\Omega)), \quad \mathbf{v}_w \in L^\infty(0,T;H^2(\Omega));$$

(**D.2**) *for every test function $\varphi \in C^\infty(\mathbb{R}^{N+1})$ with compact support, the following identity is satisfied,*

$$\int_0^\infty\int_\Omega (s\partial_t\varphi + \mathbf{v}_w\nabla\varphi)\,dtdx - \int_0^\infty\int_{\partial\Omega} h\varphi dtdx + \int_\Omega s_0(x)\varphi(0,x)dx = 0,$$

$$\int_0^\infty\int_\Omega \lambda_T(s)\nabla p\cdot\nabla\varphi - \int_0^\infty\int_{\partial\Omega}\pi\varphi dtdx = 0;$$

(**D.3**) *for every test function $\Phi \in C^\infty(\mathbb{R}^{N+1};\mathbb{R}^N)$ with compact support contained in $\mathbb{R}$, the following identity is satisfied,*

$$\mu\int_0^\infty\int_\Omega \nabla\mathbf{v}_w\cdot\nabla\Phi dtdx + \int_0^\infty\int_\Omega \mathbf{v}_w\cdot\Phi dtdx + \int_0^\infty\int_\Omega f(s)\lambda_T(s)\nabla p\cdot\Phi dtdx = 0;$$

(**D.4**) *for almost every $t > 0$*

$$\int_\Omega p(t,\mathbf{x})d\mathbf{x} = 0.$$

Due to regularity assumption (**D.2**) and the linearity of the Helmholtz operator $1-\mu\Delta$ the solutions of (2.1) solve (2.2) and vice versa.

Our main result is the following existence theorem.

Theorem 2.1. *Assume* (**H.1**), (**H.2**), (**H.3**), *and* (**H.4**). *Then, the initial boundary value problem* (2.1) *has a solution $(s, p, \mathbf{v}_w)$ in the sense of Definition* 2.1.

We use the following recursive approximation of (2.1).

We start defining

$$s_0(t,\mathbf{x}) = s_0(\mathbf{x}), \qquad t > 0,\, \mathbf{x}\in\Omega. \tag{2.7}$$

The function $p_0 = p_0(t,x)$ solves the elliptic problem with time depending Neumann boundary conditions

$$\begin{cases} -\mathrm{div}_{\mathbf{x}}\left(\lambda_T(s_0)\nabla_{\mathbf{x}}p_0\right) = 0, & t>0,\, \mathbf{x}\in\Omega,\\ \partial_\nu p_0(t,\mathbf{x}) = \pi(t,\mathbf{x}), & t>0,\, \mathbf{x}\in\partial\Omega,\\ \int_\Omega p_0(t,\mathbf{x})d\mathbf{x} = 0, & t>0, \end{cases} \tag{2.8}$$

then we define $\mathbf{v}_{w,0} = \mathbf{v}_{w,0}(t,\mathbf{x})$ as the solution of the following elliptic problem with time depending boundary conditions

$$\begin{cases} -\mu\Delta_{\mathbf{x}}\mathbf{v}_{w,0} + \mathbf{v}_{w,0} = -f(s_0)\lambda_T(s_0)\nabla_{\mathbf{x}}p, & t>0,\, \mathbf{x}\in\Omega,\\ \mathbf{v}_{w,0}(t,\mathbf{x})\cdot\nu(\mathbf{x}) = h(t,\mathbf{x}), & t>0,\, \mathbf{x}\in\partial\Omega. \end{cases} \tag{2.9}$$

The next step in the algorithm is to define $s_1 = s_1(t,\mathbf{x})$ as follows

$$s_1(t,\mathbf{x}) = s_0(\mathbf{x}) - \int_0^t \mathrm{div}_{\mathbf{x}}\left(\mathbf{v}_{w,0}\right)(\tau,\mathbf{x})d\tau, \qquad t>0,\, \mathbf{x}\in\Omega, \tag{2.10}$$

namely s_1 solves the problem

$$\begin{cases} \partial_t s_1 + \mathrm{div}_{\mathbf{x}}\left(\mathbf{v}_{w,0}\right) = 0, & t > 0,\, \mathbf{x} \in \Omega, \\ s_1(0,\mathbf{x}) = s_0(\mathbf{x}), & \mathbf{x} \in \Omega. \end{cases} \tag{2.11}$$

In general for every $n \in \mathbb{N}$ we have

$$\begin{cases} \partial_t s_{n+1} + \mathrm{div}_{\mathbf{x}}\left(\mathbf{v}_{w,n}\right) = 0, & t > 0,\, \mathbf{x} \in \Omega, \\ -\mu\Delta_{\mathbf{x}}\mathbf{v}_{w,n} + \mathbf{v}_{w,n} = -f(s_n)\lambda_T(s_n)\nabla_{\mathbf{x}} p_n, & t > 0,\, \mathbf{x} \in \Omega, \\ -\mathrm{div}_{\mathbf{x}}\left(\lambda_T(s_n)\nabla_{\mathbf{x}} p_n\right) = 0, & t > 0,\, \mathbf{x} \in \Omega, \\ \partial_\nu p_n(t,\mathbf{x}) = \pi(t,\mathbf{x}), & t > 0,\, \mathbf{x} \in \partial\Omega, \\ \mathbf{v}_{w,n}(t,\mathbf{x}) \cdot \nu(\mathbf{x}) = h(t,\mathbf{x}), & t > 0,\, \mathbf{x} \in \partial\Omega, \\ \int_\Omega p_n(t,\mathbf{x}) d\mathbf{x} = 0, & t > 0, \\ s_{n+1}(0,\mathbf{x}) = s_0(\mathbf{x}), & \mathbf{x} \in \Omega. \end{cases} \tag{2.12}$$

The above iteration scheme would be shown to converge in order to prove the existence theorem 2.1 in the next section.n

3. A Priori Estimates and Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. We begin with some a priori estimates on the solution $(s_{n+1},\, \mathbf{v}_{w,n},\, p_n)$ of (2.12).

Lemma 3.1 (H^1 estimate on p_n)**.** *Let $n \in \mathbb{N}$. We have that*

$$p_n \in L^\infty(0,T; H^1(\Omega)), \qquad T > 0.$$

In particular

$$\left\| p_n(t,\cdot) \right\|_{H^1(\Omega)} \le C_1 \left\| \pi(t,\cdot) \right\|_{L^2(\partial\Omega)}, \qquad t > 0, \tag{3.1}$$

for some positive constant C_1 independent on μ and n.

Proof. From the third equation in (2.12), (**H.3**), and the boundary conditions on p_n,

$$\begin{aligned} \lambda_* \int_\Omega |\nabla_{\mathbf{x}} p_n|^2 d\mathbf{x} &\le \int_\Omega \lambda_T(s_n) |\nabla_{\mathbf{x}} p_n|^2 d\mathbf{x} \\ &= -\int_\Omega \underbrace{\mathrm{div}_{\mathbf{x}}\left(\lambda_T(s_n)\nabla_{\mathbf{x}} p_n\right)}_{=0} p d\mathbf{x} + \int_{\partial\Omega} \lambda_T(s_n) p_n \underbrace{\partial_\nu p_n}_{=\pi} d\sigma \\ &= \int_{\partial\Omega} \lambda_T(s_n)\pi p_n d\sigma \le \frac{\left\|\lambda_T\right\|^2_{L^\infty(\mathbb{R})}}{2\alpha} \int_{\partial\Omega} \pi^2 d\sigma + \frac{\alpha}{2}\int_{\partial\Omega} p_n^2 d\sigma, \end{aligned}$$

where $\alpha > 0$ is a constant that will be chosen later. The zero mean condition on p_n and the Sobolev embeddings give

$$\begin{aligned} \lambda_* \int_\Omega |\nabla_{\mathbf{x}} p_n|^2 d\mathbf{x} \le& \frac{\left\|\lambda_T\right\|^2_{L^\infty(\mathbb{R})}}{2\alpha} \int_{\partial\Omega} \pi^2 d\sigma + \frac{\alpha}{2}\int_{\partial\Omega} p_n^2 d\sigma \\ \le& \frac{\left\|\lambda_T\right\|^2_{L^\infty(\mathbb{R})}}{2\alpha} \int_{\partial\Omega} \pi^2 d\sigma + \frac{\alpha c}{2}\int_{\Omega} |\nabla_{\mathbf{x}} p_n|^2 d\mathbf{x}, \end{aligned}$$

where c is the Sobolev embedding constant. Choosing $\alpha = \lambda_*/c$ we get

$$\int_\Omega |\nabla_{\mathbf{x}} p_n|^2 d\mathbf{x} \le \frac{c\left\|\lambda_T\right\|^2_{L^\infty(\mathbb{R})}}{\lambda_*^2} \int_{\partial\Omega} \pi^2 d\sigma.$$

The claim follows from the zero mean condition on p_n. □

Lemma 3.2. *Let $n \in \mathbb{N}$. We have that*

$$\mathbf{v}_{w,n} \in L^\infty(0,T;H^2(\Omega)),$$

for every $T > 0$. In particular

$$\|\mathbf{v}_{w,n}(t,\cdot)\|_{H^2(\Omega)} \le \frac{C_3}{\mu}\left(\|f\lambda_T\|_{L^\infty(\mathbb{R})}\|\pi(t,\cdot)\|_{L^2(\partial\Omega)} + \|h(t,\cdot)\|_{L^2(\partial\Omega)}\right), \tag{3.2}$$

for each $t > 0$ and some positive constant C_3 independent on μ and n.

Proof. The claim follows directly from classical regularity results on elliptic equations and Lemma 3.1. □

Lemma 3.3. *Let $n \in \mathbb{N}$. We have that*

$$s_n \in W^{1,\infty}(0,T;H^1(\Omega)),$$

for every $T > 0$. In particular

$$\begin{aligned} \|s_n(t,\cdot)\|_{H^1(\Omega)} &\le \|s_0\|_{H^1(\Omega)} + \frac{C_4}{\mu}\left(\|f\lambda_T\|_{L^\infty(\mathbb{R})}\|\pi(t,\cdot)\|_{L^2(\partial\Omega)} + \|h(t,\cdot)\|_{L^2(\partial\Omega)}\right),\\ \|\partial_t s_n(t,\cdot)\|_{H^1(\Omega)} &\le \frac{C_4}{\mu}\left(\|f\lambda_T\|_{L^\infty(\mathbb{R})}\|\pi(t,\cdot)\|_{L^2(\partial\Omega)} + \|h(t,\cdot)\|_{L^2(\partial\Omega)}\right), \end{aligned} \tag{3.3}$$

for each $t > 0$ and some positive constant C_4 independent on μ and n.

Proof. The claim follows directly from the first equation in (2.12) and Lemma 3.2. Indeed, we have

$$\begin{aligned} s_0(t,\mathbf{x}) &= s_0(\mathbf{x}),\\ s_{n+1}(t,\mathbf{x}) &= s_0(\mathbf{x}) - \int_0^t \mathrm{div}_{\mathbf{x}}\left(\mathbf{v}_{w,n}\right)(\tau,\mathbf{x})d\tau,\\ \partial_t s_{n+1} &= -\mathrm{div}_{\mathbf{x}}\left(\mathbf{v}_{w,n}\right). \end{aligned}$$

□

Lemma 3.4. *Let $n \in \mathbb{N}$. We have that*

$$\Delta_{\mathbf{x}} p_n \in L^\infty(0,T;L^1(\Omega)), \qquad T > 0.$$

In particular

$$\|\Delta_{\mathbf{x}} p_n(t,\cdot)\|_{L^1(\Omega)} \le C_5\left(\|\pi(t,\cdot)\|^2_{L^2(\partial\Omega)} + \|h(t,\cdot)\|^2_{L^2(\partial\Omega)}\right), \tag{3.4}$$

for each $t > 0$, where C_5 is a positive constant independent on μ and n.

Proof. From (2.12)

$$\Delta_{\mathbf{x}} p_n = -\frac{\lambda_T'(s_n)}{\lambda_T(s_n)}\nabla_{\mathbf{x}} p_n \cdot \nabla_{\mathbf{x}} s_n,$$

therefore, thanks to (**H.2**) and (**H.3**),

$$\|\Delta_{\mathbf{x}} p_n(t,\cdot)\|_{L^1(\Omega)} \le \frac{1}{2}\left\|\frac{\lambda_T'}{\lambda_T}\right\|_{L^\infty(\mathbb{R})}\left(\|\nabla_{\mathbf{x}} p_n(t,\cdot)\|^2_{L^2(\Omega)} + \|\nabla_{\mathbf{x}} s_n(t,\cdot)\|^2_{L^2(\Omega)}\right).$$

The claim follows from Lemmas 3.1 and 3.3. □

Proof of Theorem 2.1. Thanks to Lemmas 3.1, 3.2, 3.3, there exist three functions

$$s,\, p : (0,\infty)\times\Omega \longrightarrow \mathbb{R}, \qquad \mathbf{v}_w : (0,\infty)\times\Omega \longrightarrow \mathbb{R}^N, \tag{3.5}$$

such that, for every $T > 0$,

$$s \in W^{1,\infty}(0,T;H^1(\Omega)), \quad p \in L^\infty(0,T;H^1(\Omega)), \quad \mathbf{v}_w \in L^\infty(0,T;H^2(\Omega)),$$

and, passing to a subsequence,

$$\begin{aligned} s_n &\rightharpoonup s, && \text{weakly in } W^{1,\ell}(0,T;H^1(\Omega)),\ 1\le \ell<\infty,\ T>0,\\ p_n &\rightharpoonup p, && \text{weakly in } L^{\ell}(0,T;H^1(\Omega)),\ 1\le \ell<\infty,\ T>0,\\ \mathbf{v}_{w,n} &\rightharpoonup \mathbf{v}_w, && \text{weakly in } L^{\ell}(0,T;H^2(\Omega)),\ 1\le \ell<\infty,\ T>0. \end{aligned} \tag{3.6}$$

In particular, we have that

$$\begin{aligned} s_n &\to s, && \text{strongly in } L^2((0,T)\times\Omega),\ T>0,\\ s_n &\to s, && \text{a.e. in } (0,\infty)\times\Omega,\\ \nabla p_n &\rightharpoonup \nabla p, && \text{weakly in } L^2((0,T)\times\Omega),\ T>0. \end{aligned} \tag{3.7}$$

Therefore, the distributional formulation of (2.12), the Dominated Convergence Theorem and the boundedness of f and λ_T implies (**D.2**), (**D.3**), and (**D.4**).

We conclude by proving that (**D.1**) holds. Thanks to (3.5) we have only to prove that

$$p \in L^\infty(0,T;W^{2,1}(\Omega)). \tag{3.8}$$

Clearly, (3.6), Lemmas 3.1, 3.2, 3.3 and the weak lower semicontinuity if the norms give the following estimates

$$\begin{aligned} \|p(t,\cdot)\|_{H^1(\Omega)} &\le C_1 \|\pi(t,\cdot)\|_{L^2(\partial\Omega)},\\ \|\mathbf{v}_w(t,\cdot)\|_{H^2(\Omega)} &\le \frac{C_3}{\mu}\left(\|f\lambda_T\|_{L^\infty(\mathbb{R})}\|\pi(t,\cdot)\|_{L^2(\partial\Omega)} + \|h(t,\cdot)\|_{L^2(\partial\Omega)}\right),\\ \|s(t,\cdot)\|_{H^1(\Omega)} &\le \|s_0\|_{H^1(\Omega)} + \frac{C_4}{\mu}\left(\|f\lambda_T\|_{L^\infty(\mathbb{R})}\|\pi(t,\cdot)\|_{L^2(\partial\Omega)} + \|h(t,\cdot)\|_{L^2(\partial\Omega)}\right),\\ \|\partial_t s(t,\cdot)\|_{H^1(\Omega)} &\le \frac{C_4}{\mu}\left(\|f\lambda_T\|_{L^\infty(\mathbb{R})}\|\pi(t,\cdot)\|_{L^2(\partial\Omega)} + \|h(t,\cdot)\|_{L^2(\partial\Omega)}\right), \end{aligned}$$

for almost every $t > 0$.

Since, from (2.1)

$$\Delta_{\mathbf{x}} p = -\frac{\lambda_T'(s)}{\lambda_T(s)}\nabla_{\mathbf{x}} p\cdot\nabla_{\mathbf{x}} s,$$

therefore, thanks to (**H.2**) and (**H.3**),

$$\|\Delta_{\mathbf{x}} p(t,\cdot)\|_{L^1(\Omega)} \le \frac{1}{2}\left\|\frac{\lambda_T'}{\lambda_T}\right\|_{L^\infty(\mathbb{R})}\left(\|\nabla_{\mathbf{x}} p(t,\cdot)\|^2_{L^2(\Omega)} + \|\nabla_{\mathbf{x}} s(t,\cdot)\|^2_{L^2(\Omega)}\right),$$

that proves (3.8). □

Thus, we have shown that weak solutions of the Brinkman regularization of two-phase flows in a porous medium (1.12) exist. The question of uniqueness is still open.

Remark 3.1. *It must be emphasized that many of the estimates derived in the proof of the existence theorem* 2.1 *are* μ *dependent and blow up as the regularization parameter* $\mu \to 0$. *In particular, the estimate* (3.2) *on the phase velocity is* μ*-dependent as are the estimates on the saturation* (3.3). *Thus, in the limit* $\mu \to 0$, *which corresponds to the classical Darcy's law, we do not expect that the velocity field and the saturation are as regular as in the case of the Brinkman approximation. As an example, it is well known* *that the saturation contains discontinuities in the form of shocks for the classical two-phase flow problem which is inconsistent with the* H^1 *estimate in* (3.3)*. Hence, we have been unable to obtain any convergence results for the Brinkman system* (1.12) *to the classical two-phase Darcy system* (1.9) *as* $\mu \to 0$.

Remark 3.2. *Here, we have focused on the case of two-phase flows. A Brinkman regularization of multi phase flows can be obtained analogously to the derivation of the Brinkman two phase flow model in the introduction. This system for* m *(*$m \geq 3$*) phases reads as*

$$
\begin{cases}
\partial_t s_1 + \mathrm{div}_{\mathbf{x}}\left(\mathbf{v}_{w,1}\right) = 0, & t > 0,\, \mathbf{x} \in \Omega, \\
\ldots\ldots \\
\partial_t s_m + \mathrm{div}_{\mathbf{x}}\left(\mathbf{v}_{w,m}\right) = 0, & t > 0,\, \mathbf{x} \in \Omega, \\
-\mu \Delta_{\mathbf{x}} \mathbf{v}_{w,1} + \mathbf{v}_{w,1} = -\lambda_1(s_1)\nabla_{\mathbf{x}} p, & t > 0,\, \mathbf{x} \in \Omega, \\
\ldots\ldots \\
-\mu \Delta_{\mathbf{x}} \mathbf{v}_{w,m} + \mathbf{v}_{w,m} = -\lambda_m(s_m)\nabla_{\mathbf{x}} p, & t > 0,\, \mathbf{x} \in \Omega, \\
-\mathrm{div}_{\mathbf{x}}\left(\lambda_T(s_1,..,s_m)\nabla_{\mathbf{x}} p\right) = 0, & t > 0,\, \mathbf{x} \in \Omega,
\end{cases} \tag{3.9}
$$

augmented with suitable initial and boundary conditions. As in definition 2.1*, we can analogously define a suitable notion of weak solutions and prove existence of solutions by following the approximation procedure presented in section* 2 *and proving analogous estimates like those in the proof of theorem* 2.1.

4. A convergent numerical scheme for the Brinkman regularization

In this section, we will present an efficient numerical scheme to approximate the Brinkman regularization for two-phase flow (1.12). For simplicity, we consider the unit square in two space dimensions i.e, $\Omega = [0,1]^2 \subset \mathbb{R}^2$. As many interesting benchmark tests include a source in the pressure equation (to model injection of water), we consider the following modification of the Brinkman regularization (1.12),

$$
\begin{cases}
\partial_t s + \mathrm{div}_{\mathbf{x}}\left(\mathbf{v}_w\right) = 0, & t > 0,\, \mathbf{x} \in \Omega, \\
-\mu \Delta_{\mathbf{x}} \mathbf{v}_w + \mathbf{v}_w = -f(s)\lambda_T(s)\nabla_{\mathbf{x}} p, & t > 0,\, \mathbf{x} \in \Omega, \\
-\mathrm{div}_{\mathbf{x}}\left(\lambda_T(s)\nabla_{\mathbf{x}} p\right) = q, & t > 0,\, \mathbf{x} \in \Omega, \\
\partial_\nu p(t,\mathbf{x}) = \pi(t,\mathbf{x}), & t > 0,\, \mathbf{x} \in \partial\Omega, \\
\mathbf{v}_w(t,\mathbf{x}) \cdot \nu(\mathbf{x}) = h(t,\mathbf{x}), & t > 0,\, \mathbf{x} \in \partial\Omega, \\
\int_\Omega p(t,\mathbf{x})d\mathbf{x} = 0, & t > 0, \\
s(0,\mathbf{x}) = s_0(\mathbf{x}), & \mathbf{x} \in \Omega,
\end{cases} \tag{4.1}
$$

Here, $q \in L^\infty(0,T;L^2(\Omega))$ denotes a source function. Note that the existence result Theorem 2.1 can be readily extended to this case of including a source term.

For the sake of definiteness, let $\mathbf{v}_w = (u,v)$. The boundary values are $\pi(t,\mathbf{x}) = 0$, and

$$
\begin{aligned}
u(0,y) = u(1,y) = 0,\ \partial_y u(x,0) = \partial_y u(x,1) = 0 \\
v(x,0) = v(x,1) = 0,\ \partial_x v(0,y) = \partial_x v(1,y) = 0.
\end{aligned} \tag{4.2}
$$

We discretize the computational domain $[0,1]^2$ with on a Cartesian mesh with gridpoints $x_i = (i-1/2)\Delta x$, $y_j = (j-1/2)\Delta x$, $j = 1,\ldots,N$, $\Delta x = 1/N$. Let p_{ij}^n, $\mathbf{v}_{ij}^n$, and s_{ij}^n denote the approximation to p, $\mathbf{v}_w$ and s respectively, evaluated at (x_i,y_j,t_n), where $t_n = n\Delta t$. The scheme for p_{ij}^n reads

$$
-D_+^x\left(t_{i-1/2,j}^n D_-^x p_{ij}^n\right) - D_+^y\left(t_{i,j-1/2}^n D_-^y p_{ij}^n\right) = q_{ij}^n, \quad i,\, j = 1,\ldots,N, \tag{4.3}
$$

where

$$t^n_{i+1/2,j} = \frac{\lambda_T(s^n_{ij}) + \lambda_T(s^n_{i+1,j})}{2} \quad \text{and} \quad t^n_{i,j+1/2} = \frac{\lambda_T(s^n_{ij}) + \lambda_T(s^n_{i,j+1})}{2},$$

with the boundary values

$$t^n_{1/2,j} = t^n_{N+1/2,j} = t^n_{i,1/2} = t^n_{i,N+1/2} = 0, \quad \text{for } i,\, j = 1, \dots, N,\, n \geq 0,$$

and

$$q^n_{ij} = q(x_i, y_j), \quad i, j = 1, \dots, N, n \geq 0.$$

To define the scheme for $\mathbf{v}^n_{ij}$, we first define

$$f^n_{i+1/2,j} = \frac{f(s^n_{ij}) + f(s^n_{i+1,j})}{2} \quad \text{and} \quad f^n_{i,j+1/2} = \frac{f(s^n_{ij}) + f(s^n_{i,j+1})}{2}.$$

Then the scheme for $u^n_{i+1/2,j}$ reads

$$-\mu \left(D^x_+ D^x_- + D^y_+ D^y_- \right) u^n_{i+1/2,j} + u^n_{i+1/2,j} = f^n_{i+1/2,j} t^n_{i+1/2,j} D^x_+ p^n_{ij}, \tag{4.4}$$

for $i = 1, \dots, N-1$, $j = 1, \dots, N$ with boundary values

$$u^n_{1/2,j} = u^n_{N+1/2,j} = 0, \; j = 1, \dots, N.$$

Similarly, the scheme for $v^n_{i,j+1/2}$ reads

$$-\mu \left(D^x_+ D^x_- + D^y_+ D^y_- \right) v^n_{i,j+1/2} + v^n_{i,j+1/2} = f^n_{i,j+1/2} t^n_{i,j+1/2} D^y_+ p^n_{ij}, \tag{4.5}$$

for $i = 1, \dots, N$, $j = 1, \dots, N-1$ with boundary values

$$v^n_{i,1/2} = v^n_{i,N+1/2} = 0, \; i = 1, \dots, N.$$

Finally we update s^n_{ij} by

$$s^{n+1}_{ij} = \frac{1}{4} \left(s^n_{i+1,j} + s^n_{i-1,j} + s^n_{i,j+1} + s^n_{i,j-1} \right) - \Delta t \left(D^x_- u^n_{i+1/2,j} + D^y_- v^n_{i,j+1/2} \right), \tag{4.6}$$

for $n \geq 0$ and $i,\, j = 1, \dots, N$, with the initial values $s^0_{ij} = s_0(x_i, y_j)$.

4.1. **Convergence of the scheme in 2D.** We will show that the approximate solutions generated by the finite difference scheme (4.3) – (4.6) converge to a weak solution of (4.1) for a fixed μ. To do so, we mimic the estimates of Lemmas 3.1 – 3.3 in the discrete setting.

From the discrete values s^n_{ij}, $i, j = 0, \dots, N$, $n \geq 0$, we define the piecewise linear interpolant

$$\begin{aligned} s^n(x,y) &= s^n_{ij} + (x - x_i)\, D^x_+ s^n_{ij} + (y - y_j)\, D^y_+ s^n_{ij} + (x - x_i)\,(y - y_j)\, D^y_+ D^x_+ s^n_{ij} \\ s_\Delta(t; x, y) &= s^n(x,y) + (t - t_n)\, D^t_+ s^n(x,y) \\ &(t; x, y) \in [t_n, t_{n+1}) \times [x_i, x_{i+1}) \times [y_j, y_{j+1}),\, n \geq 0,\, i, j = 0, \dots, N, \end{aligned}$$

where we have denoted

$$D^t_+ s^n(x,y) = \frac{1}{\Delta t}(s^{n+1}(x,y) - s^n(x,y)),$$

the forward divided difference in the temporal direction. In a similar way, we define $p^n(x,y)$ to be the bilinear interpolation with $p^n(x_i, y_j) = p^n_{ij}$; $u^n(x,y)$ and $v^n(x,y)$ to be the piecewise quadratic splines with $u^n(x_{i+1/2}, y_j) = u^n_{i+1/2,j}$ and $v^n(x_i, y_{j+1/2}) = v^n_{i,j+1/2}$, and $p_\Delta(t; x, y)$, $u_\Delta(t; x, y)$ and $v_\Delta(t; x, y)$ to be the linear interpolations of $p^n(x,y)$, $u^n(x,y)$ and $v^n(x,y)$ in t between t_n and t_{n+1}, $n \geq 0$.

Now, we will show the following estimates on the approximate solutions:

Lemma 4.1. *Let $\Delta = (\Delta x, \Delta y, \Delta t)$, $\Delta x, \Delta y, \Delta t > 0$ and $q \in L^\infty(0,T; L^2(\Omega))$. We have $p_\Delta \in L^\infty(0,T; H^1(\Omega))$ for any $T > 0$ with*

$$\|p_\Delta(t;\cdot)\|_{H^1(\Omega)} \le \frac{\sqrt{2}}{\lambda_*}\|q(t;\cdot)\|_{L^2(\Omega)}, \quad 0 \le t \le T. \tag{4.7}$$

Proof. Using the identities

$$\sum_{i,j} t^n_{i-1/2,j}|D^x_- p^n_{ij}|^2 = -\sum_{i,j} D^x_+(t^n_{i-1/2,j}D^x_- p^n_{ij})\, p^n_{ij},$$
$$\sum_{i,j} t^n_{i,j-1/2}|D^y_- p^n_{ij}|^2 = -\sum_{i,j} D^y_+(t^n_{i,j-1/2}D^y_- p^n_{ij})\, p^n_{ij},$$

multiplying (4.3) by p_{ij} and summing over the indices i, j, we obtain

$$\sum_{i,j}\big(t^n_{i-1/2,j}|D^x_- p^n_{ij}|^2 + t^n_{i,j-1/2}|D^y_- p^n_{ij}|^2\big) = \sum_{i,j} q^n_{ij}p^n_{ij}.$$

Since $t^n_{i-1/2,j}, t^n_{i,j-1/2} \ge \lambda_*$ by assumption (**H.3**), and $(\alpha\, a^2 + b^2/\alpha)/2 \ge ab$ for $a, b \in \mathbb{R}$, $\alpha > 0$, this yields

$$\sum_{i,j}\big(|D^x_- p^n_{ij}|^2 + |D^y_- p^n_{ij}|^2\big) \le \frac{1}{2\lambda_*}\left(\alpha\sum_{i,j}(p^n_{ij})^2 + \frac{1}{\alpha}\sum_{i,j}(q^n_{ij})^2\right),$$

and hence

$$\|\partial_x p_\Delta(t;\cdot)\|^2_{L^2(\Omega)} + \|\partial_y p_\Delta(t;\cdot)\|^2_{L^2(\Omega)} \le \frac{1}{2\lambda_*}\left(\alpha\|p_\Delta(t;\cdot)\|^2_{L^2(\Omega)} + \frac{1}{\alpha}\|q(t;\cdot)\|^2_{L^2(\Omega)}\right)$$

Using Poincaré's inequality,

$$\|\nabla_{\mathbf{x}} f\|_{L^2(\Omega)} \le C(\Omega)\|f\|_{L^2(\Omega)},$$

where for $\Omega = [0,1]^2$, $C(\Omega) = 1$, we obtain by choosing $\alpha = \lambda_*$,

$$\|p_\Delta(t;\cdot)\|_{L^2(\Omega)},\ \|\nabla_{\mathbf{x}} p_\Delta(t;\cdot)\|_{L^2(\Omega)} \le \frac{1}{\lambda_*}\|q(t;\cdot)\|_{L^2(\Omega)},$$

which implies (4.8). □

Lemma 4.2. *Let $\Delta = (\Delta x, \Delta y, \Delta t)$, $\Delta x, \Delta y, \Delta t > 0$, $\mu > 0$ and $q \in L^\infty(0,T; L^2(\Omega))$. Assume furthermore that $f\lambda_T$ is bounded. Then $u_\Delta, v_\Delta \in L^\infty(0,T; H^2(\Omega))$ for $T > 0$ with*

$$\mu^2\|\nabla^2_{\mathbf{x}} u_\Delta(t;\cdot)\|^2_{L^2(\Omega)} + \mu\|\nabla_{\mathbf{x}} u_\Delta(t;\cdot)\|^2_{L^2(\Omega)} + \|u_\Delta(t;\cdot)\|^2_{L^2(\Omega)} \le C\,\|f\lambda_T\|^2_{L^\infty}\|\partial_x p_\Delta(t;\cdot)\|^2_{L^2(\Omega)} \tag{4.8a}$$

$$\mu^2\|\nabla^2_{\mathbf{x}} v_\Delta(t;\cdot)\|^2_{L^2(\Omega)} + \mu\|\nabla_{\mathbf{x}} v_\Delta(t;\cdot)\|^2_{L^2(\Omega)} + \|v_\Delta(t;\cdot)\|^2_{L^2(\Omega)} \le C\,\|f\lambda_T\|^2_{L^\infty}\|\partial_y p_\Delta(t;\cdot)\|^2_{L^2(\Omega)}, \tag{4.8b}$$

where $C > 0$ is a scaling factor, not depending on the other quantities.

Proof. We take the square of equation (4.4), sum it over the indices i and j and use the summation by parts identity

$$\sum_{i,j} u^n_{i+1/2,j}(D^x_+D^x_- + D^y_+D^y_-)u^n_{i+1/2,j} = -\sum_{i,j}\big(|D^x_- u^n_{i+1/2,j}|^2 + |D^y_- u^n_{i+1/2,j}|^2\big)$$

to obtain

$$\sum_{i,j}\big(|(D^x_+D^x_- + D^y_+D^y_-)u^n_{i+1/2,j}|^2 + 2\mu(|D^x_- u^n_{i+1/2,j}|^2 + |D^y_- u^n_{i+1/2,j}|^2) + |u^n_{i+1/2,j}|^2\big) \tag{4.9}$$

$$= \sum_{i,j} (f^n_{i+1/2,j})^2 (t^n_{i+1/2,j})^2 |D^x_+ p^n_{ij}|^2.$$

Using summation by parts twice for the first term on the right hand side of (4.9) gives

$$\sum_{i,j} |(D^x_+ D^x_- + D^y_+ D^y_-) u^n_{i+1/2,j}|^2 = \sum_{i,j} \big(|D^x_+ D^x_- u^n_{i+1/2,j}|^2 + |D^y_+ D^y_- u^n_{i+1/2,j}|^2 + 2|D^x_- D^y_- u^n_{i+1/2,j}|^2 \big),$$

which implies (4.8a). In the same way, we can show (4.8b). Since $p_\Delta \in L^\infty(0,T;H^1(\Omega))$ by Lemma 4.1, we get $u_\Delta, v_\Delta \in L^\infty(0,T;H^2(\Omega))$. □

Now it is easy to show that $s_\Delta \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$:

Lemma 4.3. *Let $\Delta = (\Delta x, \Delta y, \Delta t)$, $\Delta x, \Delta y, \Delta t > 0$ with $\Delta x/\Delta t, \Delta y/\Delta t \le K$, where $0 < K < \infty$, and let $\mu > 0$. Moreover assume $q \in L^\infty(0,T;L^2(\Omega))$, $s_0 \in H^1(\Omega)$ and that $f\lambda_T$ is bounded. Then $s_\Delta \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$ for $T > 0$ with*

$$\|s_\Delta(t;\cdot)\|_{L^2(\Omega)} \le \|s_0\|_{L^2(\Omega)} + \frac{C\,t}{\lambda_* \sqrt{\mu}} \|f\lambda_T\|_{L^\infty} \|q\|_{L^\infty(0,T;L^2(\Omega))}, \tag{4.10a}$$

$$\|\nabla_{\mathbf{x}} s_\Delta(t;\cdot)\|_{L^2(\Omega)} \le \|\nabla_{\mathbf{x}} s_0\|_{L^2(\Omega)} + \frac{C\,t}{\lambda_* \mu} \|f\lambda_T\|_{L^\infty} \|q\|_{L^\infty(0,T;L^2(\Omega))}, \tag{4.10b}$$

$$\|\partial_t s_\Delta(t;\cdot)\|_{L^2(\Omega)} \le C(\sqrt{K}+1)\Big(\|\nabla_{\mathbf{x}} s_0\|_{L^2(\Omega)} + \frac{t}{\lambda_* \mu} \|f\lambda_T\|_{L^\infty} \|q\|_{L^\infty(0,T;L^2(\Omega))} \Big), \tag{4.10c}$$

for $0 \le t \le T$ and where $C > 0$ is a constant.

Proof. We take the square of equation (4.6), sum over the indices i, j and use triangle inequality to obtain

$$\Big(\sum_{i,j} |s^{n+1}_{ij}|^2 \Big)^{1/2} \le \Big(\sum_{i,j} |s^n_{ij}|^2 \Big)^{1/2} + \Delta t \Big(\sum_{i,j} |D^x_- u^n_{i+1/2,j} + D^y_- u^n_{i,j+1/2}|^2 \Big)^{1/2},$$

which implies

$$\|s^{n+1}\|_{L^2(\Omega)} \le \|s^n\|_{L^2(\Omega)} + \Delta t \big(\|\partial_x u_\Delta(t^n;\cdot)\|_{L^2(\Omega)} + \|\partial_y v_\Delta(t^n;\cdot)\|_{L^2(\Omega)} \big).$$

Iterating over n, this yields

$$\|s^n\|_{L^2(\Omega)} \le \|s^0\|_{L^2(\Omega)} + t^n \big(\|\nabla_{\mathbf{x}} u_\Delta\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla_{\mathbf{x}} v_\Delta\|_{L^\infty(0,T;L^2(\Omega))} \big).$$

Using Lemma 4.1, we obtain

$$\begin{aligned} \|s_\Delta(t;\cdot)\|_{L^2(\Omega)} &\le \|s_0\|_{L^2(\Omega)} + \frac{C\,t}{\sqrt{\mu}} \|f\lambda_T\|_{L^\infty} \|\nabla_{\mathbf{x}} p\|_{L^\infty(0,T;L^2(\Omega))} \\ &\le \|s_0\|_{L^2(\Omega)} + \frac{C\,t}{\lambda_* \sqrt{\mu}} \|f\lambda_T\|_{L^\infty} \|q\|_{L^\infty(0,T;L^2(\Omega))}, \end{aligned}$$

where we have used (4.8) for the second inequality. In order to show that the gradient of $s_\Delta(t;\cdot)$ is in $L^2(\Omega)$, we apply the linear operators D^x_+, D^y_+ to the evolution equation for s^n_{ij}, (4.6),

$$D^x_+ s^{n+1}_{ij} = \frac{1}{4} \left(D^x_+ s^n_{i+1,j} + D^x_+ s^n_{i-1,j} + D^x_+ s^n_{i,j+1} + D^x_+ s^n_{i,j-1} \right) - \Delta t \left(D^x_+ D^x_- u^n_{i+1/2,j} + D^x_+ D^y_- v^n_{i,j+1/2} \right),$$

(and similarly for D_+^y), then take the square of the above equation, sum over the indices i,j and use again triangle inequality, to obtain

$$\Big(\sum_{i,j}|D_+^x s_{ij}^{n+1}|^2\Big)^{1/2} \leq \Big(\sum_{i,j}|D_+^x s_{ij}^n|^2\Big)^{1/2} + \Delta t\Big(\sum_{i,j}|D_+^x D_-^x u_{i+1/2,j}^n + D_+^x D_-^y u_{i,j+1/2}^n|^2\Big)^{1/2},$$

which implies after iteration over n,

$$\|\partial_x s^n\|_{L^2(\Omega)} \leq \|\partial_x s^0\|_{L^2(\Omega)} + t^n\big(\|\partial_x^2 u_\Delta\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_x\partial_y v_\Delta\|_{L^\infty(0,T;L^2(\Omega))}\big).$$

Hence, using Lemmas 4.1 and 4.2, we obtain

$$\|\partial_x s_\Delta(t;\cdot)\|_{L^2(\Omega)} \leq \|\partial_x s_0\|_{L^2(\Omega)} + \frac{C\,t}{\lambda_*\mu}\|f\lambda_T\|_{L^\infty}\|q\|_{L^\infty(0,T;L^2(\Omega))}.$$

In a similar way, we obtain an estimate for $\|\partial_y s_\Delta(t;\cdot)\|_{L^2(\Omega)}$ and thus

$$\|\nabla_{\mathbf{x}} s_\Delta(t;\cdot)\|_{L^2(\Omega)} \leq \|\nabla_{\mathbf{x}} s_0\|_{L^2(\Omega)} + \frac{C\,t}{\lambda_*\mu}\|f\lambda_T\|_{L^\infty}\|q\|_{L^\infty(0,T;L^2(\Omega))}.$$

To obtain an estimate on $\partial_t s_\Delta$, we rewrite the evolution equation for s_{ij}^n as

$$\frac{s_{ij}^{n+1}-s_{ij}^n}{\Delta t} = \frac{1}{4\Delta t}\left(\Delta x^2 D_+^x D_-^x s_{ij}^n + \Delta y^2 D_+^y D_-^y s_{ij}^n\right) - \Big(D_-^x u_{i+1/2,j}^n + D_-^y v_{i,j+1/2}^n\Big), \tag{4.11}$$

We notice that

$$\Delta x|D_+^x D_-^x s_{ij}^n| \leq |D_+^x s_{ij}^n| + |D_-^x s_{ij}^n|,$$
$$\Delta y|D_+^y D_-^y s_{ij}^n| \leq |D_+^y s_{ij}^n| + |D_-^y s_{ij}^n|,$$

which implies after taking the square of equation (4.11) and summing over i,j

$$\sum_{i,j}|D_+^t s_{ij}^n|^2 \leq K\sum_{i,j}\big(|D_+^x s_{ij}^n|^2 + |D_+^y s_{ij}^n|^2\big) + 2\sum_{i,j}\big(|D_-^x u_{i+1/2,j}^n + D_-^y u_{i,j+1/2}^n|^2\big).$$

Thus

$$\begin{aligned}\|\partial_t s_\Delta(t;\cdot)\|_{L^2(\Omega)} &\leq C\big(\sqrt{K}\|\nabla_{\mathbf{x}} s_\Delta\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_x u_\Delta\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_y v_\Delta\|_{L^\infty(0,T;L^2(\Omega))}\big)\\ &\leq C\Big(\sqrt{K}\|\nabla_{\mathbf{x}} s_0\|_{L^2(\Omega)} + \frac{(\sqrt{K}+1)t}{\lambda_*\mu}\|f\lambda_T\|_{L^\infty}\|q\|_{L^\infty(0,T;L^2(\Omega))}\Big),\end{aligned}$$

where we have used (4.10b) and Lemmas 4.1 and 4.2 for the second inequality. □

Now we are ready to prove the main convergence theorem for the finite difference scheme,

Theorem 4.1. *Fix $\mu > 0$ and assume $q \in L^\infty(0,T;L^2(\Omega))$, $s_0 \in H^1(\Omega)$ and $f,\lambda_T \in L^\infty(\mathbb{R})$. Furthermore, let $\Delta = (\Delta x, \Delta y, \Delta t) > 0$ such that $\Delta x/\Delta t, \Delta y/\Delta t \leq K < \infty$. Then a subsequence of $\{p_\Delta\}_{\Delta>0}$, $\{u_\Delta\}_{\Delta>0}$, $\{v_\Delta\}_{\Delta>0}$, $\{s_\Delta\}_{\Delta>0}$, converges to a weak solution $(p, \mathbf{v}_w, s)$ of (4.1) as $\Delta \to 0$, and*

$$s \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \quad p \in L^\infty(0,T;H^1(\Omega)), \quad \mathbf{v}_w \in L^\infty(0,T;H^2(\Omega)).$$

Proof. Due to the Lemmas 4.1, 4.2 and 4.3, we have for a subsequence

$$\begin{aligned} s_\Delta &\rightharpoonup s, && \text{weakly in } L^\ell(0,T;H^1(\Omega)),\ 1 \leq \ell < \infty,\ T > 0,\\ p_\Delta &\rightharpoonup p, && \text{weakly in } L^\ell(0,T;H^1(\Omega)),\ 1 \leq \ell < \infty,\ T > 0,\\ (u_\Delta, v_\Delta) &\rightharpoonup \mathbf{v}_w, && \text{weakly in } L^\ell(0,T;H^2(\Omega)),\ 1 \leq \ell < \infty,\ T > 0. \end{aligned} \tag{4.12}$$

The Aubin-Lions Lemma gives us the compact embedding $W^{1,\infty}(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))\subset\subset L^\ell(0,T;L^2(\Omega))$ for $1\le \ell<\infty$, and hence

$$s_\Delta \to s, \qquad \text{strongly in } L^2((0,T)\times\Omega),\ T>0, \text{ and} \tag{4.13}$$
$$s_\Delta \to s, \qquad \text{a.e. in } (0,T)\times\Omega,$$

for a subsequence. We denote

$$\Lambda^n(s_\Delta)=\begin{pmatrix}\lambda_x^n & 0\\ 0 & \lambda_y^n\end{pmatrix},\quad F^n(s_\Delta)=\begin{pmatrix} f_x^n & 0\\ 0 & f_y^n\end{pmatrix},$$

where $\lambda_x^n(s_\Delta),\lambda_y^n(s_\Delta)$ are piecewise linear interpolations of $t^n_{i+1/2,j}$ and $t^n_{i,j+1/2}$ satisfying $\lambda_x^n((s_{ij}^n+s_{i+1,j}^n)/2)=t^n_{i+1/2,j}$ and $\lambda_x^n((s_{ij}^n+s_{i,j+1}^n)/2)=t^n_{i,j+1/2}$ and similarly $f_x^n(s_\Delta), f_y^n(s_\Delta)$ are piecewise linear interpolations of $f^n_{i+1/2,j}$ and $f^n_{i,j+1/2}$ satisfying $f_x^n((s_{ij}^n+s_{i+1,j}^n)/2)=f^n_{i+1/2,j}$ and $f_x^n((s_{ij}^n+s_{i,j+1}^n)/2)=f^n_{i,j+1/2}$. Then we let

$$\Lambda_\Delta(s_\Delta,t)=\Lambda^n(s_\Delta),\quad F_\Delta(s_\Delta,t)=F^n(s_\Delta),\quad t\in[t^n,t^n+1),\ n\ge 0.$$

Than, thanks to (4.13),

$$\Lambda_\Delta(s_\Delta,t)\to\lambda_T(s), \qquad \text{a.e. in } (0,T)\times\Omega, \text{ and} \tag{4.14}$$
$$F_\Delta(s_\Delta,t)\to f(s), \qquad \text{a.e. in } (0,T)\times\Omega.$$

Thus, using in addition, that

$$\nabla_{\mathbf{x}} p_\Delta \rightharpoonup \nabla_{\mathbf{x}} p, \qquad \text{weakly in } L^2((0,T)\times\Omega),\ T>0,$$

the boundedness of f and λ_T, and the Dominated Convergence Theorem, we can pass to the limit $\Delta\to 0$ in the weak formulations

$$\int_0^T\int_\Omega\left(s_\Delta\partial_t\varphi+(u_\Delta,v_\Delta)^T\nabla_{\mathbf{x}}\varphi-s_\Delta\left(\frac{\Delta x^2}{\Delta t}\partial_x^2\varphi+\frac{\Delta y^2}{\Delta t}\partial_y^2\varphi\right)\right)d\mathbf{x}\,dt + \int_\Omega s_0(x)\varphi(0,x)\,d\mathbf{x}=0,$$

$$\int_0^T\int_\Omega\left((\Lambda_\Delta\nabla_{\mathbf{x}}p_\Delta)\cdot\nabla_{\mathbf{x}}\varphi-q_\Delta\varphi\right)d\mathbf{x}\,dt=0;$$

where $\varphi\in C^\infty([0,T)\times\Omega)$ with compact support and we have denoted by q_Δ a piecewise linear interpolation of q_{ij}^n, $i,j=0,\dots,N$, $n\ge 0$; and

$$\mu\int_0^T\int_\Omega\nabla_{\mathbf{x}}(u_\Delta,v_\Delta)^T\cdot\nabla_{\mathbf{x}}\Phi\,d\mathbf{x}\,dt+\int_0^T\int_\Omega(u_\Delta,v_\Delta)^T\cdot\Phi\,d\mathbf{x}\,dt = -\int_0^T\int_\Omega(F_\Delta\Lambda_\Delta\nabla_{\mathbf{x}}p_\Delta)\cdot\Phi\,d\mathbf{x}\,dt;$$

where $\Phi\in C^\infty([0,T)\times\Omega;\mathbb{R}^2)$ with compact support, to obtain the result. □

4.2. **Numerical experiments.** We will now show through numerical experiments that the finite difference scheme (4.3) – (4.6) is effective in computing approximate solutions of the Brinkman regularization of the two-phase flow problem (4.1). We consider the well-known *quarter five spot* problem that models water flooding in an oil reservoir. To this end, we consider

$$q(\mathbf{x})=\begin{cases}4/(\pi r^2) & |\mathbf{x}|\le r,\\ -4/(\pi r^2) & |\mathbf{x}-(1,1)|\le r,\\ 0 & \text{otherwise,}\end{cases}$$

where $r = 0.02$. This models the injection of water at $(0, 0)$ and the production of oil at $(1, 1)$. The initial water saturation was given by

$$s_0(\mathbf{x}) = \begin{cases} 1 & |\mathbf{x}| \leq r, \\ \exp\left(-150(|\mathbf{x}| - r)^2\right) & |\mathbf{x}| > r. \end{cases}$$

Furthermore, the boundary values of the saturation are given by,

$$s^n_{0,j} = s^n_{1,j}, \ s^n_{N+1,j} = s^n_{N,j}, \ s^n_{i,0} = s^n_{1,0}, \ \text{and } s^n_{i,N+1} = s^n_{i,N},$$

as well as

$$s^{n+1}_{ij} = 1 \ \text{ if } |(x_i, y_j)| \leq r. \tag{4.15}$$

4.2.1. *Convergence tests for a fixed μ.* We consider the Brinkman regularization with a fixed $\mu = 0.005$ and compute the approximate saturation with the numerical scheme (4.3) – (4.6), on a sequence of meshes ranging from 100×100 to 800×800 mesh points. The results of water saturation at $t = 1$ are shown in Figure 1. The results show that the saturation is computed in a robust manner and converges. The limit seems to consist of a series of waves emanating from the injection in the lower left corner.

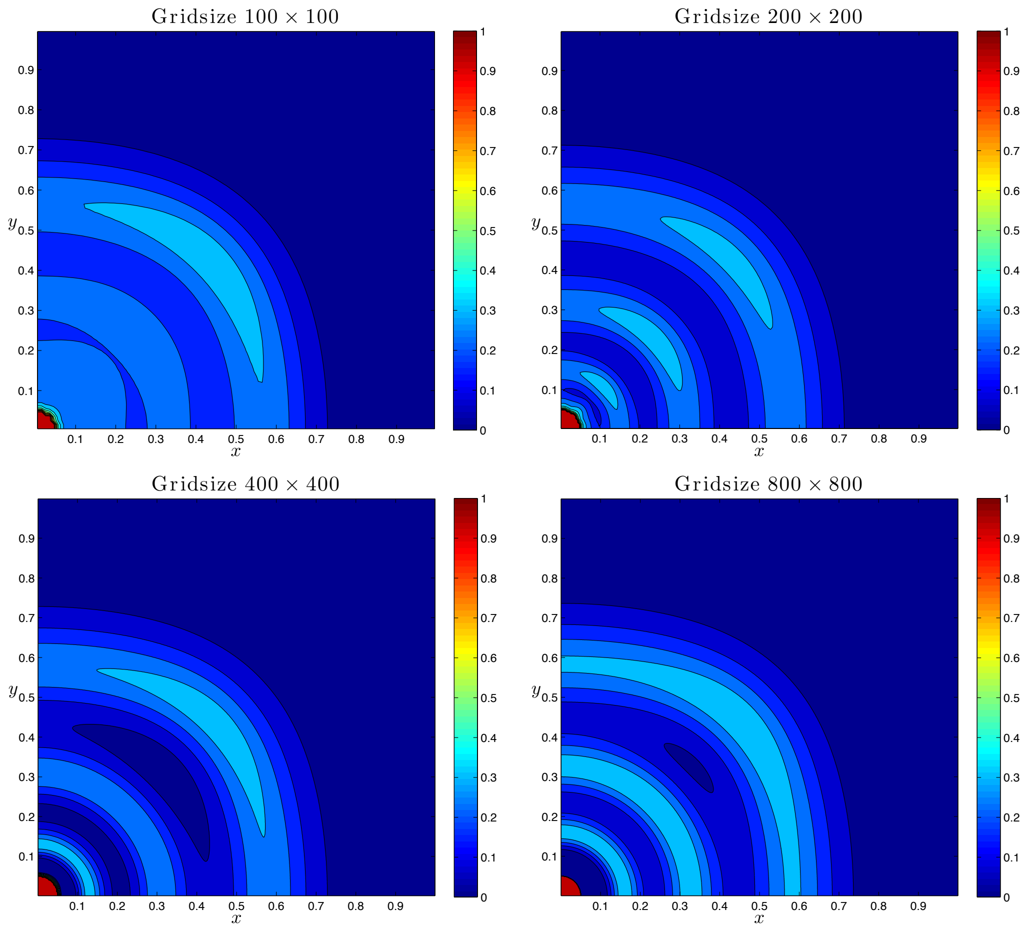

FIGURE 1. Water saturation at time $t = 1$, computed with the finite difference scheme (4.3) – (4.6) on a sequence of nested meshes with fixed regularization parameter $\mu = 0.005$.

4.2.2. *Effect of the vanishing regularization parameter μ.* The regularization parameter μ serves to indicate the deviation of the regularized problem from the classical Darcy two-phase flow problem (1.9). Formally, we can recover the classical two-phase flow problem from the regularized Brinkman approximation by letting $\mu \to 0$. On the other hand, we were unable to rigorously establish whether such a limit exists and whether it is also a weak solution of the classical two-phase flow problem (1.9), see remark 3.1. Hence, we will investigate this issue numerically by considering the quarter-five spot problem as in the previous experiment for different values of the regularization parameter μ. We present the water saturation at time $t = 0.65$ on a 2000×2000 grid, computed for four different values, $\mu = \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$. The results are shown in figure 2. Two features in the results stand out. First, the solutions become very oscillatory (atleast near the injection corner) as μ is reduced and the saturation is no longer in the physically relevant $s \in [0, 1]$ range. Second, the solutions consist of moving front between $s = 0$ and $s = 1$, followed by a train of oscillatory waves. The above results are clearly consistent

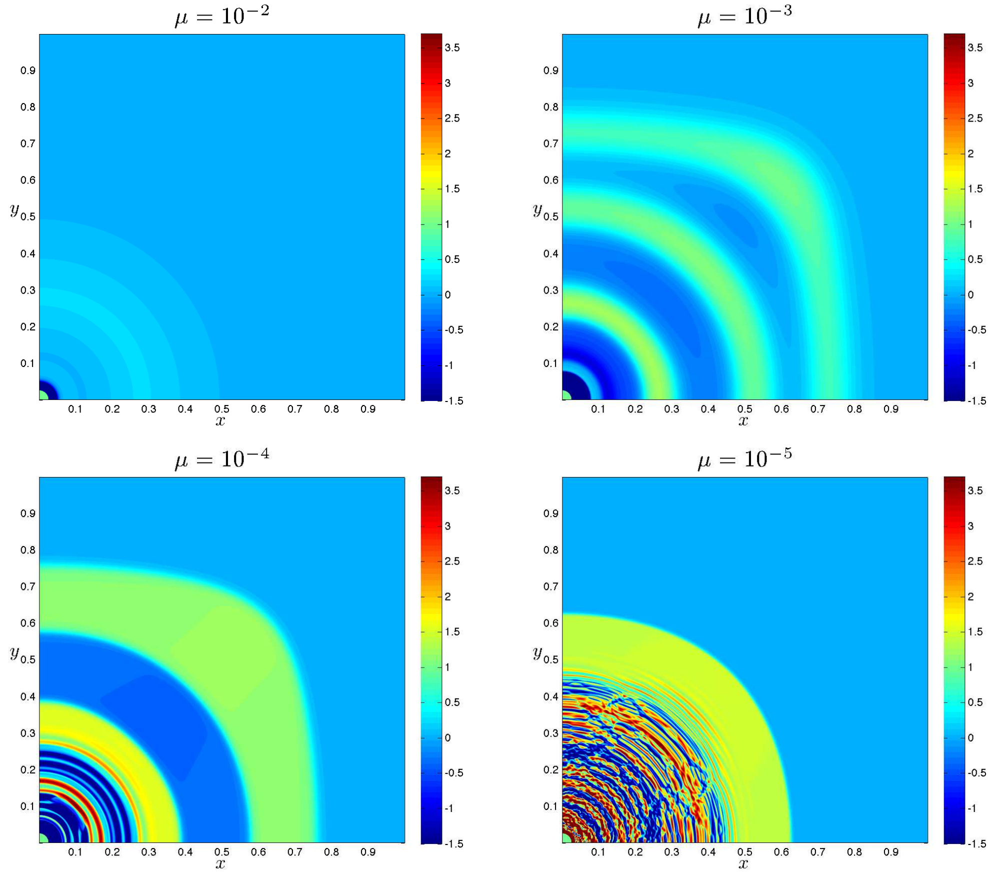

FIGURE 2. Numerical solutions of (2.1) using (4.3) – (4.15) on a 2000×2000 grid at $t = 0.65$.

with the theory. The stability estimates on the regularized saturation and velocity are μ dependent (see remark 3.1) and blow up as $\mu \to 0$. Furthermore, the convergence results for the scheme hold for any fixed no-zero μ and the stability estimates for the scheme break down as $\mu \to 0$. This break down of the estimates is perhaps reflected in the high-frequency oscillations that arise in the numerical solution as $\mu \to 0$. This clearly indicates that zero regularization limit may not be well-posed and the solutions of the Brinkman regularization may not be converge to the weak solutions of the Darcy based two-phase problem (1.9) as $\mu \to 0$.

5. Analysis in one space dimension

In order to further investigate whether the zero μ limit of the regularized Brinkman equation (1.12) converges to the Darcy two-phase flow equations (1.9), we consider the highly simplified case of one space dimension, i.e, $\Omega \subset \mathbb{R}$. In this case, the pressure equation can be solved, and the solution normalized so that $\lambda_T(s)p_x = 1$. This gives the system

$$\begin{cases} s^\mu_t + v^\mu_x = 0, \\ -\mu v^\mu_{xx} + v^\mu = f(s^\mu) \end{cases} \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}. \tag{5.1}$$

We look for traveling wave solution to this system on the form

$$s^\mu(x,t) = s\left(\frac{x-\sigma t}{\sqrt{\mu}}\right), \quad v^\mu(x,t) = v\left(\frac{x-\sigma t}{\sqrt{\mu}}\right),$$

for some functions s and v. Inserting this into (5.1),

$$-\sigma s' + v' = 0, \quad -v'' + v = f(s).$$

We want to have

$$\lim_{\xi\to-\infty} s(\xi) = s_l, \quad \lim_{\xi\to\infty} s(\xi) = s_r \quad \text{and} \quad \lim_{|\xi|\to\infty} v''(\xi) = 0.$$

Thus the first equation can be integrated to get

$$-\sigma s + v = C, \quad C = f(s_l) - \sigma s_l = f(s_r) - \sigma s_r.$$

This means that any traveling wave will travel with a speed such that the limit $\lim_{\mu\to 0} s^\mu(x,t)$ is a weak solution to the conservation law (first equation in (5.1)). We are now left with the second order equation

$$-\sigma s'' + \sigma(s - s_l) = f(s) - f(s_l),$$

or equivalently, the system of first order equations

$$\begin{aligned} s' &= w, \\ w' &= (s - s_l) - \frac{1}{\sigma}\left(f(s) - f(s_l)\right). \end{aligned} \tag{5.2}$$

This system is integrable, and the solutions are the contour lines of

$$H(s,w) = \frac{\sigma}{2}w^2 - \frac{\sigma}{s}(s-s_l)^2 + \int_{s_l}^{s} f(z) - f(s_l)\,dz.$$

Thus all fixed points are either stable centers or saddle points, located along the s-axis. Since $H_{ww} > 0$, the saddle points will be fixed points where $H_{ss} < 0$, i.e.,

$$\sigma \geq f'(s). \tag{5.3}$$

The fixed points where $\sigma < f'(s)$ will be stable centers, and cannot be left or right states of traveling waves. Since $f(s)$ is "s-shaped", for any s_l in $[0,1]$, except for the two values where f'' has extrema, there will be two other points s_1 and s_2 such that the Rankine-Hugoniot condition holds. Either one of the largest and the smallest of the three points s_l, s_1 and s_2 will be saddle points, and the middle point will be a center.

Also, independently of the shape of f, the condition (5.3) is necessary for a traveling wave. This means that the limits of such a traveling wave cannot satisfy the Lax entropy condition, $f'(s_l) \geq \sigma \geq f'(s_r)$, unless both inequalities are equalities which means that f is linear in between s_l and s_r.

If the two saddle points are on the same contour line, there is a traveling wave connecting s_l with s_r, as well as its mirror image in the (s, w) plane, connecting s_r with s_l. None of these traveling waves converge to entropic shocks as $\mu \to 0$. If there is a connecting orbit, then $H(s_l, 0) = H(s_r, 0)$, or

$$\frac{1}{2}\left(f(s_r) - f(s_l)\right)(s_r - s_l) = \int_{s_l}^{s_r} f(z) - f(s_l)\, dz. \tag{5.4}$$

Now let us assume that $1/2 - f(1/2 - \kappa) = f(1/2 + \kappa) - 1/2$, which is the case for the model flux function

$$f(s) = \frac{s^2}{s^2 + (1-s)^2}.$$

Then (5.4) implies that there is a traveling wave if and only if $|s_l - 1/2| = |s_r - 1/2|$. In particular there is a traveling wave from $s = 0$ to $s = 1$ as well as one from $s = 1$ to $s = 0$.

There is substantial evidence that the numerical schemes also converge to this non-entropic traveling wave for small μ. In Figure 3 we show a computation using the simple finite difference scheme,

$$\begin{cases} s_j^{n+1} = \frac{1}{2}\left(s_{j+1}^n + s_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(v_{j+1}^n - v_{j-1}^n\right) \\ -\frac{\mu}{\Delta x^2}\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right) + v_j^n = f\left(s_j^n\right) \end{cases} \quad \text{for } j \in 1, \dots, N,\ n \geq 0, \tag{5.5}$$

where $\Delta x = 1/N$, and $\Delta t = 0.4\Delta x$. We used initial values

$$s_j^0 = \begin{cases} 1 & j\Delta x < 0.02, \\ 0 & \text{otherwise,} \end{cases}$$

and boundary values $v_0^n = 1$, $v_{N+1}^n = 0$ and $s_0^n = 1$, $s_{N+1}^n = 0$. The figure clearly shows that even for very small $\mu = 10^{-6}$, the solution is traveling discontinuity that connects 1 and 0. On the other hand, the standard entropy solution for the limit conservation ($\mu = 0$) is given by a wave connecting 1 to some intermediate state and a shock front between this intermediate state and 0.

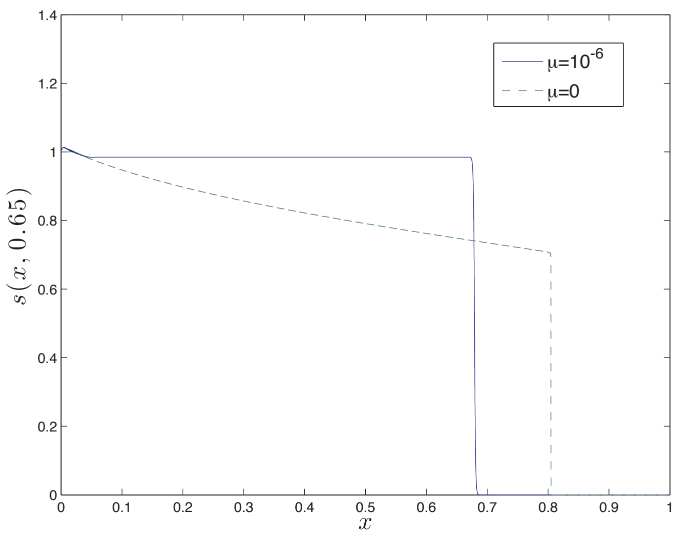

FIGURE 3. The numerical solution with $\mu = 10^{-6}$ and $\mu = 0$, and $N = 25\,000$.

Furthermore, we have also done some studied the possible convergence as $\mu \to 0$. In order to do this, we chose initial data which were not endpoints for the traveling wave solution. In Figure 4 we show the computed solutions at $t = 0.65$ using 10^4 mesh points in the interval $[0, 1]$ for three different values of μ. In this case the initial values were

$$s_0(x) = \begin{cases} 0.8 & x \leq 0.02, \\ 0.8\exp(-150(x-0.02)^2) & \text{otherwise.} \end{cases} \tag{5.6}$$

From this figure, it seems that the limit (if any such limit exists) as $\mu \to 0$ of s^μ is not

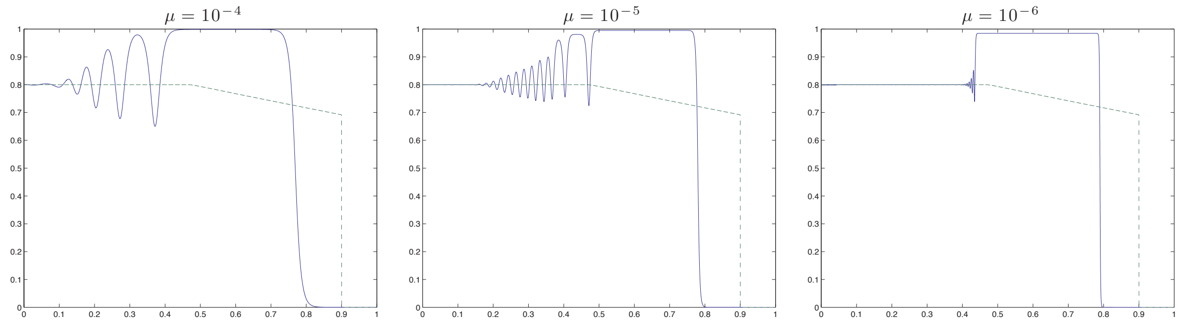

FIGURE 4. The computed solution to (5.6) at $t = 0.65$ for $\mu = 10^{-4}$ (left), $\mu = 10^{-5}$ (middle) and $\mu = 10^{-6}$ (right). In these computations, $N = 25\,000$.

the entropy solution to the conservation law. This entropy solution is also indicated in Figure 4, and differs from s^μ. As $\mu \to 0$, the computed solution seems to converge to two traveling discontinuities, one from $s = 0.8$ to 1 followed by one from 1 to 0. Only the first of these is a classical shock wave.

We have also included a test where the initial data is periodic, viz.,

$$s(x, 0) = \frac{1}{2}\left(1 + \cos(2\pi x)\right). \tag{5.7}$$

In order to check the possible convergence as $\mu \to 0$, we computed approximations with $N = 25\,000$, and $t \in [0, 1]$. In Figure 5 we show the result in the (x, t) plane for $\mu = 10^{-6}$ and $\mu = 0$. The two solutions are identical until shocks develop at $t \approx 0.05$. At this point the approximation with $\mu = 10^{-6}$ develop two shocks, the slower (and weaker) is an entropy satisfying shock wave, while the faster (and stronger) violates the entropy condition. From the figure it is visible how the characteristics "pass through" the shock. Of course, if $\mu = 0$ the scheme reduces to the Lax-Friedrichs scheme, and the approximation to the right is close to the entropy solution. The small entropic shock wave cannot be a traveling wave solution, whereas the large non-entropic shock wave is, since it is symmetric about $s = 1/2$. This follows from the previous analysis, and can be seen by the trailing oscillations in the small shock, these are absent in the large shock, see Figure 6.

Remark 5.1. *The above simulations clearly indicate that the $\mu \to 0$ limit for the Brinkman regularization results in a non-classical shock (see* [16] *for definition) of the limit conservation law ($s_t + f(s)_x = 0$). Such non-classical shocks in the context of two-phase flows in one-dimensional porous media also arise in the models with dynamic capillary pressure, see* [12, 11, 7]. *It is interesting to observe that non-classical shock waves for two-phase flows can arise with two very different regularization mechanisms, one involving dynamic capillary pressure and one with a Brinkman regularization of the Darcy's law.*

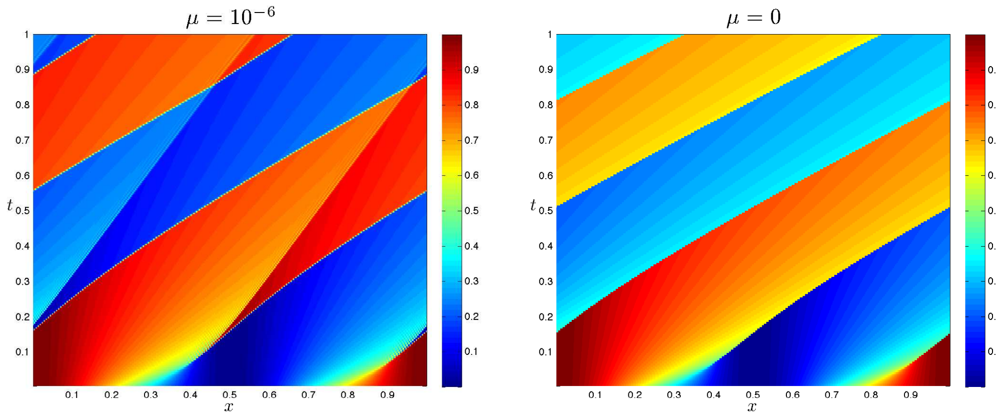

FIGURE 5. Approximations to the solution to (5.7) in the (x, t) plane, left: $\mu = 10^{-6}$, right: $\mu = 0$, $N = 25\,000$.

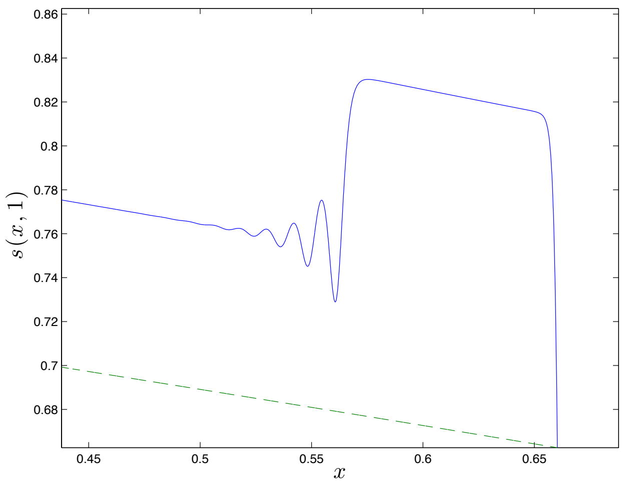

FIGURE 6. Trailing oscillations behind the entropic shock wave.

5.0.3. *Convergence of the scheme in 1D.* In order to substantiate the above one-dimensional numerical calculations, we devote a short section to prove that the scheme (5.5) produces a convergent subsequence. We note that the scheme (5.5) as is different from the two-dimensional finite difference scheme (4.3) – (4.6) for the two-dimensional case as no pressure equations are solved in the one dimensional case.

For ease of notation, we write s and v rather than s^μ and v^μ. A solution to (5.1) is defined as a pair of functions (s, v) such that

$$s \in W^{1,\infty}(0, T; L^2(\mathbb{R})), \qquad v \in L^\infty(0, T; H^2(\mathbb{R})), \tag{5.8}$$

and such that for all test functions $\varphi \in C_0^\infty(\mathbb{R} \times [0, \infty))$,

$$\int_0^\infty \int_{\mathbb{R}} s\varphi_t + v\varphi_x \, dxdt + \int_{\mathbb{R}} s_0(x)\varphi(x, 0)\, dx = 0, \tag{5.9}$$

$$\int_0^T \int_{\mathbb{R}} \mu v_x \varphi_x + v\varphi + f(s)\varphi \, dxdt = 0. \tag{5.10}$$

Using the obvious notation, (5.5) reads

$$\begin{cases} D_t^+ s_j^n + D_c v_j^n = 0, \\ \mu D_+ D_- v_j^n + v_j^n = f(s_j^n). \end{cases} \qquad s_j^0 = s_0(j\Delta x), \text{ for } j \in \mathbb{Z}. \tag{5.11}$$

From the discrete values we define the bilinear interpolant

$$\begin{aligned} s^n(x) &= s_j^n + (x - x_j)\, D_+ s_j^n \quad \text{for } x \in [x_j, x_{j+1}), \\ s_{\Delta x}(x,t) &= s^n(x) + (t - t_n)\, D_t^+ s^n(x) \text{ for } t \in [t_n, t_{n+1}). \end{aligned}$$

Regarding v_j^n, we define $v^n(x)$ to be the piecewise quadratic spline interpolation such that $v^n(x_j) = v_j^n$, and define $v_{\Delta x}(x,t)$ by a linear interpolation in t between t_n and t_{n+1}.

Since this scheme is conservative, it follows that if $s_{\Delta x} \to s$, $v_{\Delta x} \to v$ and $\partial_x v_{\Delta x} \to \partial_x v$ a.e. as $\Delta x \to 0$, then the limits s and v satisfy (5.9) and (5.10) respectively.

In order to show the strong convergence of a subsequence we square the equation for v_j^n and sum over j to find

$$\mu^2 \sum_j \left|D_- D_+ v_j^n\right|^2 + 2\mu \sum_j \left|D_- v_j^n\right|^2 + \sum_j \left|v_j^n\right|^2 = \sum_j \left|f_j^n\right|^2.$$

This means that

$$\left\|\partial_x^2 v_{\Delta x}(\cdot,t)\right\|_{L^2(\mathbb{R})}^2 + \left\|\partial_x v_{\Delta x}(\cdot,t)\right\|_{L^2(\mathbb{R})}^2 + \left\|v_{\Delta x}(\cdot,t)\right\|_{L^2(\mathbb{R})}^2 \le C \left\|f\right\|_{\mathrm{Lip}}^2 \left\|s_{\Delta x}(\cdot,t)\right\|_{L^2(\mathbb{R})}^2, \tag{5.12}$$

for some constant C which does not depend on Δx. Next, we note that

$$\begin{aligned} \left\|s_{\Delta x}(\cdot,t_{n+1})\right\|_{L^2(\mathbb{R})} &\le \left\|s_{\Delta x}(\cdot,t_n)\right\|_{L^2(\mathbb{R})} + C\Delta t \left\|\partial_x v_{\Delta x}(\cdot,t_n)\right\|_{L^2(\mathbb{R})} \\ &\le \left\|s_{\Delta x}(\cdot,t_n)\right\|_{L^2(\mathbb{R})} \left(1 + C\Delta t \left\|f\right\|_{\mathrm{Lip}}\right). \end{aligned}$$

Thus

$$\left\|s_{\Delta x}(\cdot,t)\right\|_{L^2(\mathbb{R})} \le \left\|s_0\right\|_{L^2(\mathbb{R})} e^{Ct}, \tag{5.13}$$

for some constant C which does not depend on Δx (but scales like $1/\mu$). Combining this with (5.12) we find that

$$\left\|v_{\Delta x}(\cdot,t)\right\|_{H^2(\mathbb{R})} \le C_T \tag{5.14}$$

for all $t \le T$. This means that we get a supremum bound on $s_{\Delta x}$, since

$$\left\|\partial_x v_{\Delta x}\right\|_{L^\infty(\mathbb{R})} \le \left\|v_{\Delta x}\right\|_{H^2(\mathbb{R})}.$$

Therefore

$$\left\|s_{\Delta x}(\cdot,t)\right\|_{L^\infty(\mathbb{R})} \le \left\|s_0\right\|_{L^\infty(\mathbb{R})} + tC_T. \tag{5.15}$$

In particular, this implies that we only have to demand that f is *locally* Lipschitz continuous.

Now set $r_j^n = D_t^+ s_j^n$ and $z_j^n = D_t^+ v_j^n$. Then

$$\begin{cases} D_t^+ r_j^n + D_c z_j^n = 0, \\ -\mu D_+ D_- z_j^n + z_j^n = f'\left(s_j^{n+1/2}\right) r_j^n, \end{cases} \qquad n \ge 0,$$

where $s_j^{n+1/2}$ is some value between s_j^n and s_j^{n+1}. The above holds for $n \ge 0$, and we have that

$$r_j^0 = -D_c v_j^0, \text{ or } -\mu D_+ D_- r_j^0 + r_j^0 = -f'\left(\bar{s}_j^0\right) D_c s_j^0,$$

where $\bar{s}_j^0$ is a value between s_{j-1}^0 and s_{j+1}^0. Now we can repeat the above arguments to show that

$$\|\partial_t v_{\Delta x}(\cdot,t)\|_{H^2(\mathbb{R})} \leq C\,\|f\|_{\text{Lip}}\,\|\partial_t s_{\Delta x}(\cdot,t)\|_{L^2(\mathbb{R})}\,, \tag{5.16}$$

$$\|\partial_t s_{\Delta x}(\cdot,t)\|_{L^2(\mathbb{R})} \leq \|f\|_{\text{Lip}}\,\|\partial_x s_0\|_{L^2(\mathbb{R})}\,e^{Ct}. \tag{5.17}$$

Thus, if $s_0 \in H^1(\mathbb{R})$, then $s_{\Delta x} \in \text{Lip}(0,T;L^2(\mathbb{R}))$ and $v_{\Delta x} \in \text{Lip}(0,T;H^2(\mathbb{R}))$, with Lipschitz constants independent of Δx.

Now we need to show the compactness of the two sequences $\{s_{\Delta x}\}_{\Delta x>0}$ and $\{v_{\Delta x}\}_{\Delta x>0}$. Set $\sigma_j^n = D_- s_j^n$ and $w_j^n = D_- v_j^n$, then

$$\begin{cases} D_t^+\sigma_j^n + D_c w_j^n = 0, \\ -\mu D_+ D_- w_j^n + w_j^n = f'\left(s_{j-1/2}^n\right)\sigma_j^n, \end{cases} \qquad n \geq 0,$$

where $s_{j-1/2}^n$ is an intermediate value. The initial values for the above scheme are $\sigma_j^0 = D_- v_j^0$. From this we obtain

$$\|\partial_x v_{\Delta x}(\cdot,t)\|_{H^2(\mathbb{R})} \leq C\,\|f\|_{\text{Lip}}\,\|\partial_x s_{\Delta x}(\cdot,t)\|_{L^2(\mathbb{R})}\,, \tag{5.18}$$

$$\|\partial_x s_{\Delta x}(\cdot,t)\|_{L^2(\mathbb{R})} \leq \|f\|_{\text{Lip}}\,\|\partial_x s_0\|_{L^2(\mathbb{R})}\,e^{Ct}. \tag{5.19}$$

Therefore $\{s_{\Delta x}(\cdot,t)\}_{\Delta x>0} \subset H^1(\mathbb{R}) \subset\subset L^2(\mathbb{R})$ and $\{v_{\Delta x}(\cdot,t)\}_{\Delta x>0} \subset H^3(\mathbb{R}) \subset\subset H^2(\mathbb{R})$ uniformly in t and Δx.

To sum up, we have proved

Lemma 5.1. *Assume that $s_0 \in H^1(\mathbb{R})$ and that $s_{\Delta x}$ and $v_{\Delta x}$ are defined by* (5.11). *Then there are functions s and v that are weak solutions to* (5.1), *defined by* (5.8), (5.9) *and* (5.10). *We have that*

$$\left.\begin{aligned} s_{\Delta x}(\cdot,t) &\to s(\cdot,t) \quad \text{in } L^2(\mathbb{R}), \\ v_{\Delta x}(\cdot,t) &\to v(\cdot,t) \quad \text{in } H^2(\mathbb{R}), \end{aligned}\right\} \text{along a subsequence}$$

for all $t \in [0,T]$.

6. Conclusion

Two-phase flows in a porous medium is modeled by a hyperbolic equation for the saturation, coupled with an elliptic equation for the pressure, resulting in the classical Darcy's law based equations (1.9). No existence results for the equations have been obtained till date in spite of the extensive research on these equations over the past several decades. One of the pressing issues in this context has been whether the Darcy's law is an adequate and appropriate model for flows in porous media. The Brinkman regularization of the Darcy's law [4] has been a popular alternative ([15] and references therein) for the Darcy's law in the geophysics community, atleast in the context of a single phase flow. It is natural to examine whether the Brinkman regularization is an appropriate model, also in the context of two- (and multi-) phase flows in porous media.

In this paper, we consider the Brinkman regularization of the two-phase flow equations (1.12). A suitable notion of weak solutions for these equations is proposed. We prove that these weak solutions exist. Furthermore, a simple finite difference scheme to approximate this system (1.12) is proposed and is shown to converge to the weak solutions. Numerical experiments indicate robust performance of this numerical scheme, for fixed regularization parameter μ.

Formally, we can recover the classical two-phase flow equations (1.9) by setting the regularization parameter $\mu \to 0$ in the Brinkman regularization (1.12). However, our stability estimates on the saturation and the velocity blow up as $\mu \to 0$ preventing us from rigorously showing that the limit solution of the Brinkman regularization is a weak solution of the classical Darcy problem. We investigate this question numerically using our convergent numerical scheme. Results on a benchmark quarter five-spot problem in two space dimensions show that the approximate solutions to the Brinkman regularization can become quite oscillatory as $\mu \to 0$. Furthermore, the regularized system can contain discontinuous fronts connecting full water saturation to zero water saturation. Such solutions are not included as classical entropy solutions of the Darcy problem (1.12). Hence, the numerical results indicate that the Brinkman regularization may not converge to (entropy solutions of) the Darcy limit as $\mu \to 0$.

This proposition is further investigated in the special case of one space dimension. In this case, the pressure equation is trivially solved and the saturation is modeled by a scalar conservation law. Entropy solutions (obeying Lax type entropy conditions) are widely recognized as the physically relevant solutions in this context. However, we establish using traveling wave analysis that the Brinkman limit will lead to a non-classical shock wave for the scalar conservation law. Such non-entropic solutions have been postulated for other physical models such as dynamic capillary pressure models [12, 11]. The presence of non-classical shocks for the Brinkman limit raise interesting questions, see also [9].

Summarizing, the Brinkman regularization does provide a model where existence of weak solutions can be shown rigorously and convergent numerical schemes can also be designed. Such existence and convergence results have not been possible for the Darcy problem despite several attempts. On the other hand, the Brinkman regularization may lead to limit solutions of the Darcy's equation that are not entropic and may contain non-classical shock waves. Furthermore, the question of rigorous passage to the Darcy limit for the Brinkman regularization is still wide open. Hence, this paper advocates caution in the use of Brinkman type models, atleast for two and multi-phase flows in porous media.

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